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Research Article

Algorithm for Solving a Generalized Mixed Equilibrium Problem with Perturbation in a Banach Space

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Let B be a real Banach space with the dual space B^*. Let $\phi : B \to \mathbf{R} \cup \{+\infty\}$ be a proper functional and let $\Theta : B \times B \to \mathbf{R}$ be a bifunction. In this paper, a new concept of η-proximal mapping of ϕ with respect to Θ is introduced. The existence and Lipschitz continuity of the η-proximal mapping of ϕ with respect to Θ are proved. By using properties of the η-proximal mapping of ϕ with respect to Θ, a generalized mixed equilibrium problem with perturbation (for short, GMEPP) is introduced and studied in Banach space B. An existence theorem of solutions of the GMEPP is established and a new iterative algorithm for computing approximate solutions of the GMEPP is suggested. The strong convergence criteria of the iterative sequence generated by the new algorithm are established in a uniformly smooth Banach space B, and the weak convergence criteria of the iterative sequence generated by this new algorithm are also derived in $B = H$ a Hilbert space.

1. Introduction

Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be its dual space. The value of $f \in B^*$ at $x \in B$ will be denoted by $\langle f, x\rangle$. The normalized duality mapping J from B into the family of nonempty (by Hahn-Banach theorem) weak-star compact subsets of its dual space B^* is defined by

$$J(x) = \left\{ f \in B^* : \langle f, x\rangle = \|f\|^2 = \|x\|^2 \right\}, \quad \forall x \in B. \tag{1.1}$$

It is known that the norm of B is said to be Gateaux differentiable (and B is said to be smooth) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.2}$$

exists for every x, y in $U = \{x \in B : \|x\| = 1\}$, the unit sphere of B. It is said to be uniformly Gateaux differentiable if for each $y \in U$, this limit is attained uniformly for $x \in U$. The norm of B is said to be uniformly Frechet differentiable (and B is said to be uniformly smooth) if the limit in (1.2) is attained uniformly for $(x, y) \in U \times U$. Every uniformly smooth Banach space B is reflexive and has a uniformly Gateaux differentiable norm.

Recall also that if B is smooth, then J is single-valued and continuous from the norm topology of B to the weak star topology of B^*, that is, norm-to-weak* continuous. It is also well known that if B has a uniformly Gateaux differentiable norm, then J is uniformly continuous on bounded subsets of B from the strong topology of B to the weak star topology of B^*, that is, uniformly norm-to-weak* continuous on any bounded subset of B. Moreover, if B is uniformly smooth, then J is uniformly continuous on bounded subsets of B from the strong topology of B to the strong topology of B^*, that is, uniformly norm-to-norm continuous on any bounded subset of B. See [1] for more details.

It is well known that the variational inequality theory has played an important and powerful role in the studying of a wide class of linear and nonlinear problems arising in many diverse fields of pure and applied sciences, such as mathematical programming, optimization theory, engineering, elasticity theory and equilibrium problems of mathematical economics, and game theory; see, for instance, [1–6] and the references therein.

One of the most interesting and important problems in the theory of variational inequalities is the development of an efficient iterative algorithm to compute approximate solutions. In the setting of Hilbert spaces, one of the most efficient numerical techniques is the projection method and its variant forms; see [4, 6–15]. Since the standard projection method strictly depends on the inner product property of Hilbert spaces, it can no longer be applied for general mixed type variational inequalities in Banach spaces. This fact motivates us to develop alterative methods to study the existence and iterative algorithms of solutions for general mixed variational inequalities in Banach spaces. Recently, [16–18] extended the auxiliary principle technique to study the existence of solutions and to suggest the iterative algorithms for solving various mixed type variational inequalities in Banach spaces. For some related work, we refer to [19–21] and the references therein.

Very recently, inspired by the research work going on in this field, Xia and Huang [14] first introduced a new concept of η-proximal mapping for a proper subdifferentiable functional on a Banach space. They proved an existence theorem and Lipschitz continuity of the η-proximal mapping. Using the properties of the η-proximal mapping, they proved an existence theorem of solutions for a new class of general mixed variational inequalities in a Banach space and suggested an iterative algorithm for computing approximate solutions. Moreover, they gave the strong convergence criteria of the iterative sequence generated by this algorithm. Their results include some known results in [8, 9, 11, 12, 16–18] as special cases.

Let B be a real Banach space with the dual space B^*. Let $\phi : B \to \mathbf{R} \cup \{+\infty\}$ be a proper functional and let $\Theta : B \times B \to \mathbf{R}$ be a bifunction. In this paper, motivated by Xia and Huang [14], we first introduce a new concept of η-proximal mapping of ϕ with respect to Θ. We prove an existence theorem and Lipschitz continuity of the η-proximal mapping of ϕ with respect to Θ. Utilizing the properties of the η-proximal mapping of ϕ with respect to Θ, we introduce and consider a generalized mixed equilibrium problem with perturbation (for short, GMEPP) which includes as a special case the general mixed variational inequality studied by Xia and Huang [14]. We show an existence theorem of solutions for this problem under some appropriate conditions. In order to compute approximate solutions of the GMEPP, we propose a new iterative algorithm which includes as a special case the iterative algorithm considered by Xia and Huang [14]. Finally, we establish the strong convergence criteria of the iterative sequence generated by the new algorithm in a uniformly smooth Banach space B, and also derive the weak convergence criteria of the iterative sequence generated by this new algorithm in $B = H$ a Hilbert space. Our results are new and represent the improvement, extension, and development of Xia and Huang's results in [14].

2. Preliminaries

Let B be a real Banach space with the topological dual space B^* and let $\langle f, x\rangle$ be the pairing between $f \in B^*$ and $x \in B$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. Let 2^{B^*} and $CB(B^*)$ denote the family of all subsets of B^* and the family of all nonempty closed bounded subsets of B^*, respectively. Let $\Theta : B \times B \to \mathbf{R}$ be a bifunction, let $T, A : B \to B^*$ and $g : B \to B$ be single-valued mappings, and let $\phi : B \to \mathbf{R}\cup\{+\infty\}$ be a proper lower semicontinuous functional. We consider the following generalized mixed equilibrium problem with perturbation (for short, GMEPP): find $u \in B$ such that

$$\Theta(g(u), v) + \phi(v) - \phi(g(u)) + \langle Tu - Au, v - g(u)\rangle \geq 0, \quad \forall v \in B. \tag{2.1}$$

Some special cases of problem (2.1) are the following.

(1) If $B = H$ a real Hilbert space, $g = I$ an identity mapping on H, and ϕ is a lower semicontinuous and convex functional, then GMEPP (2.1) reduces to the generalized mixed equilibrium problem with perturbation considered by Ceng et al. [22].

(2) If $B = H$ a real Hilbert space, $g = I$ an identity mapping on H, $A = 0$, and ϕ is a lower semicontinuous and convex functional, then GMEPP (2.1) reduces to the generalized mixed equilibrium problem considered by Peng and Yao [23].

(3) If $\Theta = 0$, then GMEPP (2.1) reduces to the general mixed variational inequality problem (for short, GMVIP) considered by Xia and Huang [14].

(4) If $B = H$ a real Hilbert space, $\Theta = 0$, and ϕ is a proper convex lower semicontinuous functional, then GMEPP (2.1) was studied by many authors (see, e.g., [1, 17–19]).

We first recall the following definitions and some known results.

Definition 2.1. Let $T : B \to 2^{B^*}$ be a set-valued mapping, and let $A : B \to B^*$ and $g : B \to B$ be two single-valued mappings. We say that

(i) A is α-strongly monotone with constant $\alpha > 0$ if, for any $x, y \in B$,

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2; \tag{2.2}$$

(ii) T is λ-strongly monotone if, for any $x, y \in B, u \in Tx$, and $v \in Ty$,

$$\langle u - v, x - y \rangle \geq \lambda \|x - y\|^2; \tag{2.3}$$

(iii) T is β-Lipschitz continuous with constant $\beta > 0$ if, for all $x, y \in B$,

$$H(Tx, Ty) \leq \beta \|x - y\|, \tag{2.4}$$

where $H(\cdot, \cdot)$ is the Hausdorff metric on 2^{B^*};

(iv) g is k-strongly accretive (where $k \in (0,1)$) if, for any $x, y \in B$, there exists $j(x-y) \in J(x - y)$ such that

$$\langle j(x - y), g(x) - g(y) \rangle \geq k \|x - y\|^2, \tag{2.5}$$

where $J : B \to 2^{B^*}$ is the normalized duality mapping defined by

$$J(x) = \left\{ f \in B^* : \langle f, x \rangle = \|f\| \cdot \|x\| = \|f\|^2 = \|x\|^2 \right\}, \quad \forall x \in B. \tag{2.6}$$

Definition 2.2. Let B be a Banach space with the dual space B^*, let $\Theta : B \times B \to \mathbf{R}$ be a bifunction, and let $\phi : B \to \mathbf{R} \cup \{+\infty\}$ be a proper functional. If, for $x \in B$, there exists a $f \in B^*$ such that

$$\Theta(x, y) + \phi(y) - \phi(x) \geq \langle f, y - x \rangle, \quad \forall y \in B, \tag{2.7}$$

then ϕ is said to be Θ-subdifferentiable at x. We denote by $\partial_\Theta \phi(x)$ the set of such elements $f \in B^*$, that is,

$$\partial_\Theta \phi(x) = \{ f \in B^* : \Theta(x, y) + \phi(y) - \phi(x) \geq \langle f, y - x \rangle, \ \forall y \in B \}. \tag{2.8}$$

The set $\partial_\Theta \phi(x)$ is said to be the Θ-subdifferential of ϕ at x. If there exists the Θ-subdifferential $\partial_\Theta \phi(x)$ at each $x \in B$, then ϕ is said to be Θ-subdifferentiable. The mapping $\partial_\Theta \phi : B \to 2^{B^*}$ is said to be the Θ-subdifferential of ϕ.

Definition 2.3. Let B be a Banach space with the dual space B^*, and let $\Theta : B \times B \to \mathbf{R}, \eta : B \to B^*, \phi : B \to \mathbf{R} \cup \{+\infty\}$ be a proper subdifferentiable functional. If for any given $x^* \in B^*$ and any constant $\rho > 0$, there is a unique $x \in B$ satisfying

$$\Theta(x, y) + \phi(y) - \phi(x) + \frac{1}{\rho}\langle \eta x - x^*, y - x \rangle \geq 0, \quad \forall y \in B, \tag{2.9}$$

then the mapping $x^* \mapsto x$, denoted by $x = J_\rho^{(\Theta,\phi)}(x^*)$, is said to be an η-proximal mapping of ϕ with respect to Θ.

Remark 2.4. From Definitions 2.2 and 2.3 it follows that ϕ is Θ-subdifferentiable at each $x \in R(J_\rho^{(\Theta,\phi)})$ the range of $J_\rho^{(\Theta,\phi)}$. If ϕ is additionally Θ-subdifferentiable at each $x \in B \setminus R(J_\rho^{(\Theta,\phi)})$, then there exists the Θ-subdifferential $\partial_\Theta \phi(x)$ at each $x \in B$; that is, ϕ is Θ-subdifferentiable. Observe that $x^* - \eta x \in \rho \partial_\Theta \phi(x)$ and so

$$x = J_\rho^{(\Theta,\phi)}(x^*) = (\eta + \rho \partial_\Theta \phi)^{-1}(x^*). \tag{2.10}$$

In particular, whenever $\Theta = 0$, then the concept of η-proximal mapping of ϕ with respect to Θ reduces to the one of η-proximal mapping of ϕ by Xia and Huang [14, Definition 2.2]. In this case, $J_\rho^{(\Theta,\phi)}$ is rewritten as J_ρ^ϕ. By the definition of the subdifferential, we know that $x^* - \eta x \in \rho \partial \phi(x)$ and so

$$x = J_\rho^\phi(x^*) = (\eta + \rho \partial \phi)^{-1}(x^*). \tag{2.11}$$

Lemma 2.5 (see [24]). *Let D be a nonempty convex subset of a topological vector space and let $f : D \times D \to [-\infty, +\infty]$ be such that*

(i) *for each $x \in D, y \mapsto f(x, y)$ is lower semicontinuous on each nonempty compact subset of D;*

(ii) *for each nonempty finite set $\{x_1, \ldots, x_m\} \subset D$ and for each $y = \sum_{i=1}^m \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$,*

$$\min_{1 \leq i \leq m} f(x_i, y) \leq 0; \tag{2.12}$$

(iii) *there exist a nonempty compact convex subset D_0 of D and a nonempty compact subset K of D such that for each $y \in D \setminus K$, there is an $x \in \mathrm{co}(D_0 \cup \{y\})$ with $f(x, y) > 0$.*

Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \leq 0$ for all $x \in D$.

Recall now that B satisfies Opial's property [25] provided that, for each sequence $\{x_n\}$ in B, the condition $x_n \rightharpoonup x$ implies

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|, \quad \forall y \in X, \ y \neq x. \tag{2.13}$$

It is known [25] that each l^p $(1 \leq p < \infty)$ enjoys this property, while L^p does not unless $p = 2$.

It is known [26] that any separable Banach space can be equivalently renormed so that it satisfies Opial's property.

Furthermore, recall that in a Hilbert space, there holds the following equality:

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.14}$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

In order to investigate the generalized mixed equilibrium problem (2.1) with perturbation, we will need the following conditions on mapping $\eta : B \to B^*$ and bifunction $\Theta : B \times B \to \mathbf{R}$ in the sequel:

(H1) $\Theta(x,x) = 0$, for all $x \in B$;

(H2) Θ is monotone, that is, $\Theta(x,y) + \Theta(y,x) \leq 0$, for all $x, y \in B$;

(H3) for each $x \in B, y \mapsto \Theta(x,y)$ is convex and lower semicontinuous;

(H4) the following equilibrium problem (for short, EP) or generalized equilibrium problem (for short, GEP) has a solution $\hat{y}$ in $\operatorname{dom}\phi$:

(EP) Find $\hat{y} \in B$ such that $\Theta(\hat{y}, x) \geq 0$, for all $x \in B$, or

(GEP) Find $\hat{y} \in B$ such that $\Theta(\hat{y}, x) + \langle \eta\hat{y}, x - \hat{y}\rangle \geq 0$, for all $x \in B$.

Now we give some sufficient conditions which guarantee the existence and Lipschitz continuity of an η-proximal mapping of ϕ with respect to Θ in a reflexive Banach space B.

Theorem 2.6. *Let B be a reflexive Banach space with the dual space B^*, let $\Theta : B \times B \to \mathbf{R}$ be a bifunction satisfying conditions (H1)–(H4), let $\eta : B \to B^*$ be an α-strongly monotone and continuous mapping, and let $\phi : B \to \mathbf{R} \cup \{+\infty\}$ be a lower semicontinuous subdifferentiable proper functional. Then for any given $x^* \in B^*$ and any $\rho > 0$, there exists a unique $\hat{x} \in B$ such that*

$$\Theta(\hat{x}, y) + \phi(y) - \phi(\hat{x}) + \frac{1}{\rho}\langle \eta\hat{x} - x^*, y - \hat{x}\rangle \geq 0, \quad \forall y \in B; \tag{2.15}$$

that is, $\hat{x} = J_\rho^{(\Theta,\phi)}(x^*)$ and the η-proximal mapping of ϕ with respect to Θ is well defined. If ϕ is additionally Θ-subdifferentiable at each $x \in B \setminus R(J_\rho^{(\Theta,\phi)})$, then ϕ is Θ-subdifferentiable and $J_\rho^{(\Theta,\phi)} = (\eta + \rho\partial_\Theta\phi)^{-1}$.

Proof. For any given $x^* \in B^*$ and any $\rho > 0$, define a functional $f : B \times B \to \mathbf{R} \cup \{+\infty\}$ as follows:

$$f(y,x) = \langle x^* - \eta x, y - x\rangle + \rho\phi(x) - \rho\phi(y) + \rho\Theta(y,x), \quad \forall x, y \in B. \tag{2.16}$$

By the continuity of η and the lower semicontinuity of ϕ and $x \mapsto \Theta(y,x)$, we know that the function $x \mapsto f(y,x)$ is lower semicontinuous on B for each fixed $y \in B$.

Now, let us show that $f(y,x)$ satisfies condition (ii) of Lemma 2.5. If it is false, then there exist a finite set $\{y_1, \ldots, y_m\} \subset B$ and $x_0 = \sum_{i=1}^m \lambda_i y_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$ such that

$$\langle x^* - \eta x_0, y_i - x_0\rangle + \rho\phi(x_0) - \rho\phi(y_i) + \rho\Theta(y_i, x_0) > 0, \quad \forall i = 1, \ldots, m. \tag{2.17}$$

Since ϕ is subdifferentiable at x_0, there exists a point $f^* \in B^*$ such that

$$\rho\phi(y_i) - \rho\phi(x_0) \geq \rho\langle f^*, y_i - x_0\rangle, \quad \forall i = 1, \ldots, m. \tag{2.18}$$

From (H2) it follows that

$$\Theta(x_0, y_i) \leq -\Theta(y_i, x_0), \quad \forall i = 1, \ldots, m. \tag{2.19}$$

Utilizing the convexity of $x \mapsto \Theta(y, x)$, we get from (H1)

$$\begin{aligned}
0 &= \langle x^* - \eta x_0 + \rho f^*, x_0 - x_0\rangle + \rho\Theta(x_0, x_0) \\
&= \sum_{i=1}^{m}\lambda_i\left[\langle x^* - \eta x_0, x_0 - y_i\rangle + \rho\langle f^*, y_i - x_0\rangle\right] + \rho\Theta\left(x_0, \sum_{i=1}^{m}\lambda_i y_i\right) \\
&\leq \sum_{i=1}^{m}\lambda_i\left[\langle x^* - \eta x_0, x_0 - y_i\rangle + \rho\langle f^*, y_i - x_0\rangle + \rho\Theta(x_0, y_i)\right] \\
&\leq \sum_{i=1}^{m}\lambda_i\left[\langle x^* - \eta x_0, x_0 - y_i\rangle + \rho\phi(y_i) - \rho\phi(x_0) - \rho\Theta(y_i, x_0)\right] \\
&< 0,
\end{aligned} \tag{2.20}$$

which leads to a contradiction. Therefore, $f(y, x)$ satisfies condition (ii) of Lemma 2.5.

From (H4) we know that the following equilibrium problem (for short, EP) or generalized equilibrium problem (for short, GEP) has a solution $\hat{y} \in \operatorname{dom} \phi$:

(EP) $\Theta(\hat{y}, x) \geq 0$, for all $x \in B$, or

(GEP) $\Theta(\hat{y}, x) + \langle \eta\hat{y}, x - \hat{y}\rangle \geq 0$, for all $x \in B$.

Since ϕ is subdifferentiable at $\hat{y}$, there exists a point $f^* \in B^*$ such that

$$\phi(x) - \phi(\hat{y}) \geq \langle f^*, x - \hat{y}\rangle, \quad \forall x \in B. \tag{2.21}$$

It follows that

$$\begin{aligned}
f(\hat{y}, x) &= \langle x^* - \eta x, \hat{y} - x\rangle + \rho\phi(x) - \rho\phi(\hat{y}) + \rho\Theta(\hat{y}, x) \\
&\geq \langle \eta\hat{y} - \eta x, \hat{y} - x\rangle + \langle x^* - \eta\hat{y}, \hat{y} - x\rangle + \rho\langle f^*, x - \hat{y}\rangle + \rho\Theta(\hat{y}, x).
\end{aligned} \tag{2.22}$$

Next, we discuss two cases.

Case 1. If EP has solution $\widehat{y} \in \operatorname{dom}\phi$, then from (2.22) we have

$$\begin{aligned} f(\widehat{y},x) &= \langle x^* - \eta x, \widehat{y} - x\rangle + \rho\phi(x) - \rho\phi(\widehat{y}) + \rho\Theta(\widehat{y},x) \\ &\geq \langle \eta\widehat{y} - \eta x, \widehat{y} - x\rangle + \langle x^* - \eta\widehat{y}, \widehat{y} - x\rangle + \rho\langle f^*, x - \widehat{y}\rangle \\ &\geq \alpha\|\widehat{y} - x\|^2 - \left(\|x^*\| + \|\eta\widehat{y}\| + \rho\|f^*\|\right)\|\widehat{y} - x\| \\ &= \|\widehat{y} - x\|\left[\alpha\|\widehat{y} - x\| - \left(\|x^*\| + \|\eta\widehat{y}\| + \rho\|f^*\|\right)\right] \end{aligned} \tag{2.23}$$

Let

$$r = \frac{1}{\alpha}\left(\|x^*\| + \|\eta\widehat{y}\| + \rho\|f^*\|\right), \qquad K = \left\{x \in B : \|\widehat{y} - x\| \leq r\right\}. \tag{2.24}$$

Then $D_0 = \{\widehat{y}\}$ and K are both weakly compact convex subset of B. For each $x \in B \setminus K$, there exists a point $\widehat{y} \in \operatorname{co}(D_0 \cup \{x\})$ such that $f(\widehat{y},x) > 0$ and so all conditions of Lemma 2.5 are satisfied. By Lemma 2.5, there exists a point $\widehat{x} \in B$ such that $f(y,\widehat{x}) \leq 0$ for all $y \in B$, that is,

$$\langle \eta x - \widehat{x}, y - \widehat{x}\rangle + \rho\phi(y) - \rho\phi(\widehat{x}) + \rho\Theta(\widehat{x},y) \geq 0, \quad \forall y \in B. \tag{2.25}$$

Case 2. If GEP has solution $\widehat{y} \in \operatorname{dom}\phi$, then from (2.22) we have

$$\begin{aligned} f(\widehat{y},x) &= \langle x^* - \eta x, \widehat{y} - x\rangle + \rho\phi(x) - \rho\phi(\widehat{y}) + \rho\Theta(\widehat{y},x) \\ &\geq \langle \eta\widehat{y} - \eta x, \widehat{y} - x\rangle + \langle x^* - \eta\widehat{y}, \widehat{y} - x\rangle + \rho\langle f^*, x - \widehat{y}\rangle + \rho\Theta(\widehat{y},x) \\ &\geq \langle \eta\widehat{y} - \eta x, \widehat{y} - x\rangle + \langle x^*, \widehat{y} - x\rangle + \rho\langle f^*, x - \widehat{y}\rangle \\ &\geq \alpha\|\widehat{y} - x\|^2 - \left(\|x^*\| + \rho\|f^*\|\right)\|\widehat{y} - x\| \\ &= \|\widehat{y} - x\|\left[\alpha\|\widehat{y} - x\| - \left(\|x^*\| + \rho\|f^*\|\right)\right]. \end{aligned} \tag{2.26}$$

Let

$$r = \frac{1}{\alpha}\left(\|x^*\| + \rho\|f^*\|\right), \qquad K = \left\{x \in B : \|\widehat{y} - x\| \leq r\right\}. \tag{2.27}$$

Then $D_0 = \{\widehat{y}\}$ and K are both weakly compact convex subset of B. For each $x \in B \setminus K$, there exists a point $\widehat{y} \in \operatorname{co}(D_0 \cup \{x\})$ such that $f(\widehat{y},x) > 0$ and so all conditions of Lemma 2.5 are satisfied. By Lemma 2.5, there exists a point $\widehat{x} \in B$ such that $f(y,\widehat{x}) \leq 0$ for all $y \in$ B, that is, (2.25) holds.

Now let us show that $\hat{x}$ is a unique solution of auxiliary equilibrium problem (2.15). Suppose that $x_1, x_2 \in B$ are arbitrary two solutions of auxiliary equilibrium problem (2.15). Then,

$$\Theta(x_1, y) + \phi(y) - \phi(x_1) + \frac{1}{\rho}\langle \eta x_1 - x^*, y - x_1 \rangle \geq 0, \quad \forall y \in B, \tag{2.28}$$

$$\Theta(x_2, y) + \phi(y) - \phi(x_2) + \frac{1}{\rho}\langle \eta x_2 - x^*, y - x_2 \rangle \geq 0, \quad \forall y \in B. \tag{2.29}$$

Taking $y = x_2$ in (2.28) and $y = x_1$ in (2.29) and adding these inequalities we have from the monotonicity of Θ

$$-\frac{1}{\rho}\langle \eta x_1 - \eta x_2, x_1 - x_2 \rangle \geq \Theta(x_1, x_2) + \Theta(x_2, x_1) - \frac{1}{\rho}\langle \eta x_1 - \eta x_2, x_1 - x_2 \rangle \geq 0. \tag{2.30}$$

This together with the α-strong monotonicity of η implies that

$$\alpha\|x_1 - x_2\|^2 \leq \langle \eta x_1 - \eta x_2, x_1 - x_2 \rangle \leq 0, \tag{2.31}$$

and so $x_1 = x_2$. Therefore, $\hat{x} = J_\rho^{(\Theta,\phi)}(x^*)$ and the η-proximal mapping of ϕ with respect to Θ is well defined. In the meantime, it is known that ϕ is Θ-subdifferentiable at each $x \in R(J_\rho^{(\Theta,\phi)})$. If ϕ is additionally Θ-subdifferentiable at each $x \in B \setminus R(J_\rho^{(\Theta,\phi)})$, then by Remark 2.4 we obtain that ϕ is Θ-subdifferentiable and $J_\rho^{(\Theta,\phi)} = (\eta + \rho\partial_\Theta\phi)^{-1}$. This completes the proof. □

Corollary 2.7 (see [14, Theorem 2.1]). *Let B be a reflexive Banach space with the dual space B^*, let $\phi : B \to \mathbf{R} \cup \{+\infty\}$ be a lower semicontinuous subdifferentiable proper functional, and let $\eta : B \to B^*$ be an α-strongly monotone and continuous mapping. Then for any given $x^* \in B^*$ and any $\rho > 0$, there exists a unique $x \in B$ such that*

$$\langle \eta x - x^*, y - x \rangle + \rho\phi(y) - \rho\phi(x) \geq 0, \quad \forall y \in B; \tag{2.32}$$

that is, $x = J_\rho^{\phi}(x^*)$ and the η-proximal mapping of ϕ is well defined.

Proof. Putting $\Theta = 0$ in Theorem 2.6, we obtain the desired result. □

Remark 2.8. Theorem 2.6 shows that if $\Theta : B \times B \to \mathbf{R}$ is a bifunction satisfying conditions (H1)–(H4) and $\phi : B \to \mathbf{R} \cup \{+\infty\}$ is a lower semicontinuous subdifferentiable proper functional, then for any strongly monotone and continuous mapping $\eta : B \to B^*$, the η-proximal mapping $J_\rho^{(\Theta,\phi)} : B^* \to B$ of ϕ with respect to Θ is well defined. Furthermore, if ϕ is additionally Θ-subdifferentiable at each $x \in B \setminus R(J_\rho^{(\Theta,\phi)})$, then for each $x^* \in B^*$

$$J_\rho^{(\Theta,\phi)}(x^*) = (\eta + \rho\partial_\Theta\phi)^{-1}(x^*) \tag{2.33}$$

is the unique solution of auxiliary equilibrium problem (2.15).

Theorem 2.9. *Let B be a reflexive Banach space with the dual space B^*, let $\eta : B \to B^*$ be an α-strongly monotone and continuous mapping, let $\Theta : B \times B \to \mathbf{R}$ be a bifunction satisfying conditions (H1)–(H4), and let $\phi : B \to \mathbf{R} \cup \{+\infty\}$ be a lower semicontinuous subdifferentiable proper functional. If ϕ is Θ-subdifferentiable at each $x \in B \setminus R(J_\rho^{(\Theta,\phi)})$, then ϕ is Θ-subdifferentiable and the η-proximal mapping $J_\rho^{(\Theta,\phi)} = (\eta + \rho\partial_\Theta\phi)^{-1}$ of ϕ with respect to Θ is $1/\alpha$-Lipschitz continuous. Furthermore, if additionally the Θ-subdifferential $\partial_\Theta\phi : B \to 2^{B^*}$ for ϕ is ξ-strongly monotone, then the η-proximal mapping $J_\rho^{(\Theta,\phi)} = (\eta + \rho\partial_\Theta\phi)^{-1}$ of ϕ with respect to Θ is $1/(\alpha + \rho\xi)$-Lipschitz continuous.*

Proof. First, utilizing Theorem 2.6 we know that whenever ϕ is Θ-subdifferentiable at each $x \in B \setminus R(J_\rho^{(\Theta,\phi)})$, ϕ is Θ-subdifferentiable. For any $x^*, y^* \in B^*$, let $x = J_\rho^{(\Theta,\phi)}(x^*)$ and $y = J_\rho^{(\Theta,\phi)}(y^*)$. Then $x^* - \eta x \in \rho\partial_\Theta\phi(x)$ and $y^* - \eta y \in \rho\partial_\Theta\phi(y)$. By the definition of the Θ-subdifferential of ϕ, we have

$$\Theta(x,u) + \phi(u) - \phi(x) + \frac{1}{\rho}\langle \eta x - x^*, u - x\rangle \geq 0, \quad \forall u \in B, \tag{2.34}$$

$$\Theta(y,u) + \phi(u) - \phi(y) + \frac{1}{\rho}\langle \eta y - y^*, u - y\rangle \geq 0, \quad \forall u \in B. \tag{2.35}$$

Taking $u = y$ in (2.34) and $u = x$ in (2.35) and adding these inequalities, we obtain

$$\frac{1}{\rho}\langle \eta x - \eta y, x - y\rangle \leq \frac{1}{\rho}\langle x^* - y^*, x - y\rangle + \Theta(x,y) + \Theta(y,x). \tag{2.36}$$

Utilizing condition (H2) we get $\Theta(x,y) + \Theta(y,x) \leq 0$ and hence

$$\langle \eta x - \eta y, x - y\rangle \leq \langle x^* - y^*, x - y\rangle. \tag{2.37}$$

Since η is α-strongly monotone,

$$\alpha\|x - y\|^2 \leq \langle \eta x - \eta y, x - y\rangle \leq \langle x^* - y^*, x - y\rangle \leq \|x^* - y^*\|\|x - y\|, \tag{2.38}$$

which implies that $J_\rho^{(\Theta,\phi)}$ is $1/\alpha$-Lipschitz continuous.

Now we suppose that the Θ-subdifferential $\partial_\Theta\phi : B \to 2^{B^*}$ for ϕ is ξ-strongly monotone. Then

$$\langle x^* - \eta x - (y^* - \eta y), x - y\rangle \geq \rho\xi\|x - y\|^2. \tag{2.39}$$

Since η is α-strongly monotone,

$$\langle x^* - y^*, x - y\rangle = \langle x^* - \eta x - (y^* - \eta y), x - y\rangle + \langle \eta x - \eta y, x - y\rangle \geq (\alpha + \rho\xi)\|x - y\|^2. \tag{2.40}$$

It follows that

$$\|J_\rho^{(\Theta,\phi)}(x^*) - J_\rho^{(\Theta,\phi)}(y^*)\| \leq \frac{1}{\alpha + \rho\xi}\|x^* - y^*\|. \tag{2.41}$$

Thus, $J_\rho^{(\Theta,\phi)}$ is $1/(\alpha + \rho\xi)$-Lipschitz continuous. □

Corollary 2.10 (see [14, Theorem 2.2]). *Let B be a reflexive Banach space with the dual space B^*, let $\eta : B \to B^*$ be an α-strongly monotone and continuous mapping, and let $\phi : B \to \boldsymbol{R} \cup \{+\infty\}$ be a lower semicontinuous subdifferentiable proper functional. Then the η-proximal mapping $J_\rho^\phi = (\eta + \rho\partial\phi)^{-1}$ is $1/\alpha$-Lipschitz continuous. Furthermore, if the subdifferential $\partial\phi : B \to 2^{B^*}$ for ϕ is ξ-strongly monotone, then the η-proximal mapping $J_\rho^\phi = (\eta + \rho\partial\phi)^{-1}$ is $1/(\alpha + \rho\xi)$-Lipschitz continuous.*

Proof. Putting $\Theta = 0$ in Theorem 2.9, we derive the desired result. □

3. Existence and Algorithm

We first transfer GMEPP (2.1) into a fixed point problem.

Theorem 3.1. *Let B be a reflexive Banach space with the dual space B^*, let $\eta : B \to B^*$ be an α-strongly monotone and continuous mapping, let $\Theta : B \times B \to \boldsymbol{R}$ be a bifunction satisfying conditions (H1)–(H4), and let $\phi : B \to \boldsymbol{R} \cup \{+\infty\}$ be a lower semicontinuous subdifferentiable proper functional. If ϕ is Θ-subdifferentiable at each $x \in B \setminus R(J_\rho^{(\Theta,\phi)})$, then q is a solution of the GMEPP* (2.1) *if and only if q satisfies the following relation:*

$$g(q) = J_\rho^{(\Theta,\phi)}[\eta(g(q)) - \rho(Tq - Aq)], \tag{3.1}$$

where $J_\rho^{(\Theta,\phi)} = (\eta + \rho\partial_\Theta\phi)^{-1}$ is the η-proximal mapping of ϕ with respect to Θ and $\rho > 0$ is a constant.

Proof. Since ϕ is Θ-subdifferentiable at each $x \in B \setminus R(J_\rho^{(\Theta,\phi)})$, in terms of Theorem 2.6, ϕ is Θ-subdifferentiable.

Assume that q satisfies relation (3.1). Noting $J_\rho^{(\Theta,\phi)} = (\eta + \rho\partial_\Theta\phi)^{-1}$, we know that relation (3.1) holds if and only if

$$\eta(g(q)) - \rho(Tq - Aq) \in \eta(g(q)) + \rho\partial_\Theta\phi(g(q)). \tag{3.2}$$

By the definition of the Θ-subdifferential of ϕ with respect to Θ, the above relation holds if and only if

$$\Theta(g(q), v) + \phi(v) - \phi(g(q)) \geq \langle Aq - Tq, v - g(q)\rangle, \quad \forall v \in B, \tag{3.3}$$

that is,

$$\Theta(g(q), v) + \phi(v) - \phi(g(q)) + \langle Tq - Aq, v - g(q) \rangle \geq 0, \quad \forall v \in B. \tag{3.4}$$

Thus, q is a solution of GMEPP (2.1) if and only if q satisfies (3.1). This completes the proof. □

Remark 3.2. Relation (3.1) can be written as

$$q = q - g(q) + J_\rho^{(\Theta,\phi)}[\eta(g(q)) - \rho(Tq - Aq)], \tag{3.5}$$

where $J_\rho^{(\Theta,\phi)} = (\eta + \rho\partial_\Theta\phi)^{-1}$.

Remark 3.3. By Theorem 2.6, we can choose a strongly monotone and Lipschitz continuous mapping $\eta : B \to B^*$ such that it is easy to compute the values of the η-proximal mapping $J_\rho^{(\Theta,\phi)}$ of ϕ with respect to Θ. Theorem 3.1 shows that, by using the mapping $J_\rho^{(\Theta,\phi)}$, GMEPP (2.1) can be transferred into a fixed point problem (3.5). Based on these observations, we can suggest the following new and general iterative algorithms for computing the approximate solutions of GMEPP (2.1) in reflexive Banach spaces.

Lemma 3.4 (see [27]). *Let B be a real Banach space and let $J : B \to 2^{B^*}$ be the normalized duality mapping. Then for any $x, y \in B$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y). \tag{3.6}$$

We now use Theorem 3.1 to construct the following algorithms for solving the GMEPP (2.1) in Banach spaces.

Algorithm 3.5. Let $T, A : B \to B^*$ be two single-valued mappings, let $g : B \to B$ be a single-valued mapping with $g(B) = B$, let $\eta : B \to B^*$ be an α-strongly monotone and θ-Lipschitz continuous mapping, let $\Theta : B \times B \to \mathbf{R}$ be a bifunction satisfying conditions (H1)–(H4), and let $\phi : B \to \mathrm{R} \cup \{+\infty\}$ be a lower semicontinuous subdifferentiable proper functional. For any given $x_0 \in B$, an iterative sequence $\{x_n\}$ is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\left[x_n - g(x_n) + J_\rho^{(\Theta,\phi)}(\eta(g(x_n)) - \rho(Tx_n - Ax_n))\right], \quad n = 0, 1, \ldots, \tag{3.7}$$

where $\rho > 0$ and $\alpha_n \in [0, 1]$ for all $n \geq 0$. Algorithm 3.5 is called the Mann-type iterative algorithm.

Algorithm 3.6. Let T, A, g, η, Θ, and ϕ be the same as in Algorithm 3.5. For any given $x_0 \in B$, the iterative sequences $\{x_n\}$ and $\{y_n\}$ are defined by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n\Big[x_n - g(x_n) + J_\rho^{(\Theta,\phi)}\big(\eta\big(g(x_n)\big) - \rho(Tx_n - Ax_n)\big)\Big], \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n\Big[y_n - g(y_n) + J_\rho^{(\Theta,\phi)}\big(\eta\big(g(y_n)\big) - \rho(Ty_n - Ay_n)\big)\Big] \\ n &= 0, 1, \ldots, \end{aligned} \tag{3.8}$$

where $\alpha_n, \beta_n \in [0,1]$ for all $n \geq 0$. Algorithm 3.6 is called the Ishikawa-type iterative algorithm.

Remark 3.7. If $\beta_n = 0$ for all $n \geq 0$, then Algorithm 3.6 reduces to Algorithm 3.5. Whenever $\Theta = 0$, Algorithms 3.5 and 3.6 reduce to Algorithms 3.1 and 3.2 by Xia and Huang [14], respectively.

Now we prove an existence theorem of solutions for GMEPP (2.1).

Theorem 3.8. *Let B be a reflexive Banach space with the dual space B^*, let $T, A : B \to B^*$ be two continuous mappings, and let $g : B \to B$ be a continuous mapping. Let $\eta : B \to B^*$ be α-strongly monotone and continuous, let $\Theta : B \times B \to \mathbf{R}$ be a bifunction satisfying conditions (H1)–(H4), and let $\phi : B \to \mathbf{R} \cup \{+\infty\}$ be a lower semicontinuous subdifferentiable proper functional. If ϕ is Θ-subdifferentiable at each $x \in B \setminus R(J_\rho^{(\Theta,\phi)})$, and the ranges $R(I-g), R(\eta g)$ and $R(T-A)$ are totally bounded, then there exists $q \in B$ which is a solution of GMEPP* (2.1).

Proof. Define $F : B \to B$ by

$$F(x) = x - g(x) + J_\rho^{(\Theta,\phi)}\left[\eta g(x) - \rho(Tx - Ax)\right], \quad \forall x \in B. \tag{3.9}$$

By Theorem 2.9, the mapping $J_\rho^{(\Theta,\phi)} = (\eta + \rho\partial_\Theta\phi)^{-1}$ is Lipschitz continuous. Since the ranges $R(I-g), R(\eta g)$ and $R(T-A)$ are totally bounded, we know that the range $R(F)$ is also totally bounded in B; that is, $\overline{F(B)}$ is totally bounded in B. Thus, $\overline{F(B)}$ is a compact subset of B. Since T, A, g, η, and $J_\rho^{(\Theta,\phi)}$ are continuous, so does $F : B \to B$. By Schauder fixed point theorem, $F : B \to B$ has a fixed point $q \in B$. It follows from Theorem 3.1 that q is a solution of GMEPP (2.1). This completes the proof. □

Remark 3.9. From the proof of Theorem 3.8, we know that in [14, Theorem 3.2], the assumption of the boundedness of the ranges $R(I-g)$, $R(\eta g)$, and $R(T-A)$ cannot guarantee that all the conditions of Schauder fixed point theorem are satisfied. Thus, in [14, Theorem 3.2], the assumption "the ranges $R(I-g)$, $R(\eta g)$, and $R(T-A)$ are bounded" should be replaced by the stronger condition "the ranges $R(I-g)$, $R(\eta g)$, and $R(T-A)$ are totally bounded". Here the well-known Schauder fixed point theorem is stated as follows.

Let B be a Banach space and let C be a nonempty closed convex subset of B. Assume that $F : C \to C$ is a continuous mapping such that the closure $\overline{F(C)}$ is a compact subset in B. Then F has a fixed point x in C, that is, $F(x) = x$.

In order to give some sufficient conditions, which guarantee the convergence of the iterative sequences generated by Algorithm 3.6, we will need the following lemma in the sequel.

Lemma 3.10 (see [28]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the condition*

$$s_{n+1} \leq (1-\mu_n)s_n + \mu_n\nu_n, \quad \forall n \geq 1, \tag{3.10}$$

where $\{\mu_n\}$ and $\{\nu_n\}$ are sequences of real numbers such that

(i) $\{\mu_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \mu_n = \infty$, *or equivalently,*

$$\prod_{n=1}^{\infty}(1-\mu_n) := \lim_{n\to\infty}\prod_{k=1}^{n}(1-\mu_k) = 0; \tag{3.11}$$

(ii) $\limsup_{n\to\infty}\nu_n \leq 0$, *or*

(ii)′ $\sum_{n=1}^{\infty} \mu_n\nu_n$ *is convergent.*

Then, $\lim_{n\to\infty} s_n = 0$.

Theorem 3.11. *Let B be a uniformly smooth Banach space with the dual space B^*, let $T : B \to B^*$ be δ-Lipschitz continuous, let $A : B \to B^*$ be γ-Lipschitz continuous, and let $g : B \to B$ be k-strongly accretive and ε-Lipschitz continuous. Suppose that $\eta : B \to B^*$ is α-strongly monotone and θ-Lipschitz continuous, $\Theta : B\times B \to \mathbf{R}$ is a bifunction satisfying conditions (H1)–(H4), and $\phi : B \to \mathbf{R}\cup\{+\infty\}$ is a lower semicontinuous subdifferentiable proper functional which is Θ-subdifferentiable at each $x \in B \setminus R(J_\rho^{(\Theta,\phi)})$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$ with $\alpha_n \to 0$, $\beta_n \to 0$, and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Assume that the Θ-subdifferential $\partial_\Theta\phi : B \to 2^{B^*}$ of ϕ is ξ-strongly monotone, the ranges $R(I-g)$, $R(\eta g)$, and $R(T-A)$ are totally bounded, and $\rho > (\theta\varepsilon - k\alpha)/(k\xi-(\delta+\gamma))$ where*

$$k\xi > \delta+\gamma, \qquad \theta\varepsilon > k\alpha. \tag{3.12}$$

Then for any given $x_0 \in B$, the sequence $\{x_n\}$ defined by Algorithm 3.6 converges strongly to a solution q of GMEPP (2.1).

Proof. By Theorem 3.8 and the assumptions in Theorem 3.11, we know that the solution set of GMEPP (2.1) is nonempty. Let q be a solution of GMEPP (2.1). Since $k\xi > \delta+\gamma$ and $\theta\varepsilon > k\alpha$, we can choose a constant ρ such that

$$\rho > \frac{\theta\varepsilon - k\alpha}{k\xi - (\delta+\gamma)}. \tag{3.13}$$

By Algorithm 3.6, we have

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n\Big[x_n - g(x_n) + J_\rho^{(\Theta,\phi)}\big(\eta(g(x_n)) - \rho(Tx_n - Ax_n)\big)\Big],\\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n\Big[y_n - g(y_n) + J_\rho^{(\Theta,\phi)}\big(\eta(g(y_n)) - \rho(Ty_n - Ay_n)\big)\Big]. \end{aligned} \tag{3.14}$$

Let

$$p_n = y_n - g(y_n) + J_\rho^{(\Theta,\phi)}\left(\eta(g(y_n)) - \rho(Ty_n - Ay_n)\right),$$
$$r_n = x_n - g(x_n) + J_\rho^{(\Theta,\phi)}\left(\eta(g(x_n)) - \rho(Tx_n - Ax_n)\right). \tag{3.15}$$

Then

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n p_n, \qquad y_n = (1-\beta_n)x_n + \beta_n r_n. \tag{3.16}$$

It follows from Theorem 2.9 that $J_\rho^{(\Theta,\phi)}$ is Lipschitz continuous. Since the ranges $R(I - g)$, $R(\eta g)$, and $R(T - A)$ are totally bounded, we know that the set

$$\left\{x - g(x) + J_\rho^{(\Theta,\phi)}\left(\eta(g(x)) - \rho(Tx - Ax)\right) : x \in B\right\} \tag{3.17}$$

is bounded. Let

$$M = \sup\left\{\|w - q\| : w = x - g(x) + J_\rho^{(\Theta,\phi)}\left(\eta(g(x)) - \rho(Tx - Ax)\right),\, x \in B\right\} + \|x_0 - q\| < +\infty. \tag{3.18}$$

This implies that

$$\|p_n - q\| \le M, \qquad \|r_n - q\| \le M, \quad \forall n \ge 0. \tag{3.19}$$

Since $\|x_0 - q\| \le M$,

$$\begin{aligned} \|y_0 - q\| &= \|(1-\beta_0)(x_0 - q) + \beta_0(r_0 - q)\| \\ &\le (1-\beta_0)\|x_0 - q\| + \beta_0\|r_0 - q\| \\ &\le M. \end{aligned} \tag{3.20}$$

It follows that

$$\begin{aligned} \|x_1 - q\| &\le (1-\alpha_0)\|x_0 - q\| + \alpha_0\|p_0 - q\| \le M, \\ \|y_1 - q\| &\le (1-\beta_1)\|x_1 - q\| + \beta_1\|r_1 - q\| \le M. \end{aligned} \tag{3.21}$$

By induction we can prove that

$$\|x_n - q\| \le M, \qquad \|y_n - q\| \le M, \quad \forall n \ge 0. \tag{3.22}$$

On the other hand, by Lemma 3.4,

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|(1-\alpha_n)(x_n-q)+\alpha_n(p_n-q)\|^2\\ &\le (1-\alpha_n)^2\|x_n-q\|^2+2\alpha_n\langle p_n-q, J(x_{n+1}-q)\rangle\\ &= (1-\alpha_n)^2\|x_n-q\|^2+2\alpha_n\langle p_n-q, J(y_n-q)\rangle\\ &\quad+2\alpha_n\langle p_n-q, J(x_{n+1}-q)-J(y_n-q)\rangle.\end{aligned} \tag{3.23}$$

Now we consider the third term on the right-hand side of (3.23). From (3.19) and (3.22) it follows that

$$\begin{aligned}\|(x_{n+1}-q)-(y_n-q)\| &= \|x_{n+1}-y_n\|\\ &= \|(1-\alpha_n)(x_n-y_n)+\alpha_n(p_n-y_n)\|\\ &\le (1-\alpha_n)\beta_n\|x_n-r_n\|+\alpha_n(\|p_n-q\|+\|y_n-q\|)\\ &\le (1-\alpha_n)\beta_n(\|x_n-q\|+\|r_n-q\|)+\alpha_n(\|p_n-q\|+\|y_n-q\|)\\ &\le 2((1-\alpha_n)\beta_n+\alpha_n)M\longrightarrow 0 \quad (n\longrightarrow\infty).\end{aligned} \tag{3.24}$$

Since B is a uniformly smooth Banach space, the normalized duality mapping $J: B\rightarrow B^*$ is uniformly norm-to-norm continuous on any bounded subset of B. Hence it is easy to see that

$$J(x_{n+1}-q)-J(y_n-q)\longrightarrow 0 \quad (n\longrightarrow\infty). \tag{3.25}$$

Let

$$\delta_n = |\langle p_n-q, J(x_{n+1}-q)-J(y_n-q)\rangle|. \tag{3.26}$$

Since $\{p_n-q\}$ is bounded,

$$\delta_n = |\langle p_n-q, J(x_{n+1}-q)-J(y_n-q)\rangle|\longrightarrow 0 \quad (n\longrightarrow\infty). \tag{3.27}$$

Next we consider the second term on the right-hand side of (3.23). Since q is a solution of GMEPP (2.1), by Theorem 3.1, we have

$$q = q-g(q)+J_\rho^{(\Theta,\phi)}[\eta(g(q))-\rho(T(q)-A(q))]. \tag{3.28}$$

It follows that

$$\begin{aligned}
&2\alpha_n\langle p_n - q, J(y_n - q)\rangle \\
&\quad = 2\alpha_n\Big\langle \Big(y_n - g(y_n) + J_\rho^{(\Theta,\phi)}[\eta(g(y_n)) - \rho(T(y_n) - A(y_n))]\Big) \\
&\qquad\qquad - \Big(q - g(q) + J_\rho^{(\Theta,\phi)}[\eta(g(q)) - \rho(T(q) - A(q))]\Big), J(y_n - q)\Big\rangle \\
&\quad = 2\alpha_n\langle y_n - q, J(y_n - q)\rangle - 2\alpha_n\langle g(y_n) - g(q), J(y_n - q)\rangle \\
&\qquad + 2\alpha_n\langle J_\rho^{(\Theta,\phi)}[\eta(g(y_n)) - \rho(T(y_n) - A(y_n))] \\
&\qquad\qquad - J_\rho^{(\Theta,\phi)}[\eta(g(q)) - \rho(T(q) - A(q))], J(y_n - q)\rangle \\
&\quad \le 2\alpha_n\Big\{\|y_n - q\|^2 - k\|y_n - q\|^2 + \lambda\|y_n - q\|^2\Big\} \\
&\quad = 2\alpha_n(1 - k + \lambda)\|y_n - q\|^2,
\end{aligned} \tag{3.29}$$

where

$$\lambda = \frac{1}{\alpha + \rho\xi}(\theta\varepsilon + \rho\delta + \rho\gamma). \tag{3.30}$$

Substituting (3.27) and (3.29) into (3.23), we have

$$\|x_{n+1} - q\|^2 \le (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n(1 - k + \lambda)\|y_n - q\|^2 + 2\alpha_n\delta_n. \tag{3.31}$$

Next we make an estimation for $\|y_n - q\|^2$. Indeed,

$$\begin{aligned}
\|y_n - q\|^2 &= \|(1 - \beta_n)(x_n - q) + \beta_n(r_n - q)\|^2 \\
&\le (1 - \beta_n)^2\|x_n - q\|^2 + 2\beta_n\langle r_n - q, J(y_n - q)\rangle \\
&\le (1 - \beta_n)^2\|x_n - q\|^2 + 2\beta_n\|r_n - q\|\|y_n - q\| \\
&\le \|x_n - q\|^2 + 2\beta_n M^2.
\end{aligned} \tag{3.32}$$

Substituting (3.32) into (3.31) and simplifying it, we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n(1 - k + \lambda)\Big[\|x_n - q\|^2 + 2\beta_n M^2\Big] + 2\alpha_n\delta_n \\
&= [1 - 2\alpha_n(k - \lambda)]\|x_n - q\|^2 + \alpha_n^2\|x_n - q\|^2 + 4\alpha_n\beta_n(1 - k + \lambda)M^2 + 2\alpha_n\delta_n \\
&\le [1 - 2\alpha_n(k - \lambda)]\|x_n - q\|^2 \\
&\quad + 2\alpha_n(k - \lambda)\cdot\frac{1}{2(k - \lambda)}\Big\{\alpha_n M^2 + 4\beta_n(1 - k + \lambda)M^2 + 2\delta_n\Big\}.
\end{aligned} \tag{3.33}$$

From condition (3.13), we get $k > \lambda$. Since $\alpha_n \to 0$, $\beta_n \to 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$, we deduce that $\sum_{n=0}^{\infty} 2\alpha_n (k-\lambda) = \infty$ and

$$\lim_{n\to\infty} \frac{1}{2(k-\lambda)} \left\{ \alpha_n M^2 + 4\beta_n (1-k+\lambda) M^2 + 2\delta_n \right\} = 0. \tag{3.34}$$

Thus, utilizing Lemma 3.10 we conclude that $\|x_n - q\| \to 0$ as $n \to \infty$. This completes the proof. □

Remark 3.12. By the careful analysis of the proof of Theorem 3.11, we can see readily that the following conditions are used to derive the following conclusion:

(i) the normalized duality mapping $J : B \to B^*$ is uniformly norm-to-norm continuous on any bounded subset of B;

(ii) the constant ρ in Algorithm 3.6 satisfies the inequality $\rho > (\theta\varepsilon - k\alpha)/(k\xi - (\delta+\gamma))$.

In addition, we apply Lemma 3.4 to derive the strong convergence of the iterative sequence generated by Algorithm 3.6. Moreover, we simplify the original proof by Xia and Huang [14, Theorem 3.11] to a great extent. Therefore, Theorem 3.11 is a generalization and modification of Xia and Huang's Theorem 3.3 [14].

Remark 3.13. We would like to point out that, in Theorem 3.11, the functional ϕ may not be convex, the mappings T and A may not have any monotonicity, and their domains and ranges are reflexive Banach space B and the dual space B^* of B, respectively. Hence Theorem 3.11 improves and generalizes some known results in [8, 9, 11, 16, 17]. Furthermore, the argument methods presented in this paper are quite different from those in [8–11, 13, 14, 16, 21].

Finally, we give a weak convergence theorem for the iterative sequence generated by Algorithm 3.6 in $B = H$ a real Hilbert space. However, we first recall the following lemmas.

Lemma 3.14 (see [29, page 303]). *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le a_n + b_n, \quad \forall n \ge 1. \tag{3.35}$$

If $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.

Lemma 3.15 (see [30]). *Demiclosedness Principle. Assume that T is a nonexpansive self-mapping of a nonempty closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$. Here I is the identity operator of H.*

Theorem 3.16. *Let H be a real Hilbert space, let $T : H \to H$ be δ-Lipschitz continuous, let $A : H \to H$ be γ-Lipschitz continuous, and let $g : H \to H$ be k-strongly monotone and ε-Lipschitz continuous. Suppose that $\eta : H \to H$ is α-strongly monotone and θ-Lipschitz continuous, $\Theta : H \times H \to \mathbf{R}$ is a bifunction satisfying conditions (H1)–(H4), and $\phi : H \to \mathbf{R} \cup \{+\infty\}$ is a lower semicontinuous subdifferentiable proper functional which is Θ-subdifferentiable at each $x \in B \setminus R(J_\rho^{(\Theta,\phi)})$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$ such that $0 < \liminf_{n\to\infty} \alpha_n, 0 <$*

$\liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. *Assume that the* Θ*-subdifferential* $\partial_\Theta\phi : H \to 2^H$ *of* ϕ *is* ξ*-strongly monotone, the ranges* $R(I-g)$, $R(\eta g)$, *and* $R(T-A)$ *are totally bounded, and* $\rho \geq (\theta\varepsilon - \tilde{k}\alpha)/(\tilde{k}\xi - (\delta+\gamma))$ *where*

$$\tilde{k} = 1 - (1-2k+\varepsilon^2)^{1/2}, \qquad \tilde{k}\xi > \delta+\gamma, \qquad \theta\varepsilon > \tilde{k}\alpha. \tag{3.36}$$

Then for any given $x_0 \in H$, *the sequence* $\{x_n\}$ *defined by Algorithm 3.6 converges weakly to a solution of GMEPP* (2.1).

Proof. By Theorem 3.8 and the assumptions in Theorem 3.16, we know that the solution set of GMEPP (2.1) is nonempty. Let q be a solution of GMEPP (2.1). It is easy to see that

$$\rho \geq \frac{\theta\varepsilon - \tilde{k}\alpha}{\tilde{k}\xi - (\delta+\gamma)} \Longleftrightarrow \tilde{k} \geq \lambda. \tag{3.37}$$

Define a mapping $\Gamma : H \to H$ as follows:

$$\Gamma(x) = x - g(x) + J_\rho^{(\Theta,\phi)}\left(\eta(g(x)) - \rho(Tx - Ax)\right), \quad \forall x \in H, \tag{3.38}$$

where $J_\rho^{(\Theta,\phi)} = (\eta + \rho\partial_\Theta\phi)^{-1}$. Since the Θ-subdifferential $\partial_\Theta\phi : H \to 2^H$ of ϕ is ξ-strongly monotone, by Theorem 2.9 we conclude that $J_\rho^{(\Theta,\phi)} : B^* \to B$ is $1/(\alpha+\rho\xi)$-Lipschitz continuous.

Observe that for all $x, y \in H$

$$\begin{aligned}
\|\Gamma(x) - \Gamma(y)\| &\leq \|(x-y) - (g(x) - g(y))\| \\
&\quad + \left\|J_\rho^{(\Theta,\phi)}\left(\eta(g(x)) - \rho(Tx-Ax)\right) - J_\rho^{(\Theta,\phi)}\left(\eta(g(y)) - \rho(Ty-Ay)\right)\right\| \\
&\leq (1-2k+\varepsilon^2)^{1/2}\|x-y\| \\
&\quad + \frac{1}{\alpha+\rho\xi}\left\|\left(\eta(g(x)) - \rho(Tx-Ax)\right) - \left(\eta(g(y)) - \rho(Ty-Ay)\right)\right\| \\
&\leq (1-2k+\varepsilon^2)^{1/2}\|x-y\| + \frac{1}{\alpha+\rho\xi}(\theta\varepsilon+\rho\delta+\rho\gamma)\|x-y\| \\
&= \left(1 - \tilde{k} + \lambda\right)\|x-y\| \\
&\leq \|x-y\|,
\end{aligned} \tag{3.39}$$

where $\tilde{k} = 1 - (1-2k+\varepsilon^2)^{1/2}$ and $\lambda = (1/(\alpha+\rho\xi))(\theta\varepsilon+\rho\delta+\rho\gamma)$. This implies that $\Gamma : H \to H$ is nonexpansive on H. It is easy to see that the set of GMEPP (2.1) coincides with F$ix(\Gamma)$.

Note that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|(1-\alpha_n)(x_n-q)+\alpha_n(\Gamma(y_n)-\Gamma(q))\|^2 \\
&\le (1-\alpha_n)\|x_n-q\|^2+\alpha_n\|\Gamma(y_n)-\Gamma(q)\|^2 \\
&\le (1-\alpha_n)\|x_n-q\|^2+\alpha_n\|y_n-q\|^2 \\
&= (1-\alpha_n)\|x_n-q\|^2+\alpha_n\|(1-\beta_n)(x_n-q)+\beta_n(\Gamma(x_n)-\Gamma(q))\|^2 \\
&= (1-\alpha_n)\|x_n-q\|^2+\alpha_n\Big[(1-\beta_n)\|x_n-q\|^2+\beta_n\|\Gamma(x_n)-\Gamma(q)\|^2 \\
&\qquad -\beta_n(1-\beta_n)\|\Gamma(x_n)-x_n\|^2\Big] \\
&\le (1-\alpha_n)\|x_n-q\|^2+\alpha_n\Big[(1-\beta_n)\|x_n-q\|^2+\beta_n\|x_n-q\|^2 \\
&\qquad -\beta_n(1-\beta_n)\|\Gamma(x_n)-x_n\|^2\Big] \\
&= \|x_n-q\|^2-\alpha_n\beta_n(1-\beta_n)\|\Gamma(x_n)-x_n\|^2 \\
&\le \|x_n-q\|^2.
\end{aligned} \tag{3.40}$$

This together with Lemma 3.14 implies that $\lim_{n\to\infty}\|x_n-q\|$ exists. And also, from (3.40) we have

$$\alpha_n\beta_n(1-\beta_n)\|\Gamma(x_n)-x_n\|^2 \le \|x_n-q\|^2-\|x_{n+1}-q\|^2. \tag{3.41}$$

Since $0<\liminf_{n\to\infty}\alpha_n, 0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$, we get

$$\lim_{n\to\infty}\|x_n-\Gamma(x_n)\|=0. \tag{3.42}$$

Now, let us show that $\omega_w(x_n)\subset \mathrm{Fix}(\Gamma)$. Indeed, let $\overline{x}\in\omega_w(x_n)$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}\rightharpoonup\overline{x}$. Since $(I-\Gamma)x_n\to 0$, by Lemma 3.15 we know that $\overline{x}\in Fix(\Gamma)$.

Further, let us show that $\omega_w(x_n)$ is a singleton. Indeed, let $\{x_{m_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{m_j}\rightharpoonup\widehat{x}$. Then $\widehat{x}$ is also a fixed point of Γ. If $\overline{x}\neq\widehat{x}$, by Opial's property of H, we reach the following contradiction:

$$\begin{aligned}
\lim_{n\to\infty}\|x_n-\overline{x}\| &= \lim_{i\to\infty}\|x_{n_i}-\overline{x}\| \\
&< \lim_{i\to\infty}\|x_{n_i}-\widehat{x}\| = \lim_{j\to\infty}\|x_{m_j}-\widehat{x}\| \\
&< \lim_{j\to\infty}\|x_{m_j}-\overline{x}\| \\
&= \lim_{n\to\infty}\|x_n-\overline{x}\|.
\end{aligned} \tag{3.43}$$

This implies that $\omega_w(x_n)$ is a singleton. Consequently, $\{x_n\}$ converges weakly to a fixed point of Γ, that is, a solution of GMEPP (2.1). This completes the proof. □

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