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Research Article

On the Weak Relatively Nonexpansive Mappings in Banach Spaces

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In recent years, the definition of weak relatively nonexpansive mapping has been presented and studied by many authors. In this paper, we give some results about weak relatively nonexpansive mappings and give two examples which are weak relatively nonexpansive mappings but not relatively nonexpansive mappings in Banach space l^2 and $L^p[0,1]$ $(1 < p < +\infty)$.

1. Introduction

Let E be a smooth Banach space, and let C be a nonempty closed convex subset of E. We denote by ϕ the function defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for } x,y \in E. \tag{1.1}$$

Following Alber [1], the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) = \arg\min_{y\in C} \phi(y,x), \quad \forall x \in E. \tag{1.2}$$

The generalized projection Π_C from E onto C is well defined, single value and satisfies

$$(\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|x\| + \|y\|)^2 \quad \text{for } x,y \in E. \tag{1.3}$$

If E is a Hilbert space, then $\phi(y,x) = \|y-x\|^2$, and Π_C is the metric projection of E onto C.

Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point p in C is said to be an *asymptotic fixed point* of T [2–4] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. The set of asymptotic fixed point of T will be denoted by b $\widehat{F}(T)$.

Following Matsushita and Takahashi [2], a mapping T of C into itself is said to be *relatively nonexpansive* if the following conditions are satisfied:

(1) $F(T)$ is nonempty;

(2) $\phi(u, Tx) \le \phi(u, x)$, for all $u \in F(T),\ x \in C$;

(3) $\widehat{F}(T) = F(T)$.

The hybrid algorithms for fixed point of relatively nonexpansive mappings and applications have been studied by many authors, for example [2–7]

In recent years, the definition of weak relatively nonexpansive mapping has been presented and studied by many authors [5–8], but they have not given the example which is weak relatively nonexpansive mapping but not relatively nonexpansive mapping. In this paper, we give an example which is weak relatively nonexpansive mapping but not relatively nonexpansive mapping in Banach space l^2.

A point p in C is said to be *a strong asymptotic fixed point* of T [5, 6] if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. The set of strong asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. A mapping T from C into itself is called *weak relatively nonexpansive* if

(1) $F(T)$ is nonempty;
(2) $\phi(u, Tx) \le \phi(u, x)$, for all $u \in F(T),\ x \in C$;
(3) $\widetilde{F}(T) = F(T)$.

Remark 1.1. In [6], the weak relatively nonexpansive mapping is also said to be relatively weak nonexpansive mapping.

Remark 1.2. In [7], the authors have given the definition of hemirelatively nonexpansive mapping as follows. A mapping T from C into itself is called *hemirelatively nonexpansive* if

(1) $F(T)$ is nonempty;

(2) $\phi(u, Tx) \le \phi(u, x)$, for all $u \in F(T),\ x \in C$.

The following conclusion is obvious.

Conclusion 1. A mapping is closed hemi-relatively nonexpansive if and only if it is weak relatively nonexpansive.

If E is strictly convex and reflexive Banach space, and $A \subset E \times E^*$ is a continuous monotone mapping with $A^{-1}(0) \ne \emptyset$, then it is proved in [2] that $J_r := (J + rA)^{-1}J$, for $r > 0$ is relatively nonexpansive. Moreover, if $T : E \to E$ is relatively nonexpansive, then using the definition of ϕ, one can show that $F(T)$ is closed and convex. It is obvious that relatively nonexpansive mapping is weak relatively nonexpansive mapping. In fact, for any mapping $T : C \to C$, we have $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$. Therefore, if T is relatively nonexpansive mapping, then $F(T) = \widetilde{F}(T) = \widehat{F}(T)$.

2. Results for Weak Relatively Nonexpansive Mappings in Banach Space

Theorem 2.1. *Let E be a smooth Banach space and C a nonempty closed convex and balanced subset of E. Let $\{x_n\}$ be a sequence in C such that $\{x_n\}$ converges weakly to $x_0 \neq 0$ and $\|x_n - x_m\| \geq r > 0$ for all $n \neq m$. Define a mapping $T : C \to C$ as follows:*

$$T(x) = \begin{cases} \dfrac{n}{n+1} x_n & \text{if } x = x_n \ (\exists n \geq 1), \\ -x & \text{if } x \neq x_n \ (\forall n \geq 1). \end{cases} \tag{2.1}$$

Then the following conclusions hold:

(1) *T is a weak relatively nonexpansive mapping but not relatively nonexpansive mapping;*

(2) *T is not continuous;*

(3) *T is not pseudo-contractive;*

(4) *if $\{x_n\} \subset \operatorname{int}C$, then T is also not monotone (accretive), where* int$C$ *is the interior of C.*

Proof. (1) It is obvious that T has a unique fixed point 0, that is, $F(T) = \{0\}$. Firstly, we show that x_0 is an asymptotic fixed point of T. In fact since $\{x_n\}$ converges weakly to x_0,

$$\|Tx_n - x_n\| = \left\| \frac{n}{n+1} x_n - x_n \right\| = \frac{1}{n+1} \|x_n\| \longrightarrow 0 \tag{2.2}$$

as $n \to \infty$, so, x_0 is an asymptotic fixed point of T. Secondly, we show that T has a unique strong asymptotic fixed point 0, so that, $F(T) = \widetilde{F}(T)$. In fact, for any strong convergent sequence, $\{z_n\} \subset C$ such that $z_n \to z_0$ and $\|z_n - Tz_n\| \to 0$ as $n \to \infty$, from the conditions of Theorem 2.1, there exists sufficiently large nature number N such that $z_n \neq x_m$, for any $n, m > N$. Then $Tz_n = -z_n$ for $n > N$, it follows from $\|z_n - Tz_n\| \to 0$ that $2z_n \to 0$,and hence $z_n \to z_0 = 0$. Observe that

$$\phi(0, Tx) = \|Tx\|^2 \leq \|x\|^2 = \phi(0, x), \quad \forall x \in C. \tag{2.3}$$

Then T is a weak relatively nonexpansive mapping. On the other hand, since x_0 is an asymptotic fixed point of T but not fixed point, hence T is not a relatively nonexpansive mapping.

(2) For any $x_n \neq 0$, we can take $0 \leq \lambda_m \to 0$ such that $\lambda_m x_n \overline{\in} \{x_n\}_{n=1}^{\infty}$, then we have

$$\begin{gathered} \|x_n - \lambda_m x_n\| \longrightarrow 0, \quad m \longrightarrow \infty, \\ \|Tx_n - T(\lambda_m x_n)\| = \left\| \frac{n}{n+1} x_n + \lambda_m x_n \right\| = \left(\frac{n}{n+1} + \lambda_m \right) \|x_n\| \geq \left(\frac{n}{n+1} \right) \|x_n\| > 0, \end{gathered} \tag{2.4}$$

then T is not continuous.

(3) Since $\|x_n - x_m\| \geq r > 0$ for all $n \neq m$, without loss of generality, we assume that $x_n \neq 0$ for all $n \geq 1$. In this case, we can take $1 \geq \delta_n \to 1$ such that $\delta_n x_n \overline{\in} \{x_i\}_{i=1}^{\infty}$ for all $n \geq 1$. Therefore we have

$$\begin{aligned}
\langle Tx_n - T(\delta_n x_n), J(x_n - \delta_n x_n)\rangle &= \left\langle \frac{n}{n+1}x_n + \delta_n x_n, J(x_n - \delta_n x_n)\right\rangle \\
&= \left(\frac{n}{n+1} + \delta_n\right)\langle x_n, J((1-\delta_n)x_n)\rangle \\
&= \left(\frac{n}{n+1} + \delta_n\right)\frac{1}{1-\delta_n}\langle (1-\delta_n)x_n, J((1-\delta_n)x_n)\rangle \\
&= \left(\frac{n}{n+1} + \delta_n\right)\frac{1}{1-\delta_n}\|(1-\delta_n)x_n\|^2 \\
&= \left(\frac{n}{n+1} + \delta_n\right)\frac{1}{1-\delta_n}\|x_n - \delta_n x_n\|^2.
\end{aligned} \tag{2.5}$$

Since $(n/(n+1)+\delta_n)(1/(1-\delta_n)) \to +\infty$ as $n \to \infty$, we know that T is not pseudo-contractive.

(4) In the same as (2), we can take $1 \leq \delta_n \to 1$ such that $\delta_n x_n \overline{\in} \{x_i\}_{i=1}^{\infty}$ for all $n \geq 1$. Therefore we have

$$\begin{aligned}
\langle Tx_n - T(\delta_n x_n), J(x_n - \delta_n x_n)\rangle &= \left\langle \frac{n}{n+1}x_n + \delta_n x_n, J(x_n - \delta_n x_n)\right\rangle \\
&= \left(\frac{n}{n+1} + \delta_n\right)\langle x_n, J((1-\delta_n)x_n)\rangle \\
&= \left(\frac{n}{n+1} + \delta_n\right)\frac{1}{1-\delta_n}\langle (1-\delta_n)x_n, J((1-\delta_n)x_n)\rangle \\
&= \left(\frac{n}{n+1} + \delta_n\right)\frac{1}{1-\delta_n}\|(1-\delta_n)x_n\|^2 \\
&= \left(\frac{n}{n+1} + \delta_n\right)\frac{1}{1-\delta_n}\|x_n - \delta_n x_n\|^2.
\end{aligned} \tag{2.6}$$

Since $(n/(n+1) + \delta_n)(1/(1-\delta_n)) \to -\infty$ as $n \to \infty$, we know that T is not monotone (accretive). □

3. An Example in Banach Space l^2

In this section, we will give an example which is a weak relatively nonexpansive mapping but not a relatively nonexpansive mapping.

Example 3.1. Let $E = l^2$, where

$$l^2 = \left\{ \xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots) : \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\},$$

$$\|\xi\| = \left(\sum_{n=1}^{\infty} |\xi_n|^2 \right)^{1/2}, \quad \forall \xi \in l^2, \tag{3.1}$$

$$\langle \xi, \eta \rangle = \sum_{n=1}^{\infty} \xi_n \eta_n, \quad \forall \xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots),\ \eta = (\eta_1, \eta_2, \eta_3, \ldots, \eta_n, \ldots.) \in l^2.$$

It is well known that l^2 is a Hilbert space, so that $(l^2)^* = l^2$. Let $\{x_n\} \subset E$ be a sequence defined by

$$\begin{aligned} x_0 &= (1, 0, 0, 0, \ldots), \\ x_1 &= (1, 1, 0, 0, \ldots), \\ x_2 &= (1, 0, 1, 0, 0, \ldots), \\ x_3 &= (1, 0, 0, 1, 0, 0, \ldots), \\ &\ \vdots \\ x_n &= (\xi_{n,1}, \xi_{n,2}, \xi_{n,3}, \ldots, \xi_{n,k}, \ldots), \end{aligned} \tag{3.2}$$

where

$$\xi_{n,k} = \begin{cases} 1 & \text{if } k = 1,\ n+1, \\ 0 & \text{if } k \neq 1,\ k \neq n+1, \end{cases} \tag{3.3}$$

for all $n \geq 1$. Define a mapping $T : E \longrightarrow E$ as follows:

$$T(x) = \begin{cases} \dfrac{n}{n+1} x_n & \text{if } x = x_n\ (\exists n \geq 1), \\ -x & \text{if } x \neq x_n\ (\forall n \geq 1). \end{cases} \tag{3.4}$$

Conclusion 1. $\{x_n\}$ converges weakly to x_0.

Proof. For any $f = (\zeta_1, \zeta_2, \zeta_3, \ldots, \zeta_k, \ldots) \in l^2 = (l^2)^*$, we have

$$f(x_n - x_0) = \langle f, x_n - x_0 \rangle = \sum_{k=2}^{\infty} \zeta_k \xi_{n,k} = \zeta_{n+1} \longrightarrow 0, \tag{3.5}$$

as $n \longrightarrow \infty$. That is, $\{x_n\}$ converges weakly to x_0. □

The following conclusion is obvious.

Conclusion 2. $\|x_n - x_m\| = \sqrt{2}$ for any $n \neq m$.

It follows from Theorem 2.1 and the above two conclusions that T is a weak relatively nonexpansive mapping but not relatively nonexpansive mapping. We have also the following conclusions: (1) T is not continuous; (2) T is not pseudo-contractive; (3) T is also not monotone (accretive).

4. An Example in Banach Space $L^p[0,1]$ $(1 < p < +\infty)$

Let $E = L^p[0,1]$ $(1 < p < +\infty)$, and

$$x_n = 1 - \frac{1}{2^n}, \quad n = 1,2,3,\ldots. \tag{4.1}$$

Define a sequence of functions in $L^p[0,1]$ by the following expression:

$$f_n(x) = \begin{cases} \dfrac{2}{x_{n+1} - x_n} & \text{if } x_n \leq x < \dfrac{x_{n+1} + x_n}{2}, \\ \dfrac{-2}{x_{n+1} - x_n} & \text{if } \dfrac{x_{n+1} + x_n}{2} \leq x < x_{n+1}, \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

for all $n \geq 1$. Firstly, we can see, for any $x \in [0,1]$, that

$$\int_0^x f_n(t)dt \longrightarrow 0 = \int_0^x f_0(t)dt, \tag{4.3}$$

where $f_0(x) \equiv 0$. It is wellknown that the above relation (4.3) is equivalent to $\{f_n(x)\}$ which converges weakly to $f_0(x)$ in uniformly smooth Banach space $L^p[0,1]$ $(1 < p < +\infty)$. On the other hand, for any $n \neq m$, we have

$$\begin{aligned} \|f_n - f_m\| &= \left(\int_0^1 |f_n(x) - f_m(x)|^p dx\right)^{1/p} \\ &= \left(\int_{x_n}^{x_{n+1}} |f_n(\mathrm{x}) - f_m(x)|^p dx + \int_{x_m}^{x_{m+1}} |f_n(x) - f_m(x)|^p dx\right)^{1/p} \\ &= \left(\int_{x_n}^{x_{n+1}} |f_n(x)|^p dx + \int_{x_m}^{x_{m+1}} |f_m(x)|^p dx\right)^{1/p} \\ &= \left(\left(\frac{2}{x_{n+1} - x_n}\right)^p (x_{n+1} - x_n) + \left(\frac{2}{x_{m+1} - x_m}\right)^p (x_{m+1} - x_m)\right)^{1/p} \\ &= \left(\frac{2^p}{(x_{n+1} - x_n)^{p-1}} + \frac{2^p}{(x_{m+1} - x_m)^{p-1}}\right)^{1/p} \\ &\geq (2^p + 2^p)^{1/p} > 0. \end{aligned} \tag{4.4}$$

Let

$$u_n(x) = f_n(x) + 1, \quad \forall n \geq 1. \tag{4.5}$$

It is obvious that u_n converges weakly to $u_0(x) \equiv 1$ and

$$\|u_n - u_m\| = \left\|f_n - f_m\right\| \geq (2^p + 2^p)^{1/p} > 0, \quad \forall n \geq 1. \tag{4.6}$$

Define a mapping $T : E \to E$ as follows:

$$T(x) = \begin{cases} \dfrac{n}{n+1}u_n & \text{if } x = u_n \ (\exists n \geq 1), \\ -x & \text{if } x \neq u_n \ (\forall n \geq 1). \end{cases} \tag{4.7}$$

Since (4.6) holds, by using Theorem 2.1, we know that $T : L^p[0,1] \to L^p[0,1]$ is a weak relatively nonexpansive mapping but not relatively nonexpansive mapping. We have also the following conclusions: (1) T is not continuous; (2) T is not pseudo-contractive; (3) T is also not monotone (accretive).

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