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Research Article

Quasicontraction Mappings in Modular Spaces without Δ_2-Condition

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As a generalization to Banach contraction principle, Ćirić introduced the concept of quasicontraction mappings. In this paper, we investigate these kinds of mappings in modular function spaces without the Δ_2-condition. In particular, we prove the existence of fixed points and discuss their uniqueness.

1. Introduction

Let (M,d) be a metric space. A mapping $T : M \to M$ is said to be quasicontraction if there exists $k < 1$ such that

$$d(T(x),T(y)) \leq k \max\left(d(x,y);\, d(x,T(x));\, d(y,T(y));\, d(x,T(y));\, d(y,T(x))\right), \tag{1.1}$$

for any $x, y \in M$. In 1974, Ćirić [1] introduced these mappings and proved an existence fixed point result very similar to the original Banach contraction fixed point theorem. Recently, the authors [2] tried to extend their ideas to modular spaces. Though their conclusions are very similar to Ćirić's results proved in metric spaces, they were unable to escape the Δ_2-condition. They also asked whether Ćirić's results may be proved in the modular setting without the very restrictive Δ_2-condition. In this work, we give a proof in the affirmative.

Recall that modular spaces were initiated by Nakano in 1950 [3] in connection with the theory of order spaces and redefined and generalized by Luxemburg [4–13] and Orlicz in 1959. These spaces were developed following the successful theory of Orlicz spaces, which replaces the particular, integral form of the nonlinear functional, which controls the growth of members of the space, by an abstractly given functional with some good properties. The monographic exposition of the theory of Orlicz spaces may be found in the book of Krasnosel'skii and Rutickii [14]. For a current review of the theory of Musielak-Orlicz spaces and modular spaces, the reader is referred to the books of Musielak and Orlicz [15] and Kozłowski [16].

For more information on fixed point theory in modular spaces, the reader is advised to consult [16–19], and the references therein.

2. Preliminaries

Let $\mathcal{X}$ be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A functional $\rho : \mathcal{X} \to [0, \infty]$ is called a modular, if for arbitrary f and g, elements of $\mathcal{X}$, there hold the following:

(1) $\rho(f) = 0$ if and only if $f = 0$;

(2) $\rho(\alpha f) = \rho(f)$ whenever $|\alpha| = 1$;

(3) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ whenever $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

If we replace (3) by

(3′) $\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g)$ whenever $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$,

then the modular ρ is called convex. If ρ is a modular in $\mathcal{X}$, then the set defined by

$$\mathcal{X}_\rho = \left\{h \in \mathcal{X}; \lim_{\lambda \to 0} \rho(\lambda h) = 0\right\} \tag{2.1}$$

is called a modular space. $\mathcal{X}_\rho$ is a vector subspace of $\mathcal{X}$.

Definition 2.1. A function modular is said to satisfy the Δ_2-type condition if there exists $K > 0$ such that for any $f \in \mathcal{X}_\rho$ one has $\rho(2f) \leq K\rho(f)$.

Definition 2.2. Let $(\mathcal{X}, \rho)$ be a modular space.

(1) The sequence $\{f_n\}_n \subset \mathcal{X}_\rho$ is said to be ρ-convergent to $f \in \mathcal{X}_\rho$ if

$$\rho(f_n - f) \longrightarrow 0, \tag{2.2}$$

as $n \to \infty$.

(2) The sequence $\{f_n\}_n \subset \mathcal{X}_\rho$ is said to be ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as n and m go to ∞.

(3) A subset C of $\mathcal{X}_\rho$ is called ρ-closed if the ρ-limit of a ρ-convergent sequence of C always belongs to C.

(4) A subset C of $\mathcal{X}_\rho$ is called ρ-complete if any ρ-Cauchy sequence in C is ρ-convergent and its ρ-limit is in C.

(5) A subset C of $\mathcal{X}_\rho$ is called ρ-bounded if

$$\delta_\rho(C) = \sup\left\{\rho(f - g);\ f, g \in C\right\} < \infty. \tag{2.3}$$

The above definitions are independent of any Δ_2-type conditions. In fact it is well known in the literature that many characterizations of Δ_2-condition involving (2)–(4) and vector topologies defined on $\mathcal{X}_\rho$.

The following property is crucial throughout this paper.

Definition 2.3. The modular ρ has the Fatou property if and only $\rho(f) \leq \liminf_{n\to\infty}\rho(f_n)$ whenever $\{f_n\}$ ρ-converges to f.

Note that ρ has the Fatou property if and only if the ρ-ball $B_\rho(f,r) = \{g \in \mathcal{X}_\rho;\ \rho(f-g) \leq r\}$ is ρ-closed, for any $f \in \mathcal{X}_\rho$ and $r \geq 0$.

Example 2.4. As a classical example, we consider the Orlicz' modular defined for every measurable real function f by the formula

$$\rho(f) = \int_{\mathbb{R}} \varphi(|f(t)|)\,dm(t), \tag{2.4}$$

where m denotes the Lebesgue measure in $\mathbb{R}$ and $\varphi : \mathbb{R} \to [0,\infty)$ is continuous, $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. The modular space induced by the Orlicz' modular ρ_φ is called the Orlicz space L^φ. If we take $\varphi(x) = e^x - 1$, then ρ_φ does not satisfy the Δ_2-condition. The ρ_φ-balls $B_{\rho_\varphi}(f,r)$ are ρ_φ-closed, and L^φ is ρ_φ-complete. For more on this example, the reader may consult [16, 20].

3. A fixed point theorem

Similarly to Ćirić definition, we introduce the concept of quasicontractions in modular spaces.

Definition 3.1. Let $(\mathcal{X},\rho)$ be a modular space. Let C be a nonempty subset of $\mathcal{X}_\rho$. The self-map $T : C \to C$ is said to be *quasicontraction* if there exists $k < 1$ such that

$$\rho(T(x) - T(y)) \leq k \max\left(\rho(x-y);\ \rho(x-T(x));\ \rho(y-T(y));\ \rho(x-T(y));\ \rho(y-T(x))\right), \tag{3.1}$$

for any $x, y \in C$.

In the sequel, we prove an existence fixed point theorem for such mappings. First, let T and C as in the above definition. For any $x \in C$, define the orbit

$$\mathcal{O}(x) = \{x, T(x), T^2(x), \dots\}, \tag{3.2}$$

and its ρ-diameter by

$$\delta_\rho(x) = \operatorname{diam}(\mathcal{O}(x)) = \sup\{\rho(T^n(x) - T^m(x));\ n, m \in \mathbb{N}\}. \tag{3.3}$$

Lemma 3.2. *Let $(\mathcal{X},\rho)$ be a modular space. Let C be a nonempty subset of $\mathcal{X}_\rho$ and $T : C \to C$ be quasicontraction. Let $x \in C$ such that $\delta_\rho(x) < \infty$. Then for any $n \geq 1$, one has*

$$\delta_\rho(T^n(x)) \leq k^n \delta_\rho(x), \tag{3.4}$$

where k is the constant associated with the quasicontraction definition of T. Moreover, one has

$$\rho(T^n(x) - T^{n+m}(x)) \leq k^n \delta_\rho(x), \tag{3.5}$$

for any $n \geq 1$ and $m \in \mathbb{N}$.

Proof. Let $n, m \geq 1$, we have

$$\rho(T^n(x) - T^m(y)) \leq k \max\left(\rho(T^{n-1}(x) - T^{m-1}(y));\ \rho(T^{n-1}(x) - T^n(x)); \rho(T^m(y) - T^{m-1}(y));\right.$$
$$\left.\rho(T^{n-1}(x) - T^m(y));\ \rho(T^n(x) - T^{m-1}(y))\right), \tag{3.6}$$

for any $x, y \in C$. This obviously implies the following:

$$\delta_\rho(T^n(x)) \leq k\delta_\rho(T^{n-1}(x)), \tag{3.7}$$

for any $n \geq 1$. Hence for any $n \geq 1$, we have

$$\delta_\rho(T^n(x)) \leq k^n \delta_\rho(x)\,. \tag{3.8}$$

Moreover for any $n \geq 1$ and $m \in \mathbb{N}$, we have

$$\rho(T^n(x) - T^{n+m}(x)) \leq \delta_\rho(T^n(x)) \leq k^n \delta_\rho(x)\,. \tag{3.9}$$

□

The next lemma will be helpful to prove the main result of this paper.

Lemma 3.3. *Let $(\mathcal{X}, \rho)$ be a modular space such that ρ satisfies the Fatou property. Let C be a ρ-complete nonempty subset of $\mathcal{X}_\rho$ and let $T : C \to C$ be quasicontraction. Let $x \in C$ such that $\delta_\rho(x) < \infty$. Then $\{T^n(x)\}$ ρ-converges to $\omega \in C$. Moreover, one has*

$$\rho(T^n(x) - \omega) \leq k^n \delta_\rho(x), \tag{3.10}$$

for any $n \geq 1$.

Proof. From the previous lemma, we know that $\{T^n(x)\}$ is ρ-Cauchy. Since C is ρ-complete, then there exists $\omega \in C$ such that $\{T^n(x)\}$ ρ-converges to ω. Since

$$\rho(T^n(x) - T^{n+m}(x)) \leq k^n \delta_\rho(x), \tag{3.11}$$

for any $n \geq 1$, $m \in \mathbb{N}$, and ρ satisfies the Fatou property, we let $m \to \infty$ to get

$$\rho(T^n(x) - \omega) \leq k^n \delta_\rho(x)\,. \tag{3.12}$$

□

Next, we prove that ω is in fact a fixed point of T and it is unique provided some extra assumptions.

Theorem 3.4. *Let C, T, and x be as in the previous Lemma. Assume $\rho(\omega - T(\omega)) < \infty$ and $\rho(x - T(\omega)) < \infty$. Then, the ρ-limit ω of $\{T^n(x)\}$ is a fixed point of T, that is, $T(\omega) = \omega$. Moreover, if ω^* is any fixed point of T in C such that $\rho(\omega - \omega^*) < \infty$, then one has $\omega = \omega^*$.*

Proof. We have

$$\rho(T(x) - T(\omega)) \le k \max \left(\rho(x - \omega);\ \rho(x - T(x));\ \rho(T(\omega) - \omega);\ \rho(T(x) - \omega);\ \rho(x - T(\omega))\right). \tag{3.13}$$

From the previous results, we get

$$\rho(T(x) - T(\omega)) \le k \max \left(\delta_\rho(x);\ \rho(\omega - T(\omega));\ \rho(x - T(\omega))\right). \tag{3.14}$$

Assume that for $n \ge 1$, we have

$$\rho(T^n(x) - T(\omega)) \le \max \left(k^n \delta_\rho(x);\ k\rho(\omega - T(\omega));\ k^n \rho(x - T(\omega))\right). \tag{3.15}$$

Then,

$$\begin{aligned}\rho(T^{n+1}(x) - T(\omega)) \le k \max \big(&\rho(T^n(x) - \omega);\ \rho(T^n(x) - T^{n+1}(x)); \rho(\omega - T(\omega));\\ &\rho(T^{n+1}(x) - \omega);\ \rho(T^n(x) - T(\omega))\big).\end{aligned} \tag{3.16}$$

Hence,

$$\rho(T^{n+1}(x) - T(\omega)) \le k \max \left(k^n \delta_\rho(x);\ \rho(\omega - T(\omega));\ \rho(T^n(x) - T(\omega))\right). \tag{3.17}$$

Using our previous assumption, we get

$$\rho(T^{n+1}(x) - T(\omega)) \le \max \left(k^{n+1} \delta_\rho(x);\ k\rho(\omega - T(\omega));\ k^{n+1} \rho(x - T(\omega))\right). \tag{3.18}$$

So by induction, we have

$$\rho(T^n(x) - T(\omega)) \le \max \left(k^n \delta_\rho(x);\ k\rho(\omega - T(\omega));\ k^n \rho(x - T(\omega))\right), \tag{3.19}$$

for any $n \ge 1$. Therefore, we have

$$\limsup_{n \to \infty} \rho(T^n(x) - T(\omega)) \le k\rho(\omega - T(\omega)). \tag{3.20}$$

Using the Fatou property satisfied by ρ, we get

$$\rho(\omega - T(\omega)) \le \liminf_{n \to \infty} \rho(T^n(x) - T(\omega)) \le k\rho(\omega - T(\omega)). \tag{3.21}$$

Since $k < 1$, we get $\rho(\omega - T(\omega)) = 0$ or $T(\omega) = \omega$. Let ω^* be another fixed point of T such that $\rho(\omega - \omega^*) < \infty$. Then, we have

$$\rho(\omega - \omega^*) = \rho(T(\omega) - T(\omega^*)) \le k\rho(\omega - \omega^*) \tag{3.22}$$

which implies $\rho(\omega - \omega^*) = 0$ or $\omega = \omega^*$. This completes the proof of our theorem. □

Remark 3.5. In [20], the authors initiated the theory of fixed point theory in modular function spaces. In that paper, an example is given of a contraction for the modular ρ which fails to be even nonexpansive for the associated norm. In fact, an extensive discussion is given about the importance of relaxing the Δ_2-condition and the reasons behind. Therefore, the importance of this work is in dropping this condition from the work of the authors in [2].

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