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Letter to the Editor

A Note on Strong Convergence of a Modified Halpern's Iteration for Nonexpansive Mappings

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In the paper by Hu in 2008, the author proved a strong convergence result for nonexpansive mappings using a modified Halpern's iteration algorithm. Unfortunately, the case $\lim_{n\to\infty}\beta_n = 1$ does not guarantee the strong convergence of the sequence $\{x_n\}$. In this note, we provide a counter-example to the theorem.

In [1], the author introduced a modified Halpern's iteration. For any $u, x_0 \in C$, the sequence $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T x_n, \quad n \geq 0, \tag{I}$$

where $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are three real sequences in $(0,1)$, satisfying $\alpha_n + \beta_n + \gamma_n = 1$. The author proved the following strong convergence theorem.

Theorem 1 (see [1]). *Let C be a nonempty closed convex subset of a real Banach space E which has a uniformly Gâteaux differentiable norm. Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Assume that $\{z_t\}$ converges strongly to a fixed point z of T as $t \to 0$, where z_t is the unique element of C which satisfies $z_t = tu + (1-t)Tz_t$ for any $u \in C$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three real sequences in $(0,1)$ which satisfy the following conditions: (C1) $\lim_{n\to\infty}\alpha_n = 0$ and (C2) $\sum_{n=0}^{\infty}\alpha_n = +\infty$. For any $x_0 \in C$, the sequence $\{x_n\}$ is defined by the iteration in (I). Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Counter Example

Let E be a real Banach space whose norm is uniformly Gâteaux differentiable. Let C be a nonempty closed and convex subset of E, defined by

$$C = \{x \in E : x = \lambda y,\ \lambda \in [0,3]\}, \tag{1}$$

where $y \neq 0$, with $\|y\| = 1$ a fixed element of E. Let $T : C \to C$ be a mapping defined by $Tx = 0$ for all $x \in C$. It is obvious that T is a nonexpansive mapping and $\mathrm{Fix}(T) = \{0\}$. Take $\alpha_n = 1/(n+2)$, $\beta_n = 1 - 2/(n+2)$, and $\gamma_n = 1/(n+2)$ for all $n \geq 0$ and $x_0 = y$, $u = 3y$. We also can obtain that $z_t = 3ty \to 0$ $(t \to 0)$. Observe that all conditions of Theorem 1 are satisfied. However, the iterative sequence $\{x_n\}$ does not converge strongly to the fixed point $z = 0$ of T.

Claim 1. If $\|x_n\| \leq 1$, then $\|x_{n+1}\| > \|x_n\|$.

Proof. In fact, we have

$$\begin{aligned} x_{n+1} &= \frac{1}{n+2}3y + \left(1 - \frac{2}{n+2}\right)x_n + \frac{1}{n+2}Tx_n \\ &= \frac{3}{n+2}y + \left(1 - \frac{2}{n+2}\right)x_n \\ &= \frac{3}{n+2}y + \left(1 - \frac{2}{n+2}\right)\lambda_n y, \end{aligned} \tag{2}$$

where x_n can be denoted as $x_n = \lambda_n y$. If $\|x_n\| \leq 1$, then $0 < \lambda_n = \|x_n\| \leq 1$. From the above equality we have

$$\begin{aligned} \|x_{n+1}\| &= \left\| \left[\frac{3}{n+2} + \left(1 - \frac{2}{n+2}\right)\lambda_n\right] y \right\| \\ &= \frac{3}{n+2} + \left(1 - \frac{2}{n+2}\right)\lambda_n \\ &= \frac{2}{n+2}(1 - \lambda_n) + \frac{1}{n+2} + \lambda_n \\ &> \lambda_n = \|x_n\|. \end{aligned} \tag{3}$$

Hence $\{x_n\}$ does not converge strongly to $z = 0$. □

Remark 1. Why does the proof of Theorem 1 fail? It is not difficult to check that the proof of Case 2 ($\lim_{n\to\infty}\beta_n = 1$) is not suitable.

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References

[1] L.-G. Hu, "Strong convergence of a modified Halpern's iteration for nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2008, Article ID 649162, 9 pages, 2008.