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Research Article

Modified Hybrid Algorithm for a Family of Quasi-ϕ-Asymptotically Nonexpansive Mappings

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The purpose of this paper is to propose a modified hybrid projection algorithm and prove strong convergence theorems for a family of quasi-ϕ-asymptotically nonexpansive mappings. The method of the proof is different from the original one. Our results improve and extend the corresponding results announced by Zhou et al. (2010), Kimura and Takahashi (2009), and some others.

1. Introduction

Let E be a real Banach space and C a nonempty closed convex subset of E. A mapping $T : C \to C$ is said to be asymptotically nonexpansive [1] if there exists a sequence $\{k_n\}$ of positive real numbers with $k_n \to 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \tag{1.1}$$

for all $x, y \in C$ and all $n \ge 1$.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] in 1972. They proved that if C is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping T of C has a fixed point. Further, the set $F(T)$ of fixed points of T is closed and convex. Since 1972, a host of authors have studied the weak and strong convergence problems of the iterative algorithms for such a class of mappings (see, e.g., [1–3] and the references therein).

It is well known that in an infinite-dimensional Hilbert space, the normal *Mann's* iterative algorithm has only weak convergence, in general, even for nonexpansive mappings. Consequently, in order to obtain strong convergence, one has to modify the normal *Mann's* iteration algorithm; the so-called hybrid projection iteration method is such a modification.

The hybrid projection iteration algorithm (HPIA) was introduced initially by Haugazeau [4] in 1968. For 40 years, (HPIA) has received rapid developments. For details, the readers are referred to papers in [5–11] and the references therein.

In 2003, Nakajo and Takahashi [6] proposed the following modification of the Mann iteration method for a nonexpansive mapping T in a Hilbert space H:

$$\begin{gathered}
x_0 \in C \quad \text{chosen arbitrarily}, \\
y_n = \alpha_n x_n + (1-\alpha_n) T x_n, \\
C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\
Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\
x_{n+1} = P_{C_n \cap Q_n}(x_0),
\end{gathered} \tag{1.2}$$

where C is a closed convex subset of H, P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one then the sequence $\{x_n\}$ generated by (1.2) converges strongly to $P_{F(T)}(x_0)$, where $F(T)$ denote the fixed points set of T.

In 2006, Kim and Xu [12] proposed the following modification of the Mann iteration method for asymptotically nonexpansive mapping T in a Hilbert space H:

$$\begin{gathered}
x_0 \in C \quad \text{chosen arbitrarily}, \\
y_n = \alpha_n x_n + (1-\alpha_n) T^n x_n, \\
C_n = \left\{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\right\}, \\
Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\
x_{n+1} = P_{C_n \cap Q_n}(x_0),
\end{gathered} \tag{1.3}$$

where C is bounded closed convex subset and

$$\theta_n = (1-\alpha_n)\left(k_n^2 - 1\right)(\operatorname{diam} C)^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{1.4}$$

They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.3) converges strongly to $P_{F(T)}(x_0)$.

They also proposed the following modification of the Mann iteration method for asymptotically nonexpansive semigroup $\Im$ in a Hilbert space H:

$$\begin{gathered}
x_0 \in C \quad \text{chosen arbitrarily,} \\
y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\
C_n = \left\{ z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \overline{\theta}_n \right\}, \\
Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0 \}, \\
x_{n+1} = P_{C_n \cap Q_n}(x_0),
\end{gathered} \tag{1.5}$$

where C is bounded closed convex subset and

$$\overline{\theta}_n = (1-\alpha_n)\left[\left(\frac{1}{t_n}\int_0^{t_n} L(s)ds\right)^2 - 1\right](\operatorname{diam} C)^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{1.6}$$

and $L : (0,\infty) \to [0,\infty)$ is nonincreasing in s and bounded measurable function such that, $L(s) \geq 1$ for all $s > 0, L(s) \to 1$ as $s \to \infty$, and for each $s > 0$,

$$\|T(s)x - T(s)y\| \leq L(s)\|x - y\|, \quad x, y \in C. \tag{1.7}$$

They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{F(\Im)}(x_0)$, where $F(\Im)$ denote the common fixed points set of $\Im$.

In 2006, Martinez-Yanes and Xu [7] proposed the following modification of the Ishikawa iteration method for nonexpansive mapping T in a Hilbert space H:

$$\begin{gathered}
x_0 \in C \quad \text{chosen arbitrarily,} \\
y_n = \alpha_n x_n + (1-\alpha_n)Tz_n, \\
z_n = \beta_n x_n + (1-\beta_n)Tx_n, \\
C_n = \left\{ z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1-\alpha_n)\left(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, z\rangle\right) \right\}, \\
Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0 \}, \\
x_{n+1} = P_{C_n \cap Q_n}(x_0),
\end{gathered} \tag{1.8}$$

where C is a closed convex subset of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one and $\beta_n \to 0$, then the sequence $\{x_n\}$ generated by (1.8) converges strongly to $P_{F(T)}(x_0)$.

Martinez-Yanes and Xu [7] proposed also the following modification of the Halpern iteration method for nonexpansive mapping T in a Hilbert space H:

$$\begin{aligned}
&x_0 \in C \quad \text{chosen arbitrarily,}\\
&y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n,\\
&C_n = \left\{z \in C : \left\|y_n - z\right\|^2 \le \|x_n - z\|^2 + \alpha_n\left(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle\right)\right\},\\
&Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\},\\
&x_{n+1} = P_{C_n\cap Q_n}(x_0),
\end{aligned} \tag{1.9}$$

where C is a closed convex subset of H. They proved that if the sequence $\alpha_n \to 0$, then the sequence $\{x_n\}$ generated by (1.9) converges strongly to $P_{F(T)}(x_0)$.

In 2005, Matsushita and Takahashi [8] proposed the following hybrid iteration method with generalized projection for relatively nonexpansive mapping T in a Banach space E:

$$\begin{aligned}
&x_0 \in C \quad \text{chosen arbitrarily,}\\
&y_n = J^{-1}(\alpha_n J x_0 + (1-\alpha_n)JTx_n),\\
&C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}\\
&Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\},\\
&x_{n+1} = \Pi_{C_n\cap Q_n}(x_0).
\end{aligned} \tag{1.10}$$

They proved the following convergence theorem.

Theorem MT. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by (1.10), where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}(\cdot)$ is the generalized projection from C onto $F(T)$.*

In 2009, Zhou et al. [11] proposed the following modification of the hybrid iteration method with generalized projection for a family of closed and quasi-ϕ-asymptotically nonexpansive mappings $\{T_i\}_{i\in I}$ in a Banach space E:

$$\begin{aligned}
&x_0 \in C \quad \text{chosen arbitrarily,}\\
&y_{n,i} = J^{-1}(\alpha_n J x_n + (1-\alpha_n)JT_i^n x_n),\\
&C_{n,i} = \{z \in C : \phi(z, y_{n,i}) \le \phi(z, x_n) + \zeta_{n,i}\},\\
&C_n = \bigcap_{i\in I} C_{n,i},
\end{aligned}$$

$$
\begin{gathered}
Q_0 = C, \\
Q_n = \{z \in Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\
x_{n+1} = \Pi_{C_n \cap Q_n}(x_0).
\end{gathered}
\tag{1.11}
$$

They proved the following convergence theorem.

Theorem ZGT.

Let C be a nonempty bounded closed convex subset of a uniformly convex and uniformly smooth Banach space E, and let $\{T_i\}_{i\in I} : C \to C$ be a family of quasi-ϕ*-asymptotically nonexpansive mappings such that $F = \bigcap_{i\in I} F(T_i) \neq \emptyset$. Assume that every T_i, $(i \in I)$ is asymptotically regular on C. Let $\{\alpha_n\}$ be a real sequence in* $[0,1)$ *such that* $\limsup_{n\to\infty}\alpha_n < 1$. *Define a sequence $\{x_n\}$ as given by* (1)*, then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where $\zeta_{n,i} = (1-\alpha_n)(k_{n,i}-1)M$, $M \geq \phi(z, x_n)$ for all $z \in F$, $x_n \in C$, and Π_F is the generalized projection from C onto F.*

Very recently, Kimura and Takahashi [13] established strong convergence theorems by the hybrid method for a family of relatively nonexpansive mappings as follows.

Theorem KT.

Let E be a strictly convex reflexive Banach space having the Kadec-Klee property and a Fréchet differentiable norm, and let C be a nonempty and closed convex subset of E and $\{S_\lambda : \lambda \in \Lambda\}$ a family of relatively nonexpensive mappings of C into itself having a common fixed point. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ *such that* $\liminf_{n\to\infty}\alpha_n < 1$. *For an arbitrarily chosen point $x \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C, C_1 = C$, and*

$$
\begin{aligned}
y_n(\lambda) &= J^*(\alpha_n Jx_n + (1-\alpha_n)JS_\lambda x_n), \quad \forall\, \lambda \in \Lambda, \\
C_{n+1} &= \left\{ z \in C_n : \sup_{\lambda\in\Lambda} \phi(z, y_n(\lambda)) \leq \phi(z, x_n) \right\}, \\
x_{n+1} &= P_{C_{n+1}}(x),
\end{aligned}
\tag{1.12}
$$

for every $n \in N$, then $\{x_n\}$ converges strongly to $P_F x \in C$, where $F = \bigcap_{\lambda\in\Lambda} F(S_\lambda)$ is the set of common fixed points of $\{S_\lambda\}$ and P_K is the metric projection of E onto a nonempty closed convex subset K of E.

Motivated by these results above, the purpose of this paper is to propose a Modified hybrid projection algorithm and prove strong convergence theorems for a family of quasi-ϕ-asymptotically nonexpansive mappings which are asymptotically regular on C. In order to get the strong convergence theorems for such a family of mappings, the classical hybrid projection iteration algorithm is modified and then is used to approximate the common fixed points of such a family of mappings. In the meantime, the method of the proof is different from the original one. Our results improve and extend the corresponding results announced by Zhou et al. [11], and Kimura and Takahashi [13], and some others.

2. Preliminaries

Let E be a Banach space with dual E^*. Denote by $\langle \cdot, \cdot \rangle$ the duality product. The normalize duality mapping J from E to E^* is defined by

$$Jx = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}, \tag{2.1}$$

for all $x \in E$, where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ the generalized duality pairing between E and E^*. It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E.

It is also very well known that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces C, and consequently, it is not available in more general Banach spaces. In this connection, Alber [14] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a real smooth Banach space. Let us consider the functional defined by [7, 8] as

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2 \tag{2.2}$$

for all $x, y \in E$. Observe that, in a Hilbert space H, (2.2) reduces to $\phi(y, x) = \|x - y\|^2$, x,$y \in H$.

The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$, the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \min_{y \in C} \phi(y, x). \tag{2.3}$$

Existence and uniqueness of the operator Π_C follow from the properties of the C functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, e.g., [14–18]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2 \tag{2.4}$$

for all $x, y \in E$.

Remark 2.1. If E is a reflexive strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$, then $x = y$. From (2.4), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definitions of J, we have $Jx = Jy$. That is, $x = y$; see [17, 18] for more details.

Let C be a closed convex subset of E and T a mapping from C into itself. T is said to be ϕ-asymptotically nonexpansive if there exists some real sequence $\{k_n\}$ with $k_n \ge 1$ and $k_n \to 1$ such that $\phi(T^n x, T^n y) \le k_n \phi(x, y)$ for all $n \ge 1$ and $x, y \in C$. T is said to be quasi-ϕ-asymptotically nonexpansive [9] if there exists some real sequence $\{k_n\}$ with $k_n \ge 1$ and

$k_n \to 1$ and $F(T) \neq \emptyset$ such that $\phi(p, T^n x) \leq k_n \phi(p, x)$ for all $n \geq 1$, $x \in C$, and $p \in F(T)$. $T : C \to C$ is said to be asymptotically regular on C if, for any bounded subset $\widetilde{C}$ of C, there holds the following equality:

$$\lim_{n \to \infty} \sup\left\{ \left\| T^{n+1}x - T^n x \right\| : x \in \widetilde{C} \right\} = 0. \tag{2.5}$$

We remark that a ϕ-asymptotically nonexpansive mapping with a nonempty fixed point set $F(T)$ is a quasi-ϕ-asymptotically nonexpansive mapping, but the converse may be not true.

We present some examples which are closed and quasi-ϕ-asymptotically nonexpansive.

Example 2.2. Let E be a real line. We define a mapping $T : E \to E$ by

$$T(x) = \begin{cases} \dfrac{x}{2} \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{2.6}$$

Then T is continuous quasi-nonexpansive, and hence it is closed and quasi-asymptotically nonexpansive with the constant sequence $\{1\}$ but not asymptotically nonexpansive.

Example 2.3. Let E be a uniformly smooth and strictly convex Banach space, and $A \subset E \times E^*$ is a maximal monotone mapping such that $A^{-1}0$ is nonempty. Then, $J_r = (J + rA)^{-1} J$ is a closed and quasi-ϕ-asymptotically nonexpansive mapping from E onto $D(A)$, and $F(J_r) = A^{-1}0$.

Example 2.4. Let Π_C be the generalized projection from a smooth, strictly convex, and reflexive Banach space E onto a nonempty closed convex subset C of E. Then, Π_C is a closed and quasi-ϕ-asymptotically nonexpansive mapping from E onto C with $F(\Pi_C) = C$.

Let C_n be a sequence of nonempty closed convex subsets of a reflexive Banach space E. We denote two subsets $s - Li_n C_n$ and $w - Ls_n C_n$ as follows: $x \in s - Li_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and that $x_n \in C_n$ for all $n \in N$. Similarly, $y \in w - Ls_n C_n$ if and only if there exists a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i \subset E\}$ such that $\{y_i\}$ converges weakly to y and that $y_i \in C_{n_i}$ for all $i \in N$. We define the Mosco convergence [19] of $\{C_n\}$ as follows. If C_0 satisfies that $C_0 = s - Li_n C_n = w - Ls_n C_n$, it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco, and we write $C_0 = M - \lim_{n \to \infty} C_n$. For more details, see [20].

The following theorem plays an important role in our results.

Theorem 2.5 (see Ibaraki et al. [21]). *Let E be a smooth, reflexive, and strictly convex Banach space having the Kadec-Klee property. Let $\{K_n\}$ be a sequence of nonempty closed convex subsets of E. If $K_0 = M - \lim_{n \to \infty} K_n$ exists and is nonempty, then $\{\Pi_{K_n} x\}$ converges strongly to $\Pi_{K_0} x$ for each $x \in C$.*

We also need the following lemmas for the proof of our main results.

Lemma 2.6 (Kamimura and Takahashi [16]). *Let E be a uniformly convex and smooth Banach space, and let $\{y_n\}$, $\{z_n\}$ be two sequences of E if $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Lemma 2.7 (Alber [14]). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E, and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x) \tag{2.7}$$

for all $y \in C$.

Lemma 2.8. *Let E be a uniformly convex and smooth Banach space, let C be a closed convex subset of E, and let T be a closed and quasi-ϕ-asympotically nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

3. A Modified Algorithm and Strong Convergence Theorems

Now we are in a proposition to prove the main results of this paper. In the sequel, we use the letter I to denote an index set.

Theorem 3.1. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, and let $\{T_i\}_{i\in I} : C \to C$ be a family of closed and quasi-ϕ-asymptotically nonexpansive mappings such that $F = \bigcap_{i\in I} F(T_i) \neq \emptyset$. Assume that every T_i, $(i \in I)$ is asymptotically regular on C. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be real sequences in $[0,1]$ such that $\lim_{n\to\infty}\alpha_n = 0$, $\liminf_{n\to\infty}\gamma_n > 0$. Define a sequence $\{x_n\}$ in C in the following manner:*

$$\begin{gathered}
x_0 \in C \quad \text{chosen arbitrarily,} \\
y_{n,i} = J^{-1}\left(\alpha_n J x_0 + \beta_n J x_n + \gamma_n J T_i^n x_n\right), \\
C_0 = C, \\
C_{n,i} = \left\{z \in C_{n-1} : \phi\left(z, y_{n,i}\right) \leq \alpha_n \phi(z, x_0) + \left(\beta_n + \gamma_n k_{n,i}\right)\phi(z, x_n)\right\}, \\
C_n = \bigcap_{i\in I} C_{n,i}, \\
x_{n+1} = \Pi_{C_n}(x_0).
\end{gathered} \tag{3.1}$$

Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F.

Proof. Firstly, we show that C_n is closed and convex for each $n \geq 0$.

From the definition of C_n, it is obvious that C_n is closed for each $n \geq 0$. We show that C_n is convex for each $n \geq 0$. Observe that the set

$$C_{n,i} = \left\{z \in C_{n-1} : \phi\left(z, y_{n,i}\right) \leq \alpha_n \phi(z, x_0) + \left(\beta_n + \gamma_n k_{n,i}\right)\phi(z, x_n)\right\} \tag{3.2}$$

can be written as

$$C_{n,i} = \Big\{ z \in C_{n-1} : \gamma_n(1-k_{n,i})\|z\|^2 + \big\|y_{n,i}\big\|^2 - \alpha_n\|x_0\|^2 - \big(\beta_n + \gamma_n k_{n,i}\big)\|x_n\|^2 \\ \le 2\big\langle z, Jy_{n,i} - \alpha_n Jx_0 - \big(\beta_n + \gamma_n k_{n,i}\big)Jx_n\big\rangle \Big\}. \tag{3.3}$$

For $z_1, z_2 \in C_{n,i}$ and $t \in (0,1)$, denote $z = tz_1 + (1-t)z_2$, $A = \|y_{n,i}\|^2 - \alpha_n\|x_0\|^2 - (\beta_n + \gamma_n k_{n,i})\|x_n\|^2$, and $B = Jy_{n,i} - \alpha_n Jx_0 - (\beta_n + \gamma_n k_{n,i})Jx_n$; by noting that $\|\cdot\|^2$ is convex, we have

$$\|z\|^2 = \|tz_1 + (1-t)z_2\|^2 \le t\|z_1\|^2 + (1-t)\|z_2\|^2. \tag{3.4}$$

So we obtain

$$\begin{aligned} \gamma_n(1-k_{n,i})\|z\|^2 + A &\le \gamma_n(1-k_{n,i})t\|z_1\|^2 + \gamma_n(1-k_{n,i})(1-t)\|z_2\|^2 + A \\ &= t\Big(\gamma_n(1-k_{n,i})\|z_1\|^2 + A\Big) + (1-t)\big(\gamma_n(1-k_{n,i})\|z_2\| + A\big) \\ &\le 2t\langle z_1, B\rangle + 2(1-t)\langle z_2, B\rangle \\ &= 2\langle tz_1 + (1-t)z_2, B\rangle \\ &= 2\langle z, B\rangle, \end{aligned} \tag{3.5}$$

which infers that $z \in C_{n,i}$, so we get that C_n is convex for each $n \ge 0$. Thus C_n is closed and convex for every $n \ge 0$.

Secondly, we prove that $F \subset C_n$, for all $n \ge 0$.

Indeed, by noting that $\|\cdot\|^2$ is convex and using (2.2), we have, for any $z \in F$ and all $i \in I$, that

$$\begin{aligned} \phi(z, y_{n,i}) &= \phi\Big(z, J^{-1}\big(\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JT_i^n x_n\big)\Big) \\ &= \|z\|^2 - 2\big\langle z, \big(\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JT_i^n x_n\big)\big\rangle + \big\|\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JT_i^n x_n\big\|^2 \\ &\le \|z\| - 2\big\langle z, \big(\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JT_i^n x_n\big)\big\rangle + \alpha_n\|Jx_0\|^2 + \beta_n\|Jx_n\|^2 + \gamma_n\big\|JT_i^n x_n\big\|^2 \\ &\le \alpha_n\phi(z, x_0) + \beta_n\phi(z, x_n) + \gamma_n k_{n,i}\phi(z, x_n) \\ &= \alpha_n\phi(z, x_0) + \big(\beta_n + \gamma_n k_{n,i}\big)\phi(z, x_n), \end{aligned} \tag{3.6}$$

which infers that $z \in C_{n,i}$, for all $n \ge 0$ and $i \in I$, and hence $z \in C_n = \bigcap_{i\in I} C_{n,i}$. This proves that $F \subset C_n$, for all $n \ge 0$ and $i \in I$.

Thirdly, we will show that $\lim_{n\to\infty} x_n = \overline{x} = \Pi_{\overline{C}} x_0$.

Since $\{C_n\}$ is a decreasing sequence of closed convex subsets of E such that $\overline{C} = \bigcap_{n=0}^{\infty} C_n$ is nonempty, it follows that

$$M - \lim_{n\to\infty} C_n = \overline{C} = \bigcap_{n=0}^{\infty} C_n \ne \emptyset. \tag{3.7}$$

By Theorem 2.5, $\{x_n\} = \{\Pi_{C_{n-1}} x_0\}$ converges strongly to $\{\overline{x}\} = \{\Pi_{\overline{C}} x_0\}$.

Fourthly, we prove that $\overline{x} \in F$.

Since $x_{n+1} = \Pi_{C_n}(x_0) \in C_n$, from the definition of C_n, we get

$$\phi(x_{n+1}, y_{n,i}) \le \alpha_n \phi(x_{n+1}, x_0) + (\beta_n + \gamma_n k_{n,i})\phi(x_{n+1}, x_n). \tag{3.8}$$

From $\lim_{n\to\infty} x_n = \overline{x}$, one obtains $\phi(x_{n+1}, x_n) \to 0$ as $n \to \infty$, and it follows from $\lim_{n\to\infty}\alpha_n = 0$, for every $i \in I$ that we have

$$\lim_{n\to\infty} \phi(x_{n+1}, y_{n,i}) \le \lim_{n\to\infty} \left(\alpha_n \phi(x_{n+1}, x_0) + (\beta_n + \gamma_n k_{n,i})\phi(x_{n+1}, x_n)\right) = 0, \tag{3.9}$$

and hence $x_{n+1} - y_{n,i} \to 0$ as $n \to \infty$ by Lemma 2.6. It follows that $\|y_{n,i} - x_n\| \le \|y_{n,i} - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Since J is uniformly norm-to-norm continuous on any bounded sets of E, we conclude that

$$\lim_{n\to\infty} \left\| Jx_n - Jy_{n,i} \right\| = 0, \tag{3.10}$$

for every $i \in I$. By the definition of $\{y_{n,i}\}$ and the assumption on $\{\alpha_n\}$, we deduce that

$$\begin{aligned} \left\| Jx_n - Jy_{n,i} \right\| &= \left\| Jx_n - (\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JT_i^n x_n) \right\| \\ &= \left\| \alpha_n (Jx_n - Jx_0) + \gamma_n (Jx_n - JT_i^n x_n) \right\| \\ &\ge \left\| \gamma_n (Jx_n - JT_i^n x_n) \right\| - \|\alpha_n (Jx_n - Jx_0)\| \end{aligned} \tag{3.11}$$

for every $n \in N$ and $i \in I$. So we get

$$\gamma_n \left\| Jx_n - JT_i^n x_n \right\| \le \left\| Jx_n - Jy_{n,i} \right\| + \alpha_n \|Jx_n - Jx_0\|. \tag{3.12}$$

Since $\liminf_{n\to\infty}\gamma_n > 0$, we have $Jx_n - JT_i^n x_n \to 0$ as $n \to \infty$.

Since J^{-1} is also uniformly norm-to-norm continuous on any bounded sets of E^*, we conclude that

$$\lim_{n\to\infty} \left\| x_n - T_i^n x_n \right\| = 0. \tag{3.13}$$

Noting that $x_n \to \overline{x}$ as $n \to \infty$, we have

$$T_i^n x_n \longrightarrow \overline{x}, \tag{3.14}$$

as $n \to \infty$. Observe that

$$\left\| T_i^{n+1} x_n - \overline{x} \right\| \le \left\| T_i^{n+1} x_n - T_i^n x_n \right\| + \left\| T_i^n x_n - \overline{x} \right\|. \tag{3.15}$$

By using (3.14), (3.15), and the asymptotic regularity of T_i, we have

$$T_i^{n+1} x_n \longrightarrow \overline{x} \tag{3.16}$$

as $n \to \infty$, that is, $T_i T_i^n x_n \to \overline{x}$. Now the closedness property of T_i gives that $\overline{x}$ is a common fixed point of the family $\{T_i\}_{i\in I}$, thus $\overline{x} \in F$.

Finally, since $\overline{x} = \Pi_{\overline{C}} x_0 \in F$ and F is a nonempty closed convex subset of $\overline{C} = \bigcap_{n=0}^{\infty} C_n$, we conclude that $\overline{x} = \Pi_F x_0$. This completes the proof. □

Remark 3.2. The boundedness assumption on C in Theorem ZGT can be dropped.

Remark 3.3. The asymptotic regularity assumption on T_i in Theorem 3.1 can be weakened to the assumption that $T_i^{n+1} x_n - T_i^n x_n \to 0$ as $n \to \infty$.

Recall that $T : C \to C$ is called uniformly Lipschitzian continuous if there exists some $L > 0$ such that

$$\|T^n x - T^n y\| \le L\|x - y\|, \tag{3.17}$$

for all $n \ge 1$ and $x, y \in C$.

Remark 3.4. The assumption that $T_i^{n+1} x_n - T_i^n x_n \to 0$ as $n \to \infty$ can be replaced by the uniform Lipschitz continuity of T_i.

With above observations, we have the following convergence result.

Corollary 3.5. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, and let $\{T_i\}_{i\in I} : C \to C$ be a family of uniformly Lipschitzian continuous and quasi-ϕ-asymptotically nonexpansive mappings such that $F = \bigcap_{i\in I} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be real sequences in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\liminf_{n\to\infty} \gamma_n > 0$. Define a sequence $\{x_n\}$ in C in the following manner:*

$$\begin{gathered}
x_0 \in C \quad \text{chosen arbitrarily}, \\
y_{n,i} = J^{-1}\left(\alpha_n J x_0 + \beta_n J x_n + \gamma_n J T_i^n x_n\right), \\
C_0 = C, \\
C_{n,i} = \left\{z \in C_{n-1} : \phi(z, y_{n,i}) \le \alpha_n \phi(z, x_0) + \left(\beta_n + \gamma_n k_{n,i}\right)\phi(z, x_n)\right\}, \\
C_n = \bigcap_{i\in I} C_{n,i}, \\
x_{n+1} = \Pi_{C_n}(x_0).
\end{gathered} \tag{3.18}$$

Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F.

Proof. Following the proof lines of Theorem 3.1, we can prove that F is nonempty closed convex, C_n is closed convex, $F \subset C_n$ for all $n \ge 0$ and $\lim_{n\to\infty} x_n = \overline{x}$. At this point, it is sufficient to show that $T_i^{n+1}x_n - T_i^n x_n \to 0$ as $n \to \infty$. Again, from the proof lines of Theorem 3.1, we have the following conclusions:

$$\lim_{n\to\infty} \left\| x_n - T_i^n x_n \right\| = 0. \tag{3.19}$$

Observe that

$$\begin{aligned} \left\| T_i^{n+1}x_n - T_i^n x_n \right\| &\le \left\| T_i^{n+1}x_n - T_i^{n+1}x_{n+1} \right\| + \left\| T_i^{n+1}x_{n+1} - x_{n+1} \right\| + \|x_{n+1} - x_n\| + \left\| x_n - T_i^n x_n \right\| \\ &\le (L_i + 1)\|x_{n+1} - x_n\| + \left\| T_i^{n+1}x_{n+1} - x_{n+1} \right\| + \left\| x_n - T_i^n x_n \right\|, \end{aligned} \tag{3.20}$$

so that $T_i^{n+1}x_n - T_i^n x_n \to 0$ as $n \to \infty$. By Theorem 3.1, we have the desired conclusion. This completes the proof. □

When $\alpha_n \equiv 0$ in Theorem 3.1, we obtain the following result.

Corollary 3.6. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, and let $\{T_i\}_{i\in I} : C \to C$ be a family of closed and quasi-ϕ-asymptotically nonexpansive mappings such that $F = \bigcap_{i\in I} F(T_i) \neq \emptyset$. Assume that every T_i, $(i \in I)$ is asymptotically regular on C. Let $\{\alpha_n\}$ be a real sequence in $[0,1)$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Define a sequence $\{x_n\}$ in C in the following manner:*

$$\begin{gathered} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_{n,i} = J^{-1}\left(\alpha_n J x_n + (1-\alpha_n) J T_i^n x_n\right), \\ C_0 = C, \\ C_{n,i} = \left\{ z \in C_{n-1} : \phi(z, y_{n,i}) \le \left[k_{n,i} + (1-k_{n,i})\alpha_n\right]\phi(z, x_n) \right\}, \\ C_n = \bigcap_{i\in I} C_{n,i}, \\ x_{n+1} = \Pi_{C_n}(x_0). \end{gathered} \tag{3.21}$$

Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F.

When $\beta_n \equiv 0$ in Theorem 3.1, we obtain the following result.

Corollary 3.7. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, and let $\{T_i\}_{i\in I} : C \to C$ be a family of closed and quasi-ϕ-asymptotically nonexpansive mappings such that $F = \bigcap_{i\in I} F(T_i) \neq \emptyset$. Assume that every T_i, $(i \in I)$ is asymptotically regular on C. Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. Define a sequence $\{x_n\}$*

in C in the following manner:

$$\begin{aligned} &x_0 \in C \quad \text{chosen arbitrarily}, \\ &y_{n,i} = J^{-1}\left(\alpha_n J x_0 + (1-\alpha_n) J T_i^n x_n\right), \\ &C_0 = C, \\ &C_{n,i} = \left\{z \in C_{n-1} : \phi\left(z, y_{n,i}\right) \le \alpha_n \phi(z, x_0) + k_{n,i}(1-\alpha_n)\phi(z, x_n)\right\}, \\ &C_n = \bigcap_{i \in I} C_{n,i}, \\ &x_{n+1} = \Pi_{C_n}(x_0). \end{aligned} \tag{3.22}$$

Then $\{x_n\}$ *converges strongly to* $\Pi_F x_0$*, where* Π_F *is the generalized projection from* C *onto* F*.*

In the spirit of Theorem 3.1, we can prove the following strong convergence theorem.

Theorem 3.8. *Let* C *be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space* E*, and let* $\{T_i\}_{i\in I} : C \to C$ *be a family of closed and quasi-*ϕ*-nonexpansive mappings such that* $F = \bigcap_{i\in I} F(T_i) \ne \emptyset$*. Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *be real sequences in* $[0,1]$ *such that* $\lim_{n\to\infty}\alpha_n = 0$, $\liminf_{n\to\infty}\gamma_n > 0$*. Define a sequence* $\{x_n\}$ *in* C *in the following manner:*

$$\begin{aligned} &x_0 \in C \quad \text{chosen arbitrarily}, \\ &y_{n,i} = J^{-1}\left(\alpha_n J x_0 + \beta_n J x_n + \gamma_n J T_i x_n\right), \\ &= C_0 C, \\ &C_{n,i} = \left\{z \in C_{n-1} : \phi\left(z, y_{n,i}\right) \le \alpha_n \phi(z, x_0) + (1-\alpha_n)\phi(z, x_n)\right\}, \\ &C_n = \bigcap_{i \in I} C_{n,i}, \\ &x_{n+1} = \Pi_{C_n}(x_0). \end{aligned} \tag{3.23}$$

Then $\{x_n\}$ *converges strongly to* $\Pi_F x_0$*, where* Π_F *is the generalized projection from* C *onto* F*.*

Proof. Following the proof lines of Theorem 3.1, we have the following conclusions:

(1) F is a nonempty closed convex subset of C;

(2) C_n is closed covex for all $n \ge 0$;

(3) $F \subset C_n$, for all $n \ge 0$;

(4) $\lim_{n\to\infty} x_n = \overline{x}$;

(5) $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$, for all $i \in I$.

The closedness property of T_i together with (4) and (5) implies that $\{x_n\}$ converges strongly to a common fixed point $\overline{x}$ of the family $\{T_i\}_{i\in I}$. As shown in Theorem 3.1, $\overline{x} = \Pi_F x_0$. This completes the proof. □

When $\alpha_n \equiv 0$ in Theorem 3.8, we obtain the following result.

Corollary 3.9. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space, and let $\{T_i\}_{i\in I} : C \to C$ be a family of closed and quasi-ϕ-nonexpansive mappings such that $F = \bigcap_{i\in I} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence in $[0,1)$ such that $\limsup_{n\to\infty}\alpha_n < 1$. Define a sequence $\{x_n\}$ in C in the following manner:*

$$
\begin{gathered}
x_0 \in C \quad \text{chosen arbitrarily,} \\
y_{n,i} = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T_i x_n), \\
C_0 = C, \\
C_{n,i} = \left\{ z \in C_{n-1} : \phi\left(z, y_{n,i}\right) \le \phi(z, x_n) \right\}, \\
C_n = \bigcap_{i\in I} C_{n,i}, \\
x_{n+1} = \Pi_{C_n}(x_0).
\end{gathered}
\tag{3.24}
$$

Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F.

When $\beta_n \equiv 0$ in Theorem 3.8, we obtain the following result.

Corollary 3.10. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, and let $\{T_i\}_{i\in I} : C \to C$ be a family of closed and quasi-ϕ-nonexpansive mappings such that $F = \bigcap_{i\in I} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ such that $\lim_{n\to\infty}\alpha_n = 0$. Define a sequence $\{x_n\}$ in C in the following manner:*

$$
\begin{gathered}
x_0 \in C \quad \text{chosen arbitrarily,} \\
y_{n,i} = J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J T_i x_n), \\
C_0 = C, \\
C_{n,i} = \left\{ z \in C_{n-1} : \phi\left(z, y_{n,i}\right) \le \alpha_n \phi(z, x_0) + (1-\alpha_n)\phi(z, x_n) \right\}, \\
C_n = \bigcap_{i\in I} C_{n,i}, \\
x_{n+1} = \Pi_{C_n}(x_0).
\end{gathered}
\tag{3.25}
$$

Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F.

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