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Research Article

Generalized Caristi's Fixed Point Theorems

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We present generalized versions of Caristi's fixed point theorem for multivalued maps. Our results either improve or generalize the corresponding generalized Caristi's fixed point theorems due to Bae (2003), Suzuki (2005), Khamsi (2008), and others.

1. Introduction

A number of extensions of the Banach contraction principle have appeared in literature. One of its most important extensions is known as Caristi's fixed point theorem. It is well known that Caristi's fixed point theorem is equivalent to Ekland variational principle [1], which is nowadays an important tool in nonlinear analysis. Many authors have studied and generalized Caristi's fixed point theorem to various directions. For example, see [2–8]. Kada et al. [9] and Suzuki [10] introduced the concepts of w-distance and τ-distance on metric spaces, respectively. Using these generalized distances, they improved Caristi's fixed point theorem and Ekland variational principle for single-valued maps. In this paper, using the concepts of w-distance and τ-distance, we present some generalizations of the Caristi's fixed point theorem for multivalued maps. Our results either improve or generalize the corresponding results due to Bae [4, 11], Kada et al. [9], Suzuki [8, 10], Khamsi [5], and many of others.

Let X be a metric space with metric d. We use 2^X to denote the collection of all nonempty subsets of X. A point $x \in X$ is called a fixed point of a map $f : X \to X$ ($T : X \to 2^X$) if $x = f(x)$ ($x \in T(x)$).

In 1976, Caristi [12] obtained the following fixed point theorem on complete metric spaces, known as Caristi's fixed point theorem.

Theorem 1.1. *Let X be a complete metric space with metric d. Let $\psi : X \to [0,\infty)$ be a lower semicontinuous function, and let $f : X \to X$ be a single-valued map such that for any*

$x \in X$,

$$d(x, f(x)) \leq \psi(x) - \psi(f(x)). \tag{1.1}$$

Then f has a fixed point.

To generalize Theorem 1.1, one may consider the weakening of one or more of the following hypotheses: (i) the metric d; (ii) the lower semicontinuity of the real-valued function ψ; (iii) the inequality (1.1); (iv) the function f.

In [9], Kada et al. introduced a concept of w-distance on a metric space as follows.

A function $\omega : X \times X \to [0, \infty)$ is a *w-distance* on X if it satisfies the following conditions for any $x, y, z \in X$:

(w_1) $\omega(x, z) \leq \omega(x, y) + \omega(y, z)$;

(w_2) the map $\omega(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;

(w_3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $\omega(z, x) \leq \delta$ and $\omega(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

Clearly, the metric d is a w-distance on X. Let $(Y, \|\cdot\|)$ be a normed space. Then the functions $\omega_1, \omega_2 : Y \times Y \to [0, \infty)$ defined by $\omega_1(x, y) = \|y\|$ and $\omega_2(x, y) = \|x\| + \|y\|$ for all $x, y \in Y$ are w-distances. Many other examples of w-distance are given in [9, 13]. Note that, in general, for $x, y \in X$, $\omega(x, y) \neq \omega(y, x)$, and neither of the implications $\omega(x, y) = 0 \Leftrightarrow x = y$ necessarily holds.

In the sequel, otherwise specified, we shall assume that X is a complete metric space with metric d, $\psi : X \to [0, \infty)$ is a lower semicontinuous function and ω is a w-distance on X.

Using the concept of w-distance, Kada et al. [9] generalized Caristi's fixed point theorem as follows.

Theorem 1.2. *Let f be a single-valued self map on X such that for every $x \in X$,*

$$\psi(f(x)) + \omega(x, f(x)) \leq \psi(x). \tag{1.2}$$

Then, there exists $x_0 \in X$ such that $f(x_0) = x_0$ and $\omega(x_0, x_0) = 0$.

2. The Results

Applying Theorem 1.2, first we prove the following generalization of Theorem 1.1.

Theorem 2.1. *Let $g : X \to (0, \infty)$ be any function such that for some $r > 0$,*

$$\sup\left\{g(x) : x \in X,\ \psi(x) \leq \inf_{z \in X} \psi(z) + r\right\} < \infty. \tag{2.1}$$

Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying

$$\omega(x, y) \leq g(x)(\psi(x) - \psi(y)). \tag{2.2}$$

Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Proof. Define a function $f : X \to X$ by $f(x) = y \in T(x) \subseteq X$. Note that for each $x \in X$, we have

$$\omega(x, f(x)) \leq g(x)(\psi(x) - \psi(f(x))). \tag{2.3}$$

Now, since $g(x) > 0$, it follows that $\psi(f(x)) \leq \psi(x)$. Put

$$M = \left\{x \in X : \psi(x) \leq \inf_{z\in X} \psi(z) + r\right\}, \qquad \alpha = \sup_{z\in M} g(z) < \infty. \tag{2.4}$$

Note that M is nonempty, and by the lower semicontinuity of ψ and $\omega(x,\cdot)$, M is closed subset of a complete metric space X, and hence it is complete. Now, we show that $f(M) \subseteq M$. Let $u \in M$, and $f(u) = v \in T(u)$, then we have

$$\psi(f(u)) \leq \psi(u) \leq \inf_{z\in X} \psi(z) + r, \tag{2.5}$$

and thus $f(u) \in M$, and hence f is a self map on M. Note that $\alpha\psi$ is lower semicontinuous and for each $x \in M$, we have

$$\omega(x, f(x)) \leq \alpha(\psi(x)) - \alpha(\psi(f(x))). \tag{2.6}$$

By Theorem 1.2, there exists $x_0 \in M$ such that $f(x_0) = x_0 \in T(x_0)$ and $\omega(x_0, x_0) = 0$. □

Now, applying Theorem 2.1, we obtain generalized Caristi's fixed point results.

Theorem 2.2. *Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying*

$$\omega(x, y) \leq \max\{c(\psi(x)), c(\psi(y))\}(\psi(x) - \psi(y)), \tag{2.7}$$

where $c : [0,\infty) \to (0,\infty)$ is an upper semicontinuous function from the right. Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Proof. Put $t_0 = \inf_{x\in X} \psi(x)$. By the definition of the function c, there exist some positive real numbers r, r_0 such that $c(t) \leq r_0$ for all $t \in [t_0, t_0 + r]$. Now, for all $x \in X$, we define

$$g(x) = \max\{c(\psi(x)), c(\psi(y))\}. \tag{2.8}$$

Clearly, g maps x into $(0,\infty)$. Note that for all $x \in X$, we get $\psi(y) \leq \psi(x)$, and thus for any $x \in X$ with $\psi(x) \leq t_0 + r$, we have

$$\psi(y) \leq t_0 + r. \tag{2.9}$$

Now, clearly, $g(x) \leq r_0 < \infty$ and hence we obtain

$$\sup\left\{g(x) : x \in X,\ \psi(x) \leq \inf_{z\in X} \psi(z) + r\right\} < \infty. \tag{2.10}$$

By Theorem 2.1, T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$. □

Theorem 2.3. *Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying*

$$\omega(x, y) \leq c(\psi(x))(\psi(x) - \psi(y)), \tag{2.11}$$

where $c : [0, \infty) \to (0, \infty)$ is nondecreasing function. Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Proof. For each $x \in X$, define $g(x) = c(\psi(x))$. Clearly, g does carry x into $(0, \infty)$. Now, since the function c is nondecreasing, for any real number $r > 0$ we have

$$\sup\left\{g(x) : x \in X,\ \psi(x) \leq \inf_{z\in X} \psi(z) + r\right\} \leq c\left(\inf_{z\in X} \psi(z) + r\right) < \infty. \tag{2.12}$$

Thus, by Theorem 2.1, the result follows. □

Corollary 2.4. *Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying*

$$\omega(x, y) \leq c(\psi(y))(\psi(x) - \psi(y)), \tag{2.13}$$

where $c : [0, \infty) \to (0, \infty)$ is a nondecreasing function. Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Proof. Since for each $x \in X$ there is $y \in T(x)$ such that $\psi(y) \leq \psi(x)$ and the function c is nondecreasing, we have $c(\psi(y)) \leq c(\psi(x))$. Thus the result follows from Theorem 2.3. □

Applying Theorem 2.3, we prove the following fixed point result.

Theorem 2.5. *Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying $\omega(x, y) \leq \psi(x)$ and*

$$\omega(x, y) \leq \eta(\omega(x, y))(\psi(x) - \psi(y)), \tag{2.14}$$

where $\eta : [0, \infty) \to (0, \infty)$ is an upper semicontinuous function. Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Proof. Define a function c from $[0, \infty)$ into $(0, \infty)$ by

$$c(t) = \sup\{\eta(r) : 0 \leq r \leq t\}. \tag{2.15}$$

Clearly, c is nondecreasing function. Now, since $\omega(x,y) \leq \psi(x)$, we have $c(\omega(x,y)) \leq c(\psi(x))$. Thus by Theorem 2.3, the result follows. □

The following result can be seen as a generalization of [5, Theorem 4].

Corollary 2.6. *Let $\phi : [0,\infty) \to [0,\infty)$ be a lower semicontinuous function such that*

$$\limsup_{t\to 0^+} \frac{t}{\phi(t)} < \infty. \tag{2.16}$$

Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying $\omega(x,y) \leq \psi(x)$ and

$$\phi(\omega(x,y)) \leq \psi(x) - \psi(y). \tag{2.17}$$

Then T has a fixed point $x_0 \in X$ such that $\omega(x_0,x_0) = 0$.

Proof. Define a function $\eta : [0,\infty) \to (0,\infty)$ by

$$\eta(0) = \limsup_{t\to 0^+} \frac{t}{\phi(t)}, \quad \eta(t) = \frac{t}{\phi(t)}, \quad t > 0. \tag{2.18}$$

Then η is upper semicontinuous. Also note that

$$\omega(x,y) \leq \eta(\omega(x,y))(\psi(x) - \psi(y)). \tag{2.19}$$

Thus by Theorem 2.5, T has a fixed point $x_0 \in X$ such that $\omega(x_0,x_0) = 0$. □

Now, let p be a τ distance on X [8], using the same technique as in the proof of Theorem 2.1, and applying [8, Theorem 3], we can obtain the following result.

Theorem 2.7. *Let $g : X \to (0,\infty)$ be any function such that for some $r > 0$,*

$$\sup\left\{ g(x) : x \in X,\ \psi(x) \leq \inf_{z\in X} \psi(z) + r \right\} < \infty. \tag{2.20}$$

Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying

$$p(x,y) \leq g(x)(\psi(x) - \psi(y)). \tag{2.21}$$

Then T has a fixed point $x_0 \in X$ such that $\omega(x_0,x_0) = 0$.

Now, following similar methods as in the proofs of Theorems 2.2, 2.3, 2.5, and Corollaries 2.4 and 2.6, we can obtain the following generalizations of Caristi's fixed point theorem with respect to τ-distance.

Theorem 2.8. *Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying*

$$p(x,y) \leq \max\{c(\varphi(x)), c(\varphi(y))\}(\varphi(x) - \varphi(y)), \tag{2.22}$$

where $c : [0,\infty) \to (0,\infty)$ is an upper semicontinuous from the right. Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Theorem 2.9. *Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying*

$$p(x,y) \leq c(\varphi(x))(\varphi(x) - \varphi(y)), \tag{2.23}$$

where $c : [0,\infty) \to (0,\infty)$ is a nondecreasing function. Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Corollary 2.10. *Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying*

$$p(x,y) \leq c(\varphi(y))(\varphi(x) - \varphi(y)), \tag{2.24}$$

where $c : [0,\infty) \to (0,\infty)$ is a nondecreasing function. Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Theorem 2.11. *Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying $p(x,y) \leq \varphi(x)$ and*

$$p(x,y) \leq \eta(p(x,y))(\varphi(x) - \varphi(y)), \tag{2.25}$$

where $\eta : [0,\infty) \to (0,\infty)$ is an upper semicontinuous function. Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Corollary 2.12. *Let $\phi : [0,\infty) \to [0,\infty)$ be a lower semicontinuous function such that*

$$\limsup_{t \to 0^+} \frac{t}{\phi(t)} < \infty. \tag{2.26}$$

Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying $p(x,y) \leq \varphi(x)$ and

$$\phi(p(x,y) \leq \varphi(x) - \varphi(y). \tag{2.27}$$

Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Similar generalizations of Caristi's fixed point theorem in the setting of quasi-metric spaces with respect to w-distance and with respect to Q-function are studied in [3, Theorem 5.1(iii), Theorem 5.2] and in [2, Theorem 4.1], respectively.

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