Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 264628, 13 pages
doi:10.1155/2010/264628

Research Article

Hybrid Viscosity Iterative Method for Fixed Point, Variational Inequality and Equilibrium Problems

Yi-An Chen and Yi-Ping Zhang

College of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, China

Correspondence should be addressed to Yi-An Chen, chenyian1969@sohu.com

Received 27 December 2009; Revised 3 May 2010; Accepted 1 June 2010

Academic Editor: Simeon Reich

We introduce an iterative scheme by the viscosity iterative method for finding a common element of the solution set of an equilibrium problem, the solution set of the variational inequality, and the fixed points set of infinitely many nonexpansive mappings in a Hilbert space. Then we prove our main result under some suitable conditions.

1. Introduction

Let H be a real Hilbert space with the inner product and the norm being denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty, closed, and convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of (1.1) is denoted by $\mathrm{EP}(F)$.

Let $A : C \to H$ be a mapping. The classical variational inequality, denoted by $\mathrm{VI}(A, C)$, is to find $x^* \in C$ such that

$$\langle Ax^*, v - x^* \rangle \geq 0, \quad \forall v \in C. \tag{1.2}$$

The variational inequality has been extensively studied in the literature (see, e.g., [1–3]). The mapping A is called α-inverse-strongly monotone if

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2, \quad \forall u, v \in C, \tag{1.3}$$

where α is a positive real number.

A mapping $T : C \to C$ is called strictly pseudocontractive if there exists k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.4}$$

It is easy to know that $I - T$ is $((1 - k)/2)$-inverse-strongly-monotone. If $k = 0$, then T is nonexpansive. We denote by $F(T)$ the fixed points set of T.

In 2003, for $x_0 \in C$, Takahashi and Toyoda [4] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad n \geq 0, \tag{1.5}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, A is an α-inverse-strongly monotone mapping, $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$, and P_C is the metric projection. They proved that if $F(S) \bigcap \mathrm{VI}(A, C) \neq \emptyset$, then $\{x_n\}$ converges weakly to some $z \in F(S) \bigcap \mathrm{VI}(A, C)$.

Recently, S. Takahashi and W. Takahashi [5] introduced an iterative scheme for finding a common element of the solution set of (1.1) and the fixed points set of a nonexpansive mapping in a Hilbert space. If F is bifunction which satisfies the following conditions:

- $(A1)$ $F(x, x) = 0$ for all $x \in C$;
- $(A2)$ F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
- $(A3)$ for each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y)$;
- $(A4)$ for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous,

then they proved the following strong convergence theorem.

Theorem A (see [5]). *Let C be a closed and convex subset of a real Hilbert space H. Let $F : C\times C \to \mathbb{R}$ be a bifunction which satisfies conditions $(A1)$–$(A4)$.*

Let $T : C \to H$ be a nonexpansive mapping such that $F(T) \bigcap EP(F) \neq \emptyset$ and let $f : H \to H$ be a contraction; that is, there is a constant $k \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq k\|x - y\|, \quad \forall x, y \in H, \tag{1.6}$$

and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in C$ and

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tu_n, \quad n \geq 1, \end{gathered} \tag{1.7}$$

where $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy* $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$, $\liminf_{n\to\infty} r_n > 0$, *and* $\sum_{n=1}^{\infty}|r_{n+1}-r_n| < \infty$.

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $z \in F(T)\bigcap EP(F)$, *where* $z = P_{F(T)\bigcap EP(F)}f(z)$.

Let $\{T_n\}_{n=1}^{\infty}$ *be a sequence of nonexpansive mappings of* C *into itself and* $\{\lambda_n\}_{n=1}^{\infty}$ *a sequence of nonnegative numbers in* $[0,1]$. *For each* $n \geq 1$, *define a mapping* W_n *of* C *into itself as follows:*

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I, \\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I, \\
&\vdots \\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I, \\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I, \\
&\vdots \\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I, \\
W_n = U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned} \tag{1.8}$$

Such a mapping W_n *is called the* W*-mapping generated by* $T_n, T_{n-1}, \ldots, T_1$ *and* $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$ *(see [6]).*

In this paper, we introduced a new iterative scheme generated by $x_1 \in C$ and find u_n such that

$$\begin{aligned}
F(u_n, y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle &\geq 0, \quad \forall y \in C, \\
y_n &= \beta_n f(x_n) + (1-\beta_n)x_n, \quad n \geq 1, \\
x_{n+1} &= \alpha_n y_n + (1-\alpha_n)W_n P_C(u_n - \delta_n A u_n),
\end{aligned} \tag{1.9}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, $\{r_n\}$ and $\{\delta_n\}$ are sequences in $(0,\infty)$, f is a fixed contractive mapping with contractive coefficient $k \in (0,1)$, A is an α-inverse-strongly monotone mapping of C to H, F is a bifunction which satisfies conditions $(A1)$–$(A4)$, and $\{W_n\}$ is generated by (1.8). Then we proved that the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in \bigcap_{n=1}^{\infty} F(T_n)\bigcap \mathrm{VI}(A,C)\bigcap \mathrm{EP}(F) = F$, where $x^* = P_F f(x^*)$.

2. Preliminaries

Let H be a real Hilbert space and let C be a closed and convex subset of H. P_C is the metric projection from H onto C, that is, for any $x \in H$, $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. It is easy to see that P_C is nonexpansive and

$$u \in \mathrm{VI}(A, C) \Longleftrightarrow u = P_C(u - \lambda A u), \quad \lambda > 0. \tag{2.1}$$

If A is an α-inverse-strongly monotone mapping of C to H, then it is obvious that A is $(1/\alpha)$-Lipschitz continuous. We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned} \|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2. \end{aligned} \tag{2.2}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is nonexpansive.

Lemma 2.1 (see [7]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E, and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all $n \geq 1$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|z_n - x_n\| = 0$.*

Lemma 2.2 (see [8]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 1, \tag{2.3}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

$$\sum_{n=1}^{\infty} \alpha_n = \infty; \quad \limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0 \quad \text{or} \quad \sum_{n=1}^{\infty} |\delta_n| < \infty. \tag{2.4}$$

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.3 (see [9]). *Let C be a nonempty, closed, and convex subset of H and F a bifunction of $C \times C$ into $\mathbb{R}$ that satisfies conditions $(A1)$–$(A4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.5}$$

Lemma 2.4 (see [9]). *Assume that $F : C \times C \to \mathbb{R}$ satisfies conditions $(A1)$–$(A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}. \tag{2.6}$$

Then, the following holds:

(i) T_r *is single-valued;*

(ii) T_r *is firmly nonexpansive, that is,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \quad \forall x, y \in H; \tag{2.7}$$

(iii) $F(T_r) = EP(F)$;

(iv) $EP(F)$ *is closed and convex.*

Lemma 2.5 (Opial's theorem [10]). *Each Hilbert space H satisfies Opial's condition; that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\| \tag{2.8}$$

holds for each $y \in H$ with $x \neq y$.

Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C, where C is a nonempty, closed and convex subset of a real Hilbert space H. Given a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $[0,1]$, one defines a sequence $\{W_n\}_{n=1}^{\infty}$ of self-mappings on C generated by (1.8). Then one has the following results.

Lemma 2.6 (see [6]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\{\lambda_n\}$ is a sequence in $(0,b]$ for some $b \in (0,1)$. Then, for every $x \in C$ and $k \geq 1$ the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Remark 2.7. It can be shown from Lemma 2.6 that if D is a nonempty and bounded subset of C, then for $\varepsilon > 0$ there exists $n_0 \geq k$ such that $\sup_{x\in D}\|U_{n,k}x - U_{n-1,k}x\| \leq \varepsilon$ for all $n > n_0$.

Remark 2.8. Using Lemma 2.6, we can define a mapping $W : C \to C$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x \tag{2.9}$$

for all $x \in C$. Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Since W_n is nonexpansive, $W : C \to C$ is also nonexpansive. Indeed, observe that for each $x, y \in C$,

$$\|Wx - Wy\| = \lim_{n\to\infty}\|W_n x - W_n y\| \leq \|x - y\|. \tag{2.10}$$

Let $\{x_n\}$ be a bounded sequence in C and $D = \{x_n : n \geq 0\}$. Then, it is clear from Remark 2.7 that for $\varepsilon > 0$ there exists $N_0 \geq 1$ such that for all $n > N_0$,

$$\|W_n x_n - Wx_n\| = \|U_{n,1}x_n - U_1 x_n\| \leq \sup_{x\in D}\|U_{n,1}x - U_1 x\| \leq \varepsilon. \tag{2.11}$$

This implies that $\lim_{n\to\infty}\|W_n x_n - Wx_n\| = 0$.

Lemma 2.9 (see [6]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\{\lambda_n\}$ is a sequence in $(0,b]$ for some $b \in (0,1)$. Then, $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

3. Strong Convergence Theorem

Theorem 3.1. *Let H be a Hilbert space. Let C be a nonempty, closed, and convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions $(A1)$–$(A4)$, A an α-inverse-strongly monotone mapping of C to H, f a contraction of C into itself, and $\{T_n\}_{n=1}^{\infty}$ a sequence of nonexpansive self-mappings on C such that $\mathcal{F} \neq \emptyset$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\lambda_n\}$ are sequences in $(0,1)$, and $\{r_n\}$ and $\{\delta_n\}$ are sequences in $(0,\infty)$ which satisfies the following conditions:*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$; $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(iv) $\delta_n \in [0,b]$, $b < 2\alpha$, $\lim_{n\to\infty} \delta_n = 0$;

(v) $\lambda_n \in [0,c]$, $c \in (0,1)$.

Then $\{x_n\}$ and $\{u_n\}$ generated by (1.9) converge strongly to $x^* \in \mathcal{F}$, where $x^* = P_{\mathcal{F}} f(x^*)$.

Proof. Let $p \in \mathcal{F}$. It follows from Lemma 2.4 and (1.9) that $u_n = T_{r_n} x_n$, and hence,

$$\|u_n - p\| = \|T_{r_n} x_n - T_{r_n} p\| \leq \|x_n - p\|, \tag{3.1}$$

for all $n \in \mathbb{N}$. Let $z_n = P_C(u_n - \delta_n A u_n)$. Since $I - \delta_n A$ is nonexpansive and $p = P_C(p - \delta_n A p)$, we have

$$\|z_n - p\| \leq \|u_n - \delta_n A u_n - (p - \delta_n A p)\| \leq \|u_n - p\| \leq \|x_n - p\|, \tag{3.2}$$

$$\begin{aligned} \|y_n - p\| &\leq \beta_n \|f(x_n) - p\| + (1-\beta_n)\|x_n - p\| \\ &\leq \beta_n \|f(x_n) - f(p)\| + \beta_n \|f(p) - p\| + (1-\beta_n)\|x_n - p\| \\ &\leq [1 - \beta_n(1-k)]\|x_n - p\| + \beta_n \|f(p) - p\|. \end{aligned} \tag{3.3}$$

Thus,

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n y_n + (1-\alpha_n) W_n z_n - p\| \\ &\leq \alpha_n \|y_n - p\| + (1-\alpha_n)\|z_n - p\| \\ &\leq \alpha_n [1 - \beta_n(1-k)]\|x_n - p\| + \alpha_n \beta_n \|f(p) - p\| + (1-\alpha_n)\|x_n - p\| \\ &= [1 - \alpha_n \beta_n (1-k)]\|x_n - p\| + \alpha_n \beta_n (1-k) \frac{\|f(p) - p\|}{1-k} \\ &\leq \max\left\{ \|x_n - p\|, \frac{\|f(p) - p\|}{1-k} \right\}. \end{aligned} \tag{3.4}$$

Hence $\{x_n\}$ is bounded. So $\{u_n\}$, $\{z_n\}$, $\{W_n x_n\}$, $\{W_n z_n\}$, and $\{f(x_n)\}$ are also bounded.

Next, we claim that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Indeed, assume that $x_{n+1}=\rho_n x_n+(1-\rho_n)t_n$, where $\rho_n=\alpha_n(1-\beta_n)$, $n\geq 0$. Then,

$$\begin{aligned} t_{n+1}-t_n &= \frac{\alpha_{n+1}\beta_{n+1}f(x_{n+1})+(1-\alpha_{n+1})W_{n+1}z_{n+1}}{1-\rho_{n+1}}-\frac{\alpha_n\beta_n f(x_n)+(1-\alpha_n)W_n z_n}{1-\rho_n} \\ &= \frac{\alpha_{n+1}\beta_{n+1}f(x_{n+1})}{1-\rho_{n+1}}-\frac{\alpha_n\beta_n f(x_n)}{1-\rho_n}+\frac{1-\alpha_{n+1}}{1-\rho_{n+1}}(W_{n+1}z_{n+1}-W_{n+1}z_n) \\ &\quad+\frac{1-\alpha_{n+1}}{1-\rho_{n+1}}W_{n+1}z_n-\frac{1-\alpha_n}{1-\rho_n}W_n z_n \\ &\leq \frac{\alpha_{n+1}\beta_{n+1}f(x_{n+1})}{1-\rho_{n+1}}-\frac{\alpha_n\beta_n f(x_n)}{1-\rho_n}+\frac{1-\alpha_{n+1}}{1-\rho_{n+1}}(z_{n+1}-z_n) \\ &\quad+W_{n+1}z_n-\frac{\alpha_{n+1}\beta_{n+1}}{1-\rho_{n+1}}W_{n+1}z_n-W_n z_n+\frac{\alpha_n\beta_n}{1-\rho_n}W_n z_n, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \|z_{n+1}-z_n\| &\leq \|u_{n+1}-\delta_{n+1}Au_{n+1}-(u_n-\delta_n Au_n)\| \\ &\leq \|(I-\delta_{n+1}A)u_{n+1}-(I-\delta_{n+1}A)u_n\|+\|(I-\delta_{n+1}A)u_n-(I-\delta_n A)u_n\| \\ &\leq \|u_{n+1}-u_n\|+\|\delta_{n+1}-\delta_n\|\|Au_n\|. \end{aligned} \tag{3.6}$$

Using (1.8) and the nonexpansivity of T_i, we deduce that

$$\begin{aligned} \|W_{n+1}z_n-W_n z_n\| &= \|\lambda_1 T_1 U_{n+1,2}z_n-\lambda_1 T_1 U_{n,2}z_n\| \\ &\leq \lambda_1\|U_{n+1,2}z_n-U_{n,2}z_n\| \\ &\leq \lambda_1\|\lambda_2 T_2 U_{n+1,3}z_n-\lambda_2 T_2 U_{n,3}z_n\| \\ &\leq \lambda_1\lambda_2\|U_{n+1,3}z_n-U_{n,3}z_n\| \\ &\ \ \vdots \\ &\leq \left(\prod_{i=1}^{n}\lambda_i\right)\|U_{n+1,n+1}z_n-U_{n,n+1}z_n\| \\ &\leq M\prod_{i=1}^{n}\lambda_i, \end{aligned} \tag{3.7}$$

for some constant $M\geq 0$. On the other hand, from $u_n=T_{r_n}x_n$ and $u_{n+1}=T_{r_{n+1}}x_{n+1}$, we obtain

$$F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\quad \forall y\in C, \tag{3.8}$$

$$F(u_{n+1},y)+\frac{1}{r_{n+1}}\langle y-u_{n+1},u_{n+1}-x_{n+1}\rangle\geq 0,\quad \forall y\in C. \tag{3.9}$$

Setting $y = u_{n+1}$ in (3.8) and $y = u_n$ in (3.9), we get

$$F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0,$$
$$F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0. \tag{3.10}$$

From (A_2), we have

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0, \tag{3.11}$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0. \tag{3.12}$$

Without loss of generality, we may assume that there exists a real number r such that $r_n > r > 0$ for all $n \geq 0$. Then

$$\|u_{n+1} - u_n\|^2 \leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle$$
$$\leq \|u_{n+1} - u_n\| \left(\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right), \tag{3.13}$$

and hence

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\|$$
$$\leq \|x_{n+1} - \mathrm{x}_n\| + \frac{1}{r}|r_{n+1} - r_n| L, \tag{3.14}$$

where $L = \sup\{\|u_n - x_n\| : n \geq 0\}$. It follows from (3.5), (3.6), (3.7), and (3.14) that

$$\|t_{n+1} - t_n\| - \|x_{n+1} - x_n\| \leq \frac{\alpha_{n+1}\beta_{n+1}}{1 - \rho_{n+1}}\left[\|f(x_{n+1})\| + \|W_{n+1}z_n\|\right] + \frac{\alpha_n\beta_n}{1 - \rho_n}\left[\|f(x_n)\| + \|W_n z_n\|\right]$$
$$+ \frac{1 - \alpha_{n+1}}{1 - \rho_{n+1}}\left[\|x_{n+1} - x_n\| + \frac{L}{r}|r_{n+1} - r_n| + |\delta_{n+1} - \delta_n|\|Au_n\|\right]$$
$$+ M\prod_{i=1}^{n}\lambda_i - \|x_{n+1} - x_n\|$$

$$\begin{aligned}&\leq \frac{\alpha_{n+1}\beta_{n+1}}{1-\rho_{n+1}}\left[\|f(x_{n+1})\| + \|W_{n+1}z_n\|\right] + \frac{\alpha_n\beta_n}{1-\rho_n}\left[\|f(x_n)\| + \|W_n z_n\|\right]\\ &\quad + \frac{1-\alpha_{n+1}}{1-\rho_{n+1}}\left[\frac{L}{r}|r_{n+1}-r_n| + |\delta_{n+1}-\delta_n|\|Au_n\|\right] + M\prod_{i=1}^{n}\lambda_i.\end{aligned} \tag{3.15}$$

Therefore, $\limsup_{n\to\infty}(\|t_{n+1}-t_n\| - \|x_{n+1}-x_n\|) \leq 0$.

Since $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}\alpha_n < 1$ and $\lim_{n\to\infty}\beta_n = 0$, hence,

$$0 < \liminf_{n\to\infty}\rho_n \leq \limsup_{n\to\infty}\rho_n < 1. \tag{3.16}$$

Lemma 2.1 yields that $\lim_{n\to\infty}\|t_n - x_n\| = 0$. Consequently, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\rho_n)\|t_{\mathrm{n}} - x_n\| = 0$.

For $p \in F$, we obtain

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{r_n}x_n - T_{r_n}p\|^2\\ &\leq \langle T_{r_n}x_n - T_{r_n}p, x_n - p\rangle\\ &= \langle u_n - p, x_n - p\rangle\\ &= \frac{1}{2}\left(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2\right),\end{aligned} \tag{3.17}$$

and hence

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.18}$$

This together with (3.2) yields that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \alpha_n\|y_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2\\ &\leq \alpha_n\|\beta_n(f(x_n)-p) + (1-\beta_n)(x_n - p)\|^2 + (1-\alpha_n)\|u_n - p\|^2\\ &\leq \alpha_n\beta_n\|f(x_n)-p\|^2 + \alpha_n(1-\beta_n)\|x_n - p\|^2\\ &\quad + (1-\alpha_n)\left(\|x_n - p\|^2 - \|u_n - x_n\|^2\right),\end{aligned} \tag{3.19}$$

and hence,

$$\begin{aligned}(1-\alpha_n)\|u_n - x_n\|^2 &\leq \alpha_n\beta_n\|f(x_n)-p\|^2 + (1-\alpha_n\beta_n)\|x_n - p\|^2 - \|x_{n+1}-p\|^2\\ &\leq \alpha_n\beta_n\left[\|f(x_n)-p\|^2 - \|x_n - p\|^2\right]\\ &\quad + \|x_{n+1}-x_n\|\left(\|x_n - p\| + \|x_{n+1}-p\|\right).\end{aligned} \tag{3.20}$$

So $\|u_n - x_n\| \to 0$ (note that $\lim_{n\to\infty} \beta_n = 0$ and $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$). Since

$$\begin{aligned}
\|W_n u_n - u_n\| &\le \|W_n u_n - W_n x_n\| + \|W_n x_n - x_n\| + \|x_n - u_n\| \\
&\le 2\|x_n - u_n\| + \|W_n x_n - x_n\|, \\
\|x_n - W_n x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - W_n x_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\beta_n \left\|f(x_n) - W_n x_n\right\| \\
&\quad + \alpha_n(1-\beta_n)\|x_n - W_n x_n\| + (1-\alpha_n)\|W_n z_n - W_n x_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\beta_n \left\|f(x_n) - W_n x_n\right\| \\
&\quad + \alpha_n(1-\beta_n)\|x_n - W_n x_n\| + (1-\alpha_n)\|P_C(u_n - \delta_n A u_n) - P_C x_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\beta_n \left\|f(x_n) - W_n x_n\right\| + \alpha_n(1-\beta_n)\|x_n - W_n x_n\| \\
&\quad + (1-\alpha_n)\|u_n - x_n\| + (1-\alpha_n)\delta_n\|A u_n\|,
\end{aligned} \tag{3.21}$$

we obtain $\lim_{n\to\infty}\|x_n - W_n x_n\| = 0$, and hence $\lim_{n\to\infty}\|u_n - W_n u_n\| = 0$. Thus, $\|u_n - W u_n\| \le \|u_n - W_n u_n\| + \|W_n u_n - W u_n\| \to 0$.

Let $Q = P_F$. Then Qf is a contraction of H into itself. In fact, there exists $k \in [0,1)$ such that $\|f(x) - f(y)\| \le k\|x - y\|$ for all $x, y \in H$. So

$$\left\|Qf(x) - Qf(y)\right\| \le \left\|f(x) - f(y)\right\| \le k\left\|x - y\right\| \tag{3.22}$$

for all $x, y \in H$. So Qf is a contraction by Banach contraction principle [11]. Since H is a complete space, there exists a unique element $x^* \in C \subset H$ such that $x^* = Qf(x^*)$.

Next we show that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle \le 0, \tag{3.23}$$

where $x^* = Qf(x^*)$. To show this inequality, we choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty}\left\langle f(x^*) - x^*, u_n - x^*\right\rangle = \lim_{n\to\infty}\left\langle f(x^*) - x^*, u_{n_i} - x^*\right\rangle. \tag{3.24}$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence of $\{u_{n_i}\}$ which converges weakly to some $\omega \in C$, that is, $u_{n_i} \rightharpoonup \omega$. From $\|Wu_n - u_n\| \to 0$, we obtain that $Wu_{n_i} \rightharpoonup \omega$. Now we will show that $\omega \in F(W) \bigcap \mathrm{VI}(A, C) \bigcap \mathrm{EP}(F)$. First, we will show $\omega \in \mathrm{EP}(F)$. From $u_n = T_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\left\langle y - u_n, u_n - x_n\right\rangle \ge 0, \quad \forall y \in C. \tag{3.25}$$

By $(A2)$, we also have

$$\frac{1}{r_n}\left\langle y - u_n, u_n - x_n\right\rangle \ge F(y, u_n), \tag{3.26}$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq F(y, u_{n_i}). \tag{3.27}$$

Since $((u_{n_i} - x_{n_i})/r_{n_i}) \to 0$ and $u_{n_i} \rightharpoonup \omega$, it follows from $(A4)$ that $0 \geq F(y, \omega)$ for all $y \in C$. For any $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)\omega$. Since $y \in C$ and $\omega \in C$, then we have $y_t \in C$ and hence $F(y_t, \omega) \leq 0$. This together with $(A1)$ and $(A4)$ yields that

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, \omega) \leq tF(y_t, y), \tag{3.28}$$

and thus $0 \leq F(y_t, y)$. From $(A3)$, we have $0 \leq F(\omega, y)$ for all $y \in C$ and hence $\omega \in \mathrm{EP}(F)$. Now, we show that $\omega \in F(W)$. Indeed, we assume that $\omega \notin F(W)$; from Opial's condition, we have

$$\begin{aligned}
\liminf_{i \to \infty} \|u_{n_i} - \omega\| &< \liminf_{i \to \infty} \|u_{n_i} - W\omega\| \\
&\leq \liminf_{i \to \infty} (\|u_{n_i} - Wu_{n_i}\| + \|Wu_{n_i} - W\omega\|) \\
&\leq \liminf_{i \to \infty} \|u_{n_i} - \omega\|.
\end{aligned} \tag{3.29}$$

This is a contradiction. Thus, we obtain that $\omega \in F(W)$. Finally, by the same argument as in the proof of [3, Theorem 3.1], we can show that $\omega \in \mathrm{VI}(A, C)$. Hence $\omega \in F(W) \bigcap \mathrm{VI}(A, C) \bigcap \mathrm{EP}(F)$. Hence,

$$\begin{aligned}
\limsup_{n \to \infty} \langle f(x^*) - x^*, x_n - x^* \rangle &= \limsup_{n \to \infty} \langle f(x^*) - x^*, u_n - x^* \rangle \\
&= \lim_{i \to \infty} \langle f(x^*) - x^*, u_{n_i} - x^* \rangle \\
&= \langle f(x^*) - x^*, \omega - x^* \rangle \leq 0.
\end{aligned} \tag{3.30}$$

Now we show that $\lim_{n \to \infty} \|x_n - x^*\| = 0$.

From (1.9), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \langle \alpha_n \beta_n f(x_n) + \alpha_n (1 - \beta_n) x_n + (1 - \alpha_n) W_n z_n - x^*, x_{n+1} - x^* \rangle \\
&= \alpha_n \beta_n \langle f(x_n) - x^*, x_{n+1} - x^* \rangle + \alpha_n (1 - \beta_n) \langle x_n - x^*, x_{n+1} - x^* \rangle \\
&\quad + (1 - \alpha_n) \langle W_n z_n - x^*, x_{n+1} - x^* \rangle \\
&\leq \alpha_n \beta_n k \|x_n - x^*\| \|x_{n+1} - x^*\| + \alpha_n \beta_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&\quad + \alpha_n (1 - \beta_n) \|x_n - x^*\| \|x_{n+1} - x^*\| + (1 - \alpha_n) \|x_n - x^*\| \|x_{n+1} - x^*\| \\
&\leq [1 - \alpha_n \beta_n (1 - k)] \frac{\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2}{2} \\
&\quad + \alpha_n \beta_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle,
\end{aligned} \tag{3.31}$$

and hence,

$$\begin{aligned}\|x_{n+1}-x^*\|^2 \le \left[1-\alpha_n\beta_n(1-k)\right]\|x_n-x^*\|^2 \\ + \alpha_n\beta_n(1-k)\frac{2}{1-k}\langle f(x^*)-x^*, x_{n+1}-x^*\rangle.\end{aligned} \tag{3.32}$$

Using (3.23) and Lemma 2.2, we conclude that $\{x_n\}$ converges strongly to x^*. Consequently, $\{u_n\}$ converges strongly to x^*. This completes the proof. □

Using Theorem 3.1, we prove the following theorem.

Theorem 3.2. *Let H, C, F, f, and $\{T_n\}$ be given as in Theorem 3.1 and let S be an α-strictly pseudocontractive mapping such that $F \neq \emptyset$. Suppose that $\delta_n \in [0,b]$, $b < 1-\alpha$ and $\lim_{n\to\infty}\delta_n = 0$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences and find u_n such that*

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge 0, \quad \forall y \in C, \\ y_n = \beta_n f(x_n) + (1-\beta_n)x_n, \quad n \ge 1, \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n)W_n((1-\delta_n)u_n + \delta_n S u_n), \end{gathered} \tag{3.33}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$, and $\{\lambda_n\}$ are given as in Theorem 3.1. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in F$, where $x^* = P_F f(x^*)$.

Proof. Put $A = I - S$. Then A is $((1-\alpha)/2)$-inverse-strongly-monotone. We have $F(S) = \mathrm{VI}(C, A)$ and put $P_C(u_n - \delta_n u_n) = (1-\delta_n)u_n + \delta_n S u_n$. So by Theorem 3.1 we obtain the desired result. □

Acknowledgments

The author would like to express his thanks to Professor Simeon Reich, Technion-Israel Institute of Technology, Israel, and the anonymous referees for their valuable comments and suggestions on a previous draft, which resulted in the present version of the paper. This work was supported by the Natural Science Foundation of China (10871217) and Grant KJ080725 of the Chongqing Municipal Education Commission.

References

[1] F. Liu and M. Z. Nashed, "Regularization of nonlinear ill-posed variational inequalities and convergence rates," *Set-Valued Analysis*, vol. 6, no. 4, pp. 313–344, 1998.
[2] L.-C. Zeng and J.-C. Yao, "Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems," *Taiwanese Journal of Mathematics*, vol. 10, no. 5, pp. 1293–1303, 2006.
[3] Y. Yao, Y.-C. Liou, and J.-C. Yao, "An extragradient method for fixed point problems and variational inequality problems," *Journal of Inequalities and Applications*, vol. 2007, Article ID 38752, 12 pages, 2007.
[4] W. Takahashi and M. Toyoda, "Weak convergence theorems for nonexpansive mappings and monotone mappings," *Journal of Optimization Theory and Applications*, vol. 118, no. 2, pp. 417–428, 2003.

[5] S. Takahashi and W. Takahashi, "Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 506–515, 2007.

[6] K. Shimoji and W. Takahashi, "Strong convergence to common fixed points of infinite nonexpansive mappings and applications," *Taiwanese Journal of Mathematics*, vol. 5, no. 2, pp. 387–404, 2001.

[7] T. Suzuki, "Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.

[8] H.-K. Xu, "Viscosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.

[9] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 6, no. 1, pp. 117–136, 2005.

[10] Z. Opial, "Weak convergence of the sequence of successive approximations for nonexpansive mappings," *Bulletin of the American Mathematical Society*, vol. 73, pp. 591–597, 1967.

[11] L. B. Ciric, "A generalization of Banach's contraction principle," *Proceedings of the American Mathematical Society*, vol. 45, pp. 267–273, 1974.