Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 476913, 8 pages
doi:10.1155/2010/476913

Research Article

Viscosity Approximation to Common Fixed Points of Families of Nonexpansive Mappings with Weakly Contractive Mappings

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Received 5 June 2010; Accepted 26 July 2010

Academic Editor: Brailey Sims

Let X be a reflexive Banach space which has a weakly sequentially continuous duality mapping. In this paper, we consider the following viscosity approximation sequence $x_n = \lambda_n f(x_n)+(1-\lambda_n)T_n x_n$, where $\lambda_n \in (0,1)$, $\{T_n\}$ is a uniformly asymptotically regular sequence, and f is a weakly contractive mapping. Strong convergence of the sequence $\{x_n\}$ is proved.

1. Introduction

Let C be a nonempty closed convex subset of a Banach space X. Recall that a self-mapping $T : C \to C$ is nonexpansive if

$$\|T(x) - T(y)\| \leq \|x - y\| \quad \forall x, y \in C. \tag{1.1}$$

Alber and Guerre-Delabriere [1] defined the weakly contractive maps in Hilbert spaces, and Rhoades [2] showed that the result of [1] is also valid in the complete metric spaces as follows.

Definition 1.1. Let (X, d) be a complete metric space. A mapping $T : X \to X$ is called weakly contractive if

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)), \tag{1.2}$$

where $x, y \in X$ and $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty}\psi(t) = \infty$.

Theorem 1.2. *Let $T : X \to X$ be a weakly contractive mapping, where (X, d) is a complete metric space, then T has a unique fixed point.*

In 2007, Song and Chen [3] considered the iterative sequence

$$x_n = \lambda_n f(x_n) + (1 - \lambda_n)T_n x_n, \quad n \in \{1, 2, \ldots\}. \tag{1.3}$$

They proved the strong convergence of the iterative sequence $\{x_n\}$, where f is a contraction mapping and $\{T_n\}$ is a uniformly asymptotically regular sequence of nonexpansive mappings in a reflexive Banach space X, as follows.

Theorem 1.3 (see [3, Theorem 3.1]). *Let X be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from X to X^*. Suppose that C is a nonempty closed convex subset of X and $\{T_n\}, n \in \{1, 2, \ldots\}$, is a uniformly asymptotically regular sequence of nonexpansive mappings from C into itself such that*

$$F := \bigcap_{n=1}^{\infty} \operatorname{Fix}(T_n) \neq \emptyset, \tag{1.4}$$

where $\operatorname{Fix}(T_n) := \{x \in C : x = T_n x\}, n \in \{1, 2, \ldots\}$. Let $\{x_n\}$ be defined by (1.3) and $\lambda_n \in (0, 1)$, such that $\lim_{n\to\infty}\lambda_n = 0$. Then as $n \to \infty$, the sequence $\{x_n\}$ converges strongly to p, such that p is the unique solution, in F, to the variational inequality:

$$\langle f(p) - p, J(y - p)\rangle \leq 0, \quad \forall y \in F. \tag{1.5}$$

In this paper, inspired by the above results, strong convergence of sequence (1.3) is proved, where f is a weakly contractive mapping.

2. Preliminaries

A Banach space X is called strictly convex if

$$\|x\| = \|y\| = 1, \quad x \neq y \text{ implies } \frac{\|x + y\|}{2} < 1. \tag{2.1}$$

A Banach space X is called uniformly convex, if for all $\varepsilon \in [0, 2]$, there exist $\delta_\varepsilon > 0$ such that

$$\|x\| = \|y\| = 1 \quad \text{with } \|x - y\| \geq \varepsilon \text{ implies that } \frac{\|x + y\|}{2} < 1 - \delta_\varepsilon. \tag{2.2}$$

The following results are well known which can be founded in [4].

(1) A uniformly convex Banach space X is reflexive and strictly convex.

(2) If C is a nonempty convex subset of a strictly convex Banach space X and $T : C \to C$ is a nonexpansive mapping, then the fixed point set $F(T)$ of T is a closed convex subset of C.

By a gauge function we mean a continuous strictly increasing function φ defined on $[0,\infty)$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty}\varphi(r) = \infty$. The mapping $J_\varphi : X \to 2^{X^*}$ defined by

$$J_\varphi(x) = \left\{x^* \in X^* : \langle x, x^*\rangle = \|x\|\|x^*\|, \|x^*\| = \varphi(\|x\|)\right\}, \quad \text{for each } x \in X, \tag{2.3}$$

is called the duality mapping with gauge function φ. In the case where $\varphi(t) = t$, then $J_\varphi = J$ which is the normalized duality mapping.

Proposition 2.1 (see [5]). *(1) $J = I$ if and only if X is a Hilbert space.*

(2) J is surjective if and only if X is reflexive.

(3) $J_\varphi(\lambda x) = \operatorname{sign}\lambda(\varphi(|\lambda| \cdot \|x\|)/\|x\|)J(x)$ for all $x \in X \setminus \{0\}, \lambda \in R$; in particular $J(-x) = -J(x)$, for all $x \in X$.

We say that a Banach space X has a weakly sequentially continuous duality mapping if there exists a gauge function φ such that the duality mapping J_φ is single-valued and continuous from the weak topology to the weak* topology of X.

We recall [6] that a Banach space X is said to satisfy Opial's condition, if for any sequence $\{x_n\}$ in X, which converges weakly to $x \in X$, we have

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\left\|x_n - y\right\| \quad \forall y \in X,\ y \neq x. \tag{2.4}$$

It is known [7] that any separable Banach space can be equivalently renormed such that it satisfies Opial's condition. A space with a weakly sequentially continuous duality mapping is easily seen to satisfy Opial's condition [8].

Lemma 2.2 (see [9, Lemma 4]). *Let X be a Banach space satisfying Opial's condition and C a nonempty, closed, and convex subset of X. Suppose that $T : C \to C$ is a nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is, if $\{x_n\}$ is a sequence in C which converges weakly to x and if the sequence $x_n - Tx_n$ converges strongly to zero, then $x - Tx = 0$.*

Definition 2.3 (see [3]). Let C be a nonempty closed convex subset of a Banach space X and $T_n : C \to C$, where $n \in \{1,2,\ldots\}$. Then the mapping sequence $\{T_n\}$ is called uniformly asymptotically regular on C, if for all $m \in \{1,2,\ldots\}$ and any bounded subset K of C we have

$$\lim_{n\to+\infty}\sup_{x\in K}\|T_m(T_nx) - T_nx\| = 0. \tag{2.5}$$

3. Main Result

In this section, we prove a new version of Theorem 1.3.

Theorem 3.1. *Let X be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from X to X^*. Suppose that C is a nonempty closed convex subset of X and $T_m : C \to C, m \in \{1,2,\ldots\}$, is a uniformly asymptotically regular sequence of nonexpansive mappings such that*

$$F := \bigcap_{m=1}^{\infty} \operatorname{Fix}(T_m) \neq \emptyset. \tag{3.1}$$

Let $f : C \to C$ be a weakly contractive mapping. Suppose that $\{t_m\}$ is a sequence of positive numbers in $(0,1)$ satisfying $\lim_{m\to\infty} t_m = 0$. Assume that $\{x_m\}$ is defined by the following iterative process:

$$x_m = t_m f(x_m) + (1-t_m)T_m x_m, \quad m \in \{1,2,\ldots\}. \tag{3.2}$$

Then the above sequence $\{x_m\}$ converges strongly to a common fixed point p of $\{T_m\}, m \in \{1,2,\ldots\}$ such that p is the unique solution, in F, to the variational inequality

$$\langle f(p) - p, J(y-p)\rangle \leq 0, \quad \forall y \in F. \tag{3.3}$$

Proof.

Step 1. We prove the uniqueness of the solution to the variational inequality (3.3). Suppose that $p, q \in F$ are distinct solutions to (3.3). Then

$$\begin{aligned} &\langle f(p) - p, J(q-p)\rangle \leq 0, \\ &\langle f(q) - q, J(p-q)\rangle \leq 0. \end{aligned} \tag{3.4}$$

By adding up the above relations, we get

$$\begin{aligned} 0 &\geq \langle (p - f(p)) - (q - f(q)), J(p-q)\rangle \\ &\geq \langle p-q, J(p-q)\rangle - \langle f(p) - f(q), J(p-q)\rangle \\ &\geq \|p-q\|^2 - \|f(p) - f(q)\| \|J(p-q)\| \\ &\geq \|p-q\|^2 - \|p-q\|^2 + \psi(\|p-q\|)\|p-q\|. \end{aligned} \tag{3.5}$$

Thus $\psi(\|p-q\|)\|p-q\| \leq 0$, hence $p = q$. We denote by p the unique solution, in F, to(3.3).

Step 2. We show that the sequence $\{x_m\}$ is bounded. Let $q \in F$; from (3.2) we get then that

$$\begin{aligned}\|x_m - q\|^2 &= \langle t_m(f(x_m) - q) + (1-t_m)(T_m x_m - q), J(x_m - q)\rangle \\ &= t_m\langle (f(x_m) - f(q)) + (f(q) - q), J(x_m - q)\rangle \\ &\quad + (1-t_m)\langle T_m x_m - T_m q, J(x_m - q)\rangle \\ &\le t_m\|f(x_m) - f(q)\|\|J(x_m - q)\| + t_m\langle f(q) - q, J(x_m - q)\rangle \\ &\quad + (1-t_m)\|T_m x_m - T_m q\|\|J(x_m - q)\| \\ &\le t_m[(\|x_m - q\| - \psi(\|x_m - q\|))\|x_m - q\| + \langle f(q) - q, J(x_m - q)\rangle] \\ &\quad + (1-t_m)\|T_m x_m - T_m q\|\|J(x_m - q)\| \\ &\le t_m\Big[\|x_m - q\|^2 - \psi(\|x_m - q\|)\|x_m - q\| + \langle f(q) - q, J(x_m - q)\rangle\Big] \\ &\quad + (1-t_m)\|x_m - q\|^2 \\ &\le \|x_m - q\|^2 - t_m\|x_m - q\|\psi(\|x_m - q\|) + t_m\|f(q) - q\|\|x_m - q\|.\end{aligned} \tag{3.6}$$

Thus

$$\|x_m - q\|\psi(\|x_m - q\|) \le \|f(q) - q\|\|x_m - q\|, \tag{3.7}$$

or

$$\psi(\|x_m - q\|) \le \|f(q) - q\|. \tag{3.8}$$

Therefore $\{x_m\}$ is bounded.

Step 3. We prove that $\lim_{m\to+\infty}\|x_m - T_n x_m\| = 0$, for all $n \in \{1,2,\ldots\}$. Since the sequence $\{x_m\}$ is bounded, so $\{f(x_m)\}$ and $\{T_m x_m\}$ are bounded. Hence $\lim_{m\to\infty} t_m\|T_m x_m - f(x_m)\| = 0$, thus $\lim_{m\to\infty}\|x_m - T_m x_m\| = 0$. Let K be a bounded subset of C which contains $\{x_m\}$. Since the sequence $\{T_m\}$ is uniformly asymptotically regular, we can obtain

$$\lim_{m\to\infty}\|T_n(T_m x_m) - T_m x_m\| \le \lim_{m\to\infty}\sup_{x\in K}\|T_n(T_m x) - T_m x\| = 0. \tag{3.9}$$

Let $m \to \infty$, then

$$\begin{aligned}\|x_m - T_n x_m\| &\le \|x_m - T_m x_m\| + \|T_m x_m - T_n(T_m x_m)\| + \|T_n(T_m x_m) - T_n x_m\| \\ &\le 2\|x_m - T_m x_m\| + \|T_m x_m - T_n(T_m x_m)\| \longrightarrow 0.\end{aligned} \tag{3.10}$$

Hence $\lim_{m\to\infty}\|x_m - T_n x_m\| = 0$, for all $n \in \{1,2,\ldots\}$.

Step 4. We show that the sequence $\{x_m\}$ is sequentially compact. Since X is reflexive and $\{x_m\}$ is bounded, there exists a subsequence $\{x_{m_k}\}$ of $\{x_m\}$ such that $\{x_{m_k}\}$ is weakly convergent to $q \in C$ as $k \to \infty$. Since $\lim_{k\to\infty}\|x_{m_k} - T_n x_{m_k}\| = 0$ for all $n \in \{1, 2, \ldots\}$, by Lemma 2.2, we have $q = T_n q$ for all $n \in \{1, 2, \ldots\}$. Thus $q \in F$.

Step 2 implies that

$$\begin{aligned} \|x_{m_k} - q\|^2 \le{}& t_{m_k}\left[\left(\|x_{m_k} - q\| - \psi(\|x_{m_k} - q\|)\right)\|x_{m_k} - q\| + \langle f(q) - q, J(x_{m_k} - q)\rangle\right] \\ &+ (1 - t_{m_k})\|x_{m_k} - q\|^2. \end{aligned} \tag{3.11}$$

Hence

$$t_{m_k}\|x_{m_k} - q\|\psi(\|x_{m_k} - q\|) \le t_{m_k}\langle f(q) - q, J(x_{m_k} - q)\rangle. \tag{3.12}$$

Since J is single valued and weakly sequentially continuous from X to X^*, we have

$$\limsup_{k\to\infty}\|x_{m_k} - q\|\psi(\|x_{m_k} - q\|) \le \lim_{k\to\infty}\langle f(q) - q, J(x_{m_k} - q)\rangle = 0. \tag{3.13}$$

Thus $\lim_{k\to\infty} x_{m_k} = q$. Hence the sequence $\{x_m\}$ is sequentially compact.

Step 5. We now prove that $q \in F$ is a solution to the variational inequality (3.3). Suppose that $y \in F$, then

$$\begin{aligned} \|x_m - y\|^2 &= t_m\langle (f(x_m) - x_m) + (x_m - y), J(x_m - y)\rangle \\ &\quad + (1 - t_m)\langle T_m x_m - T_m y, J(x_m - y)\rangle \\ &\le t_m\langle (f(x_m) - x_m), J(x_m - y)\rangle + \|x_m - y\|^2. \end{aligned} \tag{3.14}$$

Hence

$$\langle (f(x_m) - x_m), J(y - x_m)\rangle \le 0 \quad \text{for each } m \in \{1, 2, \ldots\}. \tag{3.15}$$

Since $\{x_{m_k}\} \to q$ as $k \to \infty$, we have

$$\begin{aligned} &\|(x_{m_k} - f(x_{m_k})) - (q - f(q))\| \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \\ &|\langle (x_{m_k} - f(x_{m_k})), J(x_{m_k} - y)\rangle - \langle (q - f(q)), J(q - y)\rangle| \\ &\quad = |\langle (x_{m_k} - f(x_{m_k})) - (q - f(q)), J(x_{m_k} - y)\rangle + \langle (q - f(q)), J(x_{m_k} - y) - J(q - y)\rangle| \\ &\quad \le \|(x_{m_k} - f(x_{m_k})) - (q - f(q))\|\|x_{m_k} - y\| \\ &\qquad + |\langle (q - f(q)), J(x_{m_k} - y) - J(q - y)\rangle| \longrightarrow 0, \end{aligned} \tag{3.16}$$

as $k \to \infty$. Hence

$$\langle f(q) - q, J(y - q)\rangle = \lim_{k\to\infty}\langle f(x_{m_k}) - x_{m_k}, J(y - x_{m_k})\rangle \le 0. \tag{3.17}$$

Thus $q \in F$ is a solution to the variational inequality (3.3). By uniqueness, $q = p$. Since the sequence $\{x_m\}$ is sequentially compact and each cluster point of it is equal to p, then $\{x_m\} \to p$ as $m \to \infty$. The proof is completed. □

It is known that [10, Example 2] in a uniformly convex Banach space E, the Cesàro means $T_n = (1/n)\sum_{j=0}^{n-1} T^j$ for nonexpansive mapping T is uniformly asymptotically regular. So we have the following corollary, which is a new version of [10, Theorem 3.2].

Corollary 3.2. *Let X be a real uniformly convex Banach space which admits a weakly sequentially continuous duality mapping J from X to X^* and C a nonempty closed convex subset of X. Suppose that $T : C \to C$ is a nonexpansive mapping, $F(T) \neq \emptyset$ and $f : C \to C$ is a weakly contractive mapping. Let $\{z_m\}$ be defined by*

$$z_m = t_m f(z_m) + (1 - t_m)\frac{1}{m+1}\Sigma_{j=0}^{m} T^j z_m, \quad m \ge 0, \tag{3.18}$$

where $t_m \in (0,1)$ and $\lim_{m\to\infty} t_m = 0$. Then as $m \to \infty$, $\{z_m\}$ converges strongly to a fixed point p of T, where p is the unique solution in $F(T)$ to the following variational inequality:

$$\langle f(p) - p, j(u - p)\rangle \le 0 \quad \forall u \in F(T). \tag{3.19}$$

Acknowledgment

A. Razani would like to thank the School of Mathematics of the Institute for Research in Fundamental Sciences, Teheran, Iran for supporting this paper (Grant no.89470126).

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