

*Research Article*

# **Browder's Convergence for Uniformly Asymptotically Regular Nonexpansive Semigroups in Hilbert Spaces**

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We give a sufficient and necessary condition concerning a Browder's convergence type theorem for uniformly asymptotically regular one-parameter nonexpansive semigroups in Hilbert spaces.

## **1. Introduction**

Let  $C$  be a closed convex subset of a Hilbert space  $E$ . A mapping  $T$  on  $C$  is called a *nonexpansive* mapping if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in C$ . We denote by  $F(T)$  the set of fixed points of  $T$ . Browder, see [1], proved that  $F(T)$  is nonempty provided that  $C$  is, in addition, bounded. Kirk in a very celebrated paper, see [2], extended this result to the setting of reflexive Banach spaces with normal structure.

Browder [3] initiated the investigation of an implicit method for approximating fixed points of nonexpansive self-mappings defined on a Hilbert space. Fix  $u \in C$ , he studied the implicit iterative algorithm

$$z_t = tu + (1 - t)Tz_t. \quad (1.1)$$

Namely,  $z_t$ ,  $t \in (0, 1)$ , is the unique fixed point of the contraction  $x \mapsto tu + (1 - t)Tx$ ,  $x \in C$ . Browder proved that  $\lim_{t \rightarrow +0} z_t = Pu$ , where  $Pu$  is the element of  $F(T)$  nearest to  $u$ . Extensions to the framework of Banach spaces of Browder's convergence results have been done by many authors, including Reich [4], Takahashi and Ueda [5], and O'Hara et al. [6].

A family of mappings  $\{T(t) : t \geq 0\}$  is called a *one-parameter strongly continuous semigroup of nonexpansive mappings* (*nonexpansive semigroup*, for short) on  $C$  if the following are satisfied.

(NS1) For each  $t \geq 0$ ,  $T(t)$  is a nonexpansive mapping on  $C$ .

(NS2)  $T(s+t) = T(s) \circ T(t)$  for all  $s, t \geq 0$ .

(NS3) For each  $x \in C$ , the mapping  $t \mapsto T(t)x$  from  $[0, \infty)$  into  $C$  is strongly continuous.

There are many papers concerning the existence of common fixed points of  $\{T(t) : t \geq 0\}$ ; see, for instance, [7–13]. As a matter of fact, Browder [8] proved that if  $C$  is bounded, then  $\bigcap_{t \geq 0} F(T(t))$  is nonempty.

Browder's type convergence theorem for nonexpansive semigroups is proved in [11, 14–18] and others. For example, the following theorem is proved in [17].

**Theorem 1.1** (see [17]). *Let  $C$  be a closed convex subset of a Hilbert space  $E$ . Let  $\{T(t) : t \geq 0\}$  be a nonexpansive semigroup on  $C$  such that  $\bigcap_{t \geq 0} F(T(t)) \neq \emptyset$ . Let  $\{\alpha_n\}$  and  $\{t_n\}$  be sequences in  $\mathbb{R}$  satisfying*

(C1)  $0 < \alpha_n < 1$  and  $0 \leq t_n$ ;

(C2)  $\lim_n t_n = \lim_n \alpha_n / t_n = 0$ , where  $1/0 = \infty$ .

Fix  $u \in C$  and define a sequence  $\{x_n\}$  in  $C$  by

$$x_n = \alpha_n u + (1 - \alpha_n) T(t_n) x_n. \quad (1.2)$$

Then  $\{x_n\}$  converges strongly to the element of  $\bigcap_{t \geq 0} F(T(t))$  nearest to  $u$ .

We note that (C1) is needed to define  $\{x_n\}$ .

A nonexpansive semigroup  $\{T(t) : t \geq 0\}$  on  $C$  is said to be *uniformly asymptotically regular* (*u.a.r.*) if for every  $t \geq 0$  and for every bounded subset  $K$  of  $C$ ,

$$\lim_{s \rightarrow \infty} \sup_{x \in K} \|T(s+t)x - T(s)x\| = 0 \quad (1.3)$$

holds. The following is proved by Domínguez Benavides et al. [16]; see also [15].

**Theorem 1.2** (see [16]). *Let  $E, C$ , and  $\{T(t) : t \geq 0\}$  be as in Theorem 1.1. Assume that  $\{T(t) : t \geq 0\}$  is u.a.r. Let  $\{\alpha_n\}$  and  $\{t_n\}$  be sequences in  $\mathbb{R}$  satisfying (C1) and*

(D2)  $\lim_n \alpha_n = 0$  and  $\lim_n t_n = \infty$ .

Fix  $u \in C$  and define a sequence  $\{x_n\}$  in  $C$  by (1.2). Then  $\{x_n\}$  converges strongly to the element of  $\bigcap_{t \geq 0} F(T(t))$  nearest to  $u$ .

There is an interesting difference between Theorems 1.1 and 1.2, that is,  $\{t_n\}$  in Theorem 1.1 converges to 0 and  $\{t_n\}$  in Theorem 1.2 diverges to  $\infty$ . By the way, very recently, Akiyama and Suzuki [14] generalized Theorem 1.1. They replaced (C2) of Theorem 1.1 by

the following:

(C2')  $\{t_n\}$  is bounded;

(C3')  $\lim_n \alpha_n / (t_n - \tau) = 0$  for all  $\tau \in [0, \infty)$ .

They also showed that the conjunction of (C2') and (C3') is best possible; see also [18].

In this paper, motivated by the previous considerations, we generalize Theorem 1.2 concerning  $\{\alpha_n\}$  and  $\{t_n\}$ . Also, we will show that our new condition is best possible.

## 2. Main Results

We denote by  $\mathbb{N}$  the set of all positive integers and by  $\mathbb{R}$  the set of all real numbers. For  $t \in \mathbb{R}$ , we denote by  $[t]$  the maximum integer not exceeding  $t$ .

The following proposition plays an important role in this paper.

**Proposition 2.1.** *Let  $C$  be a set of a separated topological vector space  $E$ . Let  $\{T(t) : t \geq 0\}$  be a family of mappings on  $C$  such that  $T(s) \circ T(t) = T(s+t)$  for all  $s, t \in [0, \infty)$ . Assume that  $\{T(t) : t \geq 0\}$  is asymptotic regular, that is,*

$$\lim_{s \rightarrow \infty} (T(t+s)x - T(s)x) = 0 \quad (2.1)$$

for all  $t \in [0, \infty)$  and  $x \in C$ . Then

$$F(T(t)) = \bigcap_{s \geq 0} F(T(s)) \quad (2.2)$$

holds for all  $t \in (0, \infty)$ .

*Proof.* Fix  $t \in (0, \infty)$ . It is obvious that  $F(T(t)) \supset \bigcap_s F(T(s))$  holds. Let  $z \in C$  be a fixed point of  $T(t)$ . For every  $h \in [0, \infty)$ , we have

$$\begin{aligned} T(h)z - z &= \lim_{n \rightarrow \infty} (T(h) \circ T(t)^n z - T(t)^n z) \\ &= \lim_{n \rightarrow \infty} (T(h+nt)z - T(nt)z) \\ &= \lim_{s \rightarrow \infty} (T(h+s)z - T(s)z) \\ &= 0, \end{aligned} \quad (2.3)$$

and hence  $z$  is a common fixed point of  $\{T(t) : t \geq 0\}$ . □

It is well known that every Hilbert space has the Opial property.

**Proposition 2.2** (Opial [19]). *Let  $E$  be a Hilbert space. Let  $\{x_n\}$  be a sequence in  $E$  converging weakly to  $z_0 \in H$ . Then the inequality  $\liminf_n \|x_n - z\| \leq \liminf_n \|x_n - z_0\|$  implies  $z = z_0$ .*

We generalize Theorem 1.2.

**Theorem 2.3.** *Let  $C$  be a closed convex subset of a Hilbert space  $E$ . Let  $\{T(t) : t \geq 0\}$  be a u.a.r. nonexpansive semigroup on  $C$  such that  $\bigcap_{t \geq 0} F(T(t)) \neq \emptyset$ . Let  $\{\alpha_n\}$  and  $\{t_n\}$  be sequences in  $\mathbb{R}$  satisfying (C1) and*

$$(D2') \lim_n \alpha_n = \lim_n \alpha_n / t_n = 0.$$

*Fix  $u \in C$  and define a sequence  $\{x_n\}$  in  $C$  by (1.2). Then  $\{x_n\}$  converges strongly to the element of  $\bigcap_{t \geq 0} F(T(t))$  nearest to  $u$ .*

*Proof.* Put  $F(\mathcal{T}) = \bigcap_{t \geq 0} F(T(t))$ . Let  $v$  be the element of  $F(\mathcal{T})$  nearest to  $u$ . Since

$$\begin{aligned} \|x_n - v\| &= \|(1 - \alpha_n)T(t_n)x_n + \alpha_n u - v\| \\ &\leq (1 - \alpha_n)\|T(t_n)x_n - v\| + \alpha_n\|u - v\| \\ &\leq (1 - \alpha_n)\|x_n - v\| + \alpha_n\|u - v\|, \end{aligned} \quad (2.4)$$

we have  $\|x_n - v\| \leq \|u - v\|$ . Therefore  $\{x_n\}$  is bounded. Hence  $\{T(t)x_n : n \in \mathbb{N}, t \geq 0\}$  is also bounded.

We put

$$M := \sup\{\|T(t)x_n - u\| : n \in \mathbb{N}, t \geq 0\} < \infty. \quad (2.5)$$

Let  $\{f(n)\}$  be an arbitrary subsequence of  $\{n\}$ . Then there exists a subsequence  $\{g(n)\}$  of  $\{n\}$  such that  $\{x_{f \circ g(n)}\}$  converges weakly to  $x$ . We choose a subsequence  $\{h(n)\}$  of  $\{n\}$  such that

$$\tau := \lim_{n \rightarrow \infty} t_{f \circ g \circ h(n)} = \limsup_{n \rightarrow \infty} t_{f \circ g(n)}. \quad (2.6)$$

Put  $y_j = x_{f \circ g \circ h(j)}$ ,  $\beta_j = \alpha_{f \circ g \circ h(j)}$ , and  $s_j = t_{f \circ g \circ h(j)}$ . We will show  $x \in F(\mathcal{T})$ , dividing the following three cases:

- (i)  $\tau = \infty$ ,
- (ii)  $0 < \tau < \infty$ ,
- (iii)  $\tau = 0$ .

In the first case, we fix  $t \geq 0$ . For sufficiently large  $j \in \mathbb{N}$ , we have

$$\begin{aligned} \|T(t)x - y_j\| &\leq \|T(t)x - T(t)y_j\| + \|T(t)y_j - y_j\| \\ &\leq \|x - y_j\| + \beta_j\|T(t)y_j - u\| + (1 - \beta_j)\|T(t)y_j - T(s_j)y_j\| \\ &\leq \|x - y_j\| + \beta_j M + (1 - \beta_j)\|T(s_j - t)y_j - y_j\| \\ &\leq \|x - y_j\| + \beta_j M + (1 - \beta_j)\beta_j\|T(s_j - t)y_j - u\| + (1 - \beta_j)^2\|T(s_j - t)y_j - T(s_j)y_j\| \\ &\leq \|x - y_j\| + \beta_j(2 - \beta_j)M + (1 - \beta_j)^2\|T(s_j - t + t)y_j - T(s_j - t)y_j\|, \end{aligned} \quad (2.7)$$

and hence

$$\liminf_{j \rightarrow \infty} \|T(t)x - y_j\| \leq \liminf_{j \rightarrow \infty} \|x - y_j\|. \quad (2.8)$$

By the Opial property, we obtain  $T(t)x = x$ . Thus  $x \in F(\mathcal{T})$ .

In the second case, we have

$$\begin{aligned} \|T(\tau)x - y_j\| &\leq \|T(\tau)x - T(s_j)x\| + \|T(s_j)x - T(s_j)y_j\| + \|T(s_j)y_j - y_j\| \\ &\leq \|T(\tau)x - T(s_j)x\| + \|x - y_j\| + \beta_j \|T(s_j)y_j - u\| \\ &\leq \|T(|\tau - s_j|)x - T(0)x\| + \|x - y_j\| + \beta_j M, \end{aligned} \quad (2.9)$$

and hence

$$\liminf_{j \rightarrow \infty} \|T(\tau)x - y_j\| \leq \liminf_{j \rightarrow \infty} \|x - y_j\|. \quad (2.10)$$

By the Opial property, we obtain  $T(\tau)x = x$ . By Proposition 2.1, we obtain  $x \in F(\mathcal{T})$ .

In the third case, we fix  $t \geq 0$ . For sufficiently large  $j \in \mathbb{N}$ , we have

$$\begin{aligned} \|T(t)x - y_j\| &\leq \|T(t)x - T([t/s_j]s_j)x\| + \|T([t/s_j]s_j)x - T([t/s_j]s_j)y_j\| \\ &\quad + \sum_{k=0}^{[t/s_j]-1} \|T(ks_j)y_j - T((k+1)s_j)y_j\| + \|T(0)y_j - y_j\| \\ &\leq \|T(t - [t/s_j]s_j)x - T(0)x\| + \|x - y_j\| \\ &\quad + [t/s_j] \|T(s_j)y_j - y_j\| + \|T(0)y_j - T(s_j)y_j\| + \|T(s_j)y_j - y_j\| \\ &\leq \|T(t - [t/s_j]s_j)x - T(0)x\| + \|x - y_j\| \\ &\quad + [t/s_j] \|T(s_j)y_j - y_j\| + \|y_j - T(s_j)y_j\| + \|T(s_j)y_j - y_j\| \\ &= \|T(t - [t/s_j]s_j)x - T(0)x\| + \|x - y_j\| + ([t/s_j] + 2) \|T(s_j)y_j - y_j\| \\ &= \|T(t - [t/s_j]s_j)x - T(0)x\| + \|x - y_j\| + ([t/s_j] + 2)\beta_j \|T(s_j)y_j - u\| \\ &\leq \max\{\|T(s)x - T(0)x\| : 0 \leq s \leq s_j\} + \|x - y_j\| + (t\beta_j/s_j + 2\beta_j)M. \end{aligned} \quad (2.11)$$

Hence (2.8) holds. Thus we obtain  $x \in F(\mathcal{T})$ .

We next prove that  $\{y_j\}$  converges strongly to  $v$ . Since

$$\begin{aligned} &\beta_j \|y_j - v\|^2 + (1 - \beta_j) \langle (y_j - T(s_j)y_j) - (v - T(s_j)v), y_j - v \rangle \\ &= \beta_j \langle u - v, y_j - v \rangle, \\ &\langle (y_j - T(s_j)y_j) - (v - T(s_j)v), y_j - v \rangle \\ &\geq \|y_j - v\|^2 - \|T(s_j)y_j - T(s_j)v\| \|y_j - v\| \geq 0, \end{aligned} \quad (2.12)$$

we obtain  $\|y_j - v\|^2 \leq \langle u - v, y_j - v \rangle$ . Since  $\langle u - v, x - v \rangle \leq 0$ , we have

$$\begin{aligned} \|y_j - v\|^2 &\leq \langle u - v, y_j - v \rangle \\ &= \langle u - v, y_j - x \rangle + \langle u - v, x - v \rangle \\ &\leq \langle u - v, y_j - x \rangle, \end{aligned} \quad (2.13)$$

and hence  $\{y_j\}$  converges strongly to  $v$ . Since  $\{x_{f(n)}\}$  is arbitrary, we obtain that  $\{x_n\}$  converges strongly to  $v$ .  $\square$

Using [20, Theorem 7], we obtain the following Moudafi's type convergence theorem; see [21].

**Corollary 2.4.** *Let  $E, C, \{T(t) : t \geq 0\}, \{\alpha_n\}$ , and  $\{t_n\}$  be as in Theorem 2.3. Let  $\Phi$  be a contraction on  $C$ ; that is, there exists  $r \in [0, 1)$  such that  $\|\Phi x - \Phi y\| \leq r\|x - y\|$  for  $x, y \in C$ . Define a sequence  $\{x_n\}$  in  $C$  by*

$$x_n = \alpha_n \Phi x_n + (1 - \alpha_n) T(t_n) x_n. \quad (2.14)$$

*Then  $\{x_n\}$  converges strongly to the unique point  $z \in C$  satisfying  $P \circ \Phi z = z$ , where  $P$  is the metric projection from  $C$  onto  $\bigcap_{t \geq 0} F(T(t))$ .*

We will show that (D2') is best possible.

*Example 2.5.* Put  $E = \ell^2(\mathbb{N})$ , that is,  $E$  is a Hilbert space consisting of all the functions  $x$  from  $\mathbb{N}$  into  $\mathbb{R}$  satisfying  $\sum_{k \in \mathbb{N}} |x(k)|^2 < \infty$  with inner product  $\langle x, y \rangle = \sum_{k \in \mathbb{N}} x(k)y(k)$ . Define a bounded closed convex subset  $C$  of  $E$  by

$$C = \{x \in E : 0 \leq x(k) \leq p_k\}, \quad (2.15)$$

where  $p_k = 2^{-k/2}$ . Define a u.a.r. nonexpansive semigroup  $\{T(t) : t \geq 0\}$  on  $C$  by

$$(T(t)x)(k) = \max\{x(k) - tp_k^2, 0\}. \quad (2.16)$$

Let  $\{e_k\}$  be the canonical basis of  $E$  and put  $u = \sum_{k=1}^{\infty} p_k e_k$ . Let  $\{\alpha_n\}$  and  $\{t_n\}$  be sequences in  $\mathbb{R}$  satisfying (C1) and define  $\{x_n\}$  in  $C$  by (1.2). Then  $\{x_n\}$  converges to a common fixed point of  $\{T(t) : t \geq 0\}$  only if  $\lim_n \alpha_n = \lim_n \alpha_n / t_n = 0$ .

*Proof.* For  $\alpha \in (0, 1)$  and  $t \geq 0$ , we define  $x(\alpha, t)$  by

$$x(\alpha, t) = \alpha u + (1 - \alpha) T(t)x(\alpha, t). \quad (2.17)$$

We note

$$x(\alpha, t)(k) = \begin{cases} \alpha p_k, & \text{if } \alpha \leq t p_k, \\ \left(1 + t p_k - \frac{t p_k}{\alpha}\right) p_k, & \text{if } \alpha \geq t p_k. \end{cases} \quad (2.18)$$

So,  $x(\alpha, t)(k) \geq \alpha p_k$ . It is obvious that  $\bigcap_{t \geq 0} F(T(t)) = \{0\}$ . We assume  $\lim_n x_n = \lim_n x(\alpha_n, t_n) = Pu = 0$ . Then

$$0 = \lim_{n \rightarrow \infty} \frac{x_n(1)}{p_1} \geq \lim_{n \rightarrow \infty} \alpha_n. \quad (2.19)$$

Arguing by contradiction, we assume  $\limsup_n \alpha_n / t_n > 0$ . Then there exist  $\kappa \in \mathbb{N}$  and a subsequence  $\{f(n)\}$  of  $\{n\}$  such that

$$\frac{\alpha_{f(n)}}{t_{f(n)}} \geq 2p_\kappa. \quad (2.20)$$

Since  $\lim_n x_{f(n)}(\kappa) = 0$ , we have

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \frac{x_{f(n)}(\kappa)}{p_\kappa} = \lim_{n \rightarrow \infty} \left(1 + t_{f(n)} p_\kappa - \frac{t_{f(n)} p_\kappa}{\alpha_{f(n)}}\right) \\ &\geq \limsup_{n \rightarrow \infty} \left(1 - \frac{t_{f(n)} p_\kappa}{\alpha_{f(n)}}\right) \geq \frac{1}{2} > 0, \end{aligned} \quad (2.21)$$

which is a contradiction. Therefore we obtain  $\lim_n \alpha_n / t_n = 0$ .  $\square$

By Theorem 2.3 and Example 2.5, we obtain the following.

**Theorem 2.6.** *Let  $E$  be an infinite-dimensional Hilbert space. Let  $\{\alpha_n\}$  and  $\{t_n\}$  be sequences in  $\mathbb{R}$  satisfying (C1). Then the following are equivalent:*

- (i)  $\lim_n \alpha_n = \lim_n \alpha_n / t_n = 0$ ,
- (ii) *if  $C$  is a bounded closed convex subset  $C$  of  $E$ ,  $\{T(t) : t \geq 0\}$  is a u.a.r. nonexpansive semigroup on  $C$ ,  $u \in C$ , and  $\{x_n\}$  is a sequence in  $C$  defined by (1.2), then  $\{x_n\}$  converges strongly to the element of  $\bigcap_{t \geq 0} F(T(t))$  nearest to  $u$ .*

Compare (D2') with the conjunction of (C2') and (C3'). We can tell that the difference between both conditions is u.a.r.

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