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Research Article

Convergence Theorems for Common Fixed Points of Nonself Asymptotically Quasi-Non-Expansive Mappings

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We introduce a new three-step iterative scheme with errors. Several convergence theorems of this scheme are established for common fixed points of nonself asymptotically quasi-non-expansive mappings in real uniformly convex Banach spaces. Our theorems improve and generalize recent known results in the literature.

1. Introduction

Let K be a nonempty closed convex subset of real normed linear space E. Recall that a mapping $T : K \to K$ is called asymptotically nonexpansive if there exists a sequence$\{r_n\} \subset [0,\infty)$, with $\lim_{n\to\infty} r_n = 0$ such that $\|T^n x - T^n y\| \leq (1 + r_n)\|x - y\|$, for all $x, y \in K$ and $n \geq 1$. Moreover, it is uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|$, for all $x, y \in K$ and each $n \geq 1$. Denote and define by $F(T) = \{x \in K : Tx = x\}$ the set of fixed points of T. Suppose $F(T) \neq \varnothing$. A mapping T is called asymptotically quasi-non-expansive if there exists a sequence $\{r_n\} \subset [0,\infty)$, with $\lim_{n\to\infty} r_n = 0$ such that $\|T^n x - p\| \leq (1 + r_n)\|x - p\|$, for all $x, y \in K$, $p \in F(T)$, and $n \geq 1$.

It is clear from the above definitions that an asymptotically nonexpansive mapping must be uniformly L-Lipschitzian as well as asymptotically quasi-non-expansive, but the converse does not hold. Iterative technique for asymptotically nonexpansive self-mapping in Hilbert spaces and Banach spaces including Mann-type and Ishikawa-type iteration processes has been studied extensively by many authors; see, for example, [1–6].

Recently, Chidume et al. [7] have introduced the concept of nonself asymptotically nonexpansive mappings, which is the generalization of asymptotically nonexpansive mappings. Similarly, the concept of nonself asymptotically quasi-non-expansive mappings can also be defined as the generalization of asymptotically quasi-non-expansive mappings and nonself asymptotically nonexpansive mappings. These mappings are defined as follows.

Definition 1.1. Let K be a nonempty closed convex subset of real normed linear space E, let $P : E \to K$ be the nonexpansive retraction of E onto K, and let $T : K \to E$ be a nonself mapping.

(i) T is said to be a nonself asymptotically nonexpansive mapping if there exists a sequence $\{r_n\} \subset [0,\infty)$, with $\lim_{n\to\infty} r_n = 0$ such that

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \leq (1+r_n)\|x-y\|, \tag{1.1}$$

for all $x, y \in K$ and $n \geq 1$.

(ii) T is said to be a nonself uniformly L-Lipschitzian mapping if there exists a constant $L > 0$ such that

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \leq L\|x-y\|, \tag{1.2}$$

for all $x, y \in K$ and $n \geq 1$.

(iii) T is said to be a nonself asymptotically quasi-non-expansive mapping if $F(T) \neq \varnothing$ and there exists a sequence $\{r_n\} \subset [0,\infty)$, with $\lim_{n\to\infty} r_n = 0$ such that

$$\left\|T(PT)^{n-1}x - p\right\| \leq (1+r_n)\|x-p\|, \tag{1.3}$$

for all $x, y \in K$, $p \in F(T)$, and $n \geq 1$.

By studying the following iteration process (Mann-type iteration):

$$x_1 \in K, \qquad x_{n+1} = P\left((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n\right), \quad \forall n \geq 1, \tag{1.4}$$

where $\{\alpha_n\} \subset [0,1]$, Chidume et al. [7] obtained many convergence theorems for the fixed points of nonself asymptotically nonexpansive mapping T. Later on, Wang [8] generalized the iteration process (1.4) as follows (Ishikawa-type iteration):

$$\begin{gathered} x_1 \in K, \\ x_{n+1} = P\left((1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n\right), \\ y_n = P\left((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n\right), \quad \forall n \geq 1 \end{gathered} \tag{1.5}$$

where $T_1, T_2 : K \to E$ are nonself asymptotically nonexpansive mappings and $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. Also, he got several convergence theorems of the iterative scheme (1.5) under proper conditions.

In 2000, Noor [9] first introduced a three-step iterative sequence and studied the approximate solutions of variational inclusion in Hilbert spaces by using the techniques of updating the solution and the auxiliary principle. Glowinski and Tallec [10] showed that the three-step iterative schemes perform better than the Mann-type and Ishikawa-type iterative schemes. On the other hand, Xu and Noor [11] introduced and studied a three-step scheme to approximate fixed points of asymptotically nonexpansive mappings in Banach spaces. Cho et al. [12] and Plubtieng et al. [13] extended the work of Xu and Noor to the three-step iterative scheme with errors, and gave weak and strong convergence theorems for asymptotically nonexpansive mappings in Banach spaces.

Inspired and motivated by these facts, a new class of three-step iterative schemes with errors, for three nonself asymptotically quasi-non-expansive mappings, is introduced and studied in this paper. This scheme can be viewed as an extension for (1.4), (1.5), and others. This scheme is defined as follows.

Let K be a nonempty convex subset of real normed linear space X, let $P : E \to K$ be the nonexpansive retraction of E onto K, and let $T_1, T_2, T_3 : K \to E$ be three nonself asymptotically quasi-non-expansive mappings. Compute the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ by

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= P\left(\alpha_n T_1 (PT_1)^{n-1} y_n + \beta_n x_n + \gamma_n w_n\right), \\ y_n &= P\left(\alpha_n' T_2 (PT_2)^{n-1} z_n + \beta_n' x_n + \gamma_n' v_n\right), \\ z_n &= P\left(\alpha_n'' T_3 (PT_3)^{n-1} x_n + \beta_n'' x_n + \gamma_n'' u_n\right), \quad \forall n \geq 1 \end{aligned} \tag{1.6}$$

where $\{\alpha_n\}$, $\{\alpha_n'\}$, $\{\alpha_n''\}$, $\{\beta_n\}$, $\{\beta_n'\}$, $\{\beta_n''\}$, $\{\gamma_n\}$, $\{\gamma_n'\}$, and $\{\gamma_n''\}$ are real sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = \alpha_n'' + \beta_n'' + \gamma_n'' = 1$, and $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ are bounded sequences in K.

Remark 1.2. (i) If $T_1 = T_2 = T_3 := T$, $\gamma_n = \gamma_n' = \gamma_n'' = 0$, and $\alpha_n' = \alpha_n'' = 0$, then scheme (1.6) reduces to the Mann-type iteration (1.4).

(ii) If $T_2 = T_3$, $\gamma_n = \gamma_n' = \gamma_n'' = 0$, and $\alpha_n'' = 0$, then scheme (1.6) reduces to the Ishikawa-type iteration (1.5).

(iii) If T_1, T_2, and T_3 are three self-asymptotically nonexpansive mappings, then scheme (1.6) reduces to the three-step iteration with errors defined by [12, 13], and others.

The purpose of this paper is to study the iterative sequences (1.6) to converge to a common fixed point of three nonself asymptotically quasi-non-expansive mappings in real uniformly convex Banach spaces. Our results extend and improve the corresponding results in [5, 7, 8, 11–13], and many others.

2. Preliminaries and lemmas

In this section, we first recall some well-known definitions.

A real Banach space E is said to be uniformly convex if the modulus of convexity of E:

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1,\ \|x-y\| = \varepsilon\right\} > 0, \tag{2.1}$$

for all $0 < \varepsilon \leq 2$ (i.e., $\delta_E(\varepsilon)$ is a function $(0,2] \to (0,1)$).

A subset K of E is said to be a retract if there exists continuous mapping $P : E \to K$ such that $Px = x$, for all $x \in K$, and every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : E \to E$ is said to be a retraction if $P^2 = P$.

A mapping $T : K \to E$ with $F(T) \neq \varnothing$ is said to satisfy condition (A) (see [14]) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, for all $r \in (0,\infty)$, such that

$$\|x - Tx\| \geq f\left(d(x, F(T))\right), \tag{2.2}$$

for all $x \in K$, where $d(x, F(T)) = \inf\{\|x - x^*\| : x^* \in F(T)\}$.

We modify this condition for three mappings $T_1, T_2, T_3 : K \to E$ as follows. Three mappings $T_1, T_2, T_3 : K \to E$, where K is a subset of E, are said to satisfy condition (B) if there exist a real number $\alpha > 0$ and a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, for all $r \in (0,\infty)$, such that

$$\|x - T_1x\| \geq \alpha f(d(x,F)) \quad \text{or} \quad \|x - T_2x\| \geq \alpha f(d(x,F)) \quad \text{or} \quad \|x - T_3x\| \geq \alpha f(d(x,F)), \tag{2.3}$$

for all $x \in K$, where $F = F(T_1)\cap F(T_2)\cap F(T_3) \neq \emptyset$. Note that condition (B) reduces to condition (A) when $T_1 = T_2 = T_3$ and $\alpha = 1$.

A mapping $T : K \to E$ is said to be semicompact if, for any sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ $(n \to \infty)$, there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x^* \in K$.

Next we state the following useful lemmas.

Lemma 2.1 (see [5]). *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + c_n)a_n + b_n, \quad \forall n \geq 1. \tag{2.4}$$

If $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.2 (see [15]). *Let E be a real uniformly convex Banach space and $0 \leq k \leq t_n \leq q < 1$, for all positive integer $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty}\|x_n\| \leq r$, $\limsup_{n\to\infty}\|y_n\| \leq r$, and $\lim_{n\to\infty}\|t_nx_n + (1 - t_n)y_n\| = r$ hold, for some $r \geq 0$; then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

3. Main results

In this section, we will prove the strong convergence of the iteration scheme (1.6) to a common fixed point of nonself asymptotically quasi-non-expansive mappings T_1, T_2, and T_3. We first prove the following lemmas.

Lemma 3.1. *Let K be a nonempty closed convex subset of a real normed linear space E. Let $T_1, T_2, T_3 : K \to E$ be nonself asymptotically quasi-non-expansive mappings with sequences $\{r_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$, for all $i = 1,2,3$. Suppose that $\{x_n\}$ is defined by* (1.6) *with $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\gamma_n' < \infty$, and $\sum_{n=1}^{\infty}\gamma_n'' < \infty$. If $F = F(T_1)\cap F(T_2)\cap F(T_3) \neq \emptyset$, then $\lim_{n\to\infty}\|x_n - p\|$ exists, for all $p \in F$.*

Proof. Let $p \in F$. Since $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ are bounded sequences in K, therefore there exists $M > 0$ such that

$$M = \max\left\{\sup_{n\geq1}\|u_n - p\|, \sup_{n\geq1}\|v_n - p\|, \sup_{n\geq1}\|w_n - p\|\right\}. \tag{3.1}$$

Let $r_n = \max\{r_n^{(1)}, r_n^{(2)}, r_n^{(3)}\}$ and $k_n = \max\{\gamma_n, \gamma_n', \gamma_n''\}$. Then $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} k_n < \infty$. By (1.6), we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|P[\alpha_nT_1(PT_1^{n-1})y_n + \beta_nx_n + \gamma_nw_n] - P(p)\| \\ &\leq \|\alpha_nT_1(PT_1^{n-1})y_n + \beta_nx_n + \gamma_nw_n - (\alpha_n + \beta_n + \gamma_n)p\| \\ &\leq \|\alpha_n[T_1(PT_1^{n-1})y_n - p] + \beta_n(x_n - p) + \gamma_n(w_n - p)\| \\ &\leq \alpha_n(1 + r_n)\|y_n - p\| + \beta_n\|x_n - p\| + k_n\|w_n - p\|,\end{aligned} \tag{3.2}$$

$$\begin{aligned}\|y_n - p\| &= \|P[\alpha_n'T_2(PT_2^{n-1})z_n + \beta_n'x_n + \gamma_n'v_n] - P(p)\| \\ &\leq \|\alpha_n'T_2(PT_2^{n-1})z_n + \beta_n'x_n + \gamma_n'v_n - (\alpha_n' + \beta_n' + \gamma_n')p\| \\ &\leq \alpha_n'(1 + r_n)\|z_n - p\| + \beta_n'\|x_n - p\| + k_n\|v_n - p\|,\end{aligned} \tag{3.3}$$

and similarly, we also have

$$\|z_n - p\| \le \alpha''_n(1+r_n)\|x_n - p\| + \beta''_n\|x_n - p\| + k_n\|u_n - p\|. \tag{3.4}$$

Substituting (3.4) into (3.3), we obtain

$$\begin{aligned}
\|y_n - p\| &\le \alpha'_n(1+r_n)\left[\alpha''_n(1+r_n)\|x_n - p\| + \beta''_n\|x_n - p\| + k_n\|u_n - p\|\right] \\
&\quad + \beta'_n\|x_n - p\| + k_n\|v_n - p\| \\
&\le \alpha'_n\alpha''_n(1+r_n)^2\|x_n - p\| + \alpha'_n\beta''_n(1+r_n)\|x_n - p\| + \beta'_n\|x_n - p\| \\
&\quad + \alpha'_n k_n(1+r_n)\|u_n - p\| + k_n\|v_n - p\| \\
&\le (1-\beta'_n-\gamma'_n)\alpha''_n(1+r_n)^2\|x_n - p\| + (1-\beta'_n-\gamma'_n)\beta''_n(1+r_n)\|x_n - p\| \\
&\quad + \beta'_n\|x_n - p\| + k_n(1+r_n)\|u_n - p\| + k_n\|v_n - p\| \\
&\le (1-\beta'_n-\gamma'_n)(\alpha''_n+\beta''_n)(1+r_n)^2\|x_n - p\| + \beta'_n\|x_n - p\| + m_n \\
&\le (1-\beta'_n)(1+r_n)^2\|x_n - p\| + \beta'_n(1+r_n)^2\|x_n - p\| + m_n \\
&\le (1+r_n)^2\|x_n - p\| + m_n,
\end{aligned} \tag{3.5}$$

where $m_n = k_n(2+r_n)M$. Since $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} k_n < \infty$, then $\sum_{n=1}^{\infty} m_n < \infty$. Substituting (3.5) into (3.2), we have

$$\begin{aligned}
\|x_{n+1} - p\| &\le \alpha_n(1+r_n)\left[(1+r_n^2)\|x_n - p\| + m_n\right] + \beta_n\|x_n - p\| + \gamma_n\|w_n - p\| \\
&\le \left[\alpha_n(1+r_n)^3 + \beta_n\right]\|x_n - p\| + \alpha_n(1+r_n)m_n + \gamma_n\|w_n - p\| \\
&\le (\alpha_n+\beta_n)(1+r_n)^3\|x_n - p\| + (1+r_n)m_n + k_n\|w_n - p\| \\
&\le (1+r_n)^3\|x_n - p\| + (1+r_n)m_n + k_n M \\
&\le (1+c_n)\|x_n - p\| + b_n,
\end{aligned} \tag{3.6}$$

where $c_n = (1+r_n)^3 - 1$ and $b_n = (1+r_n)m_n + k_n M$. Since $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} k_n < \infty$, and $\sum_{n=1}^{\infty} m_n < \infty$, then $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$. It follows from Lemma 2.1 that $\lim_{n\to\infty}\|x_n - p\|$ exists. This completes the proof. □

Lemma 3.2. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. Let $T_1, T_2, T_3 : K \to E$ be uniformly L-Lipschitzian nonself asymptotically quasi-non-expansive mappings with sequences $\{r_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$, for all $i = 1,2,3$. Suppose that $\{x_n\}$ is defined by* (1.6) *with $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\gamma'_n < \infty$, and $\sum_{n=1}^{\infty}\gamma''_n < \infty$, where α_n, α'_n, and α''_n are three sequences in $[\varepsilon, 1-\varepsilon]$, for some $\varepsilon > 0$. If $F = F(T_1)\cap F(T_2)\cap F(T_3) \ne \emptyset$, then*

$$\lim_{n\to\infty}\|x_n - T_1x_n\| = \lim_{n\to\infty}\|x_n - T_2x_n\| = \lim_{n\to\infty}\|x_n - T_3x_n\| = 0. \tag{3.7}$$

Proof. For any $p \in F$, by Lemma 3.1, we see that $\lim_{n\to\infty}\|x_n - p\|$ exists. Assume $\lim_{n\to\infty}\|x_n - p\| = a$, for some $a \ge 0$. For all $n \ge 1$, let $r_n = \max\{r_n^{(1)}, r_n^{(2)}, r_n^{(3)}\}$ and $k_n = \max\{\gamma_n, \gamma'_n, \gamma''_n\}$.

Then, $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} k_n < \infty$. From (3.5), we have

$$\|y_n - p\| \leq (1 + r_n)^2 \|x_n - p\| + m_n. \tag{3.8}$$

Taking $\limsup_{n\to\infty}$ on both sides in (3.8), since $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} m_n < \infty$, we obtain

$$\limsup_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \|x_n - p\| = a \tag{3.9}$$

so that

$$\limsup_{n\to\infty} \|T_1(PT_1)^{n-1} y_n - p\| \leq \limsup_{n\to\infty} (1 + r_n) \|y_n - p\| = \limsup_{n\to\infty} \|y_n - p\| \leq a. \tag{3.10}$$

Next consider

$$\|T_1(PT_1)^{n-1} y_n - p + \gamma_n(w_n - x_n)\| \leq \|T_1(PT_1)^{n-1} y_n - p\| + k_n \|w_n - x_n\|. \tag{3.11}$$

Since $\lim_{n\to\infty} k_n = 0$, we have

$$\limsup_{n\to\infty} \|T_1(PT_1)^{n-1} y_n - p + \gamma_n(w_n - x_n)\| \leq a. \tag{3.12}$$

In addition,

$$\|x_n - p + \gamma_n(w_n - x_n)\| \leq \|x_n - p\| + k_n \|w_n - x_n\|. \tag{3.13}$$

This implies that

$$\limsup_{n\to\infty} \|x_n - p + \gamma_n(w_n - x_n)\| \leq a. \tag{3.14}$$

Further, observe that

$$\begin{aligned}
a &= \lim_{n\to\infty} \|x_n - p\| \\
&= \lim_{n\to\infty} \|\alpha_n T_1(PT_1)^{n-1} y_n + \beta_n x_n + \gamma_n w_n - p\| \\
&= \lim_{n\to\infty} \|\alpha_n T_1(PT_1)^{n-1} y_n + (1 - \alpha_n) x_n - \gamma_n x_n + \gamma_n w_n - (1 - \alpha_n) p - \alpha_n p\| \\
&= \lim_{n\to\infty} \|\alpha_n T_1(PT_1)^{n-1} y_n - \alpha_n p + \alpha_n \gamma_n w_n - \alpha_n \gamma_n x_n + (1 - \alpha_n) x_n \\
&\qquad - (1 - \alpha_n) p - \gamma_n x_n + \gamma_n w_n - \alpha_n \gamma_n w_n + \alpha_n \gamma_n x_n\| \\
&= \lim_{n\to\infty} \|\alpha_n [T_1(PT_1)^{n-1} y_n - p + \gamma_n(w_n - x_n)] + (1 - \alpha_n) [x_n - p + \gamma_n(w_n - x_n)]\|.
\end{aligned} \tag{3.15}$$

By Lemma 2.2, (3.12), (3.14), and (3.15), we have

$$\lim_{n\to\infty} \|T_1(PT_1)^{n-1} y_n - x_n\| = 0. \tag{3.16}$$

Next we will prove that $\lim_{n\to\infty}\|T_2(PT_2)^{n-1}z_n - x_n\| = 0$. Since

$$\begin{aligned}\|x_n - p\| &\le \|T_1(PT_1)^{n-1}y_n - x_n\| + \|T_1(PT_1)^{n-1}y_n - p\| \\ &\le \|T_1(PT_1)^{n-1}y_n - x_n\| + (1+r_n)\|y_n - p\|\end{aligned} \tag{3.17}$$

and $\lim_{n\to\infty}\|T_1(PT_1)^{n-1}y_n - x_n\| = 0 = \lim_{n\to\infty} r_n$, we obtain

$$a = \lim_{n\to\infty}\|x_n - p\| \le \liminf_{n\to\infty}\|y_n - p\|. \tag{3.18}$$

Thus, it follows from (3.10) and (3.18) that

$$\lim_{n\to\infty}\|y_n - p\| = a. \tag{3.19}$$

On the other hand, from (3.4), we have

$$\begin{aligned}\|z_n - p\| &\le [\alpha''_n(1+r_n) + \beta''_n]\|x_n - p\| + k_n\|u_n - p\| \\ &\le (1+r_n)\|x_n - p\| + k_n\|u_n - p\|.\end{aligned} \tag{3.20}$$

By boundedness of the sequence $\{u_n\}$ and by $\lim_{n\to\infty} r_n = \lim_{n\to\infty} k_n = 0$, we have

$$\limsup_{n\to\infty}\|z_n - p\| \le \limsup_{n\to\infty}\|x_n - p\| = a \tag{3.21}$$

so that

$$\limsup_{n\to\infty}\|T_2(PT_2)^{n-1}z_n - p\| \le \limsup_{n\to\infty}(1+r_n)\|z_n - p\| \le a. \tag{3.22}$$

Next consider

$$\|T_2(PT_2)^{n-1}z_n - p + \gamma'_n(v_n - x_n)\| \le \|T_2(PT_2)^{n-1}z_n - p\| + k_n\|v_n - x_n\|. \tag{3.23}$$

Thus, we have

$$\begin{aligned}&\limsup_{n\to\infty}\|T_2(PT_2)^{n-1}z_n - p + \gamma'_n(v_n - x_n)\| \le a, \\ &\|x_n - p + \gamma'_n(v_n - x_n)\| \le \|x_n - p\| + k_n\|v_n - x_n\|.\end{aligned} \tag{3.24}$$

This implies that

$$\limsup_{n\to\infty}\|x_n - p + \gamma'_n(v_n - x_n)\| \le a. \tag{3.25}$$

Note that

$$\begin{aligned}a &= \lim_{n\to\infty}\|y_n - p\| \\ &= \lim_{n\to\infty}\|\alpha'_n T_2(PT_2)^{n-1}z_n + \beta'_n x_n + \gamma'_n v_n - p\| \\ &= \lim_{n\to\infty}\|\alpha'_n[T_2(PT_2)^{n-1}z_n - p + \gamma'_n(v_n - x_n)] + (1-\alpha'_n)[x_n - p + \gamma'_n(v_n - x_n)]\|.\end{aligned} \tag{3.26}$$

It follows from Lemma 2.2, (3.24), and (3.25) that

$$\lim_{n\to\infty}\left\|T_2(PT_2)^{n-1}z_n - x_n\right\| = 0. \tag{3.27}$$

Similarly, by using the same argument as in the proof above, we obtain

$$\lim_{n\to\infty}\left\|T_3(PT_3)^{n-1}x_n - x_n\right\| = 0. \tag{3.28}$$

Hence,

$$\lim_{n\to\infty}\left\|T_1(PT_1)^{n-1}y_n - x_n\right\| = \lim_{n\to\infty}\left\|T_2(PT_2)^{n-1}z_n - x_n\right\| = \lim_{n\to\infty}\left\|T_3(PT_3)^{n-1}x_n - x_n\right\| = 0, \tag{3.29}$$

and this implies that

$$\left\|x_{n+1} - x_n\right\| \leq \alpha_n\left\|T_1(PT_1)^{n-1}y_n - x_n\right\| + k_n\left\|w_n - x_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.30}$$

Since T_1 is uniformly L-Lipschitzian mapping, then we have

$$\begin{aligned}
&\left\|T_1(PT_1)^{n-1}x_n - x_n\right\| \\
&\quad\leq \left\|T_1(PT_1)^{n-1}x_n - T_1(PT_1)^{n-1}y_n\right\| + \left\|T_1(PT_1)^{n-1}y_n - x_n\right\| \\
&\quad\leq L\left\|x_n - y_n\right\| + \left\|T_1(PT_1)^{n-1}y_n - x_n\right\| \\
&\quad\leq L\left\|x_n - \alpha'_nT_2(PT_2)^{n-1}z_n - \beta'_nx_n - \gamma'_nv_n\right\| + \left\|T_1(PT_1)^{n-1}y_n - x_n\right\| \\
&\quad\leq L\alpha'_n\left\|T_2(PT_2)^{n-1}z_n - x_n\right\| + Lk_n\left\|v_n - x_n\right\| + \left\|T_1(PT_1)^{n-1}y_n - x_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
&\left\|x_n - T_1x_n\right\| \\
&\quad\leq \left\|x_{n+1}-x_n\right\| + \left\|x_{n+1}-T_1(PT_1)^nx_{n+1}\right\| + \left\|T_1(PT_1)^nx_{n+1}-T_1(PT_1)^nx_n\right\| + \left\|T_1(PT_1)^nx_n-T_1x_n\right\| \\
&\quad\leq \left\|x_{n+1} - x_n\right\| + \left\|x_{n+1} - T_1(PT_1)^nx_{n+1}\right\| + L\left\|x_{n+1} - x_n\right\| + L\left\|T_1(PT_1)^{n-1}x_n - x_n\right\|.
\end{aligned} \tag{3.32}$$

It follows from (3.30), (3.31), and (3.32) that

$$\lim_{n\to\infty}\left\|x_n - T_1x_n\right\| = 0. \tag{3.33}$$

Next consider

$$\begin{aligned}
&\left\|T_2(PT_2)^{n-1}x_n - x_n\right\| \\
&\quad\leq \left\|T_2(PT_2)^{n-1}x_n - T_2(PT_2)^{n-1}z_n\right\| + \left\|T_2(PT_2)^{n-1}z_n - x_n\right\| \\
&\quad\leq L\left\|x_n - z_n\right\| + \left\|T_2(PT_2)^{n-1}z_n - x_n\right\| \\
&\quad\leq L\alpha''_n\left\|T_3(PT_3)^{n-1}x_n - x_n\right\| + Lk_n\left\|u_n - x_n\right\| + \left\|T_2(PT_2)^{n-1}z_n - x_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
&\left\|x_n - T_2x_n\right\| \\
&\quad\leq \left\|x_{n+1}-x_n\right\| + \left\|x_{n+1}-T_2(PT_2)^nx_{n+1}\right\| + \left\|T_2(PT_2)^nx_{n+1}-T_2(PT_2)^nx_n\right\| + \left\|T_2(PT_2)^nx_n-T_2x_n\right\| \\
&\quad\leq \left\|x_{n+1} - x_n\right\| + \left\|x_{n+1} - T_2(PT_2)^nx_{n+1}\right\| + L\left\|x_{n+1} - x_n\right\| + L\left\|T_2(PT_2)^{n-1}x_n - x_n\right\|.
\end{aligned} \tag{3.35}$$

It follows from (3.30), (3.34), and (3.35) that

$$\lim_{n\to\infty} \|x_n - T_2 x_n\| = 0. \tag{3.36}$$

Finally, we consider

$$\begin{aligned}
&\|x_n - T_3 x_n\| \\
&\quad \le \|x_{n+1} - x_n\| + \|x_{n+1} - T_3(PT_3)^n x_{n+1}\| + \|T_3(PT_3)^n x_{n+1} - T_3(PT_3)^n x_n\| + \|T_3(PT_3)^n x_n - T_3 x_n\| \\
&\quad \le \|x_{n+1} - x_n\| + \|x_{n+1} - T_3(PT_3)^n x_{n+1}\| + L\|x_{n+1} - x_n\| + L\|T_3(PT_3)^{n-1} x_n - x_n\|.
\end{aligned} \tag{3.37}$$

It follows from (3.29), (3.30), and (3.37) that

$$\lim_{n\to\infty} \|x_n - T_3 x_n\| = 0. \tag{3.38}$$

Therefore,

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = \lim_{n\to\infty} \|x_n - T_2 x_n\| = \lim_{n\to\infty} \|x_n - T_3 x_n\| = 0. \tag{3.39}$$

This completes the proof. □

Now, we give our main theorems of this paper.

Theorem 3.3. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. Let $T_1, T_2, T_3 : K \to E$ be uniformly L-Lipschitzian and nonself asymptotically quasi-non-expansive mappings with sequences $\{r_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$, for all $i = 1, 2, 3$, satisfying condition (B). Suppose that $\{x_n\}$ is defined by (1.6) with $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$, and $\sum_{n=1}^{\infty} \gamma_n'' < \infty$, where α_n, α_n', and α_n'' are three sequences in $[\varepsilon, 1-\varepsilon]$, for some $\varepsilon > 0$. If $F = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2, and T_3.*

Proof. It follows from Lemma 3.2 that $\lim_{n\to\infty}\|x_n - T_1 x_n\| = \lim_{n\to\infty}\|x_n - T_2 x_n\| = \lim_{n\to\infty}\|x_n - T_3 x_n\| = 0$. Since T_1, T_2, and T_3 satisfy condition (B), we have $\lim_{n\to\infty} d(x_n, F) = 0$.

From Lemma 3.1 and the proof of Qihou [5], we can obtain that $\{x_n\}$ is a Cauchy sequence in K. Assume that $\lim_{n\to\infty} x_n = p \in K$. Since $\lim_{n\to\infty}\|x_n - T_1 x_n\| = \lim_{n\to\infty}\|x_n - T_2 x_n\| = \lim_{n\to\infty}\|x_n - T_3 x_n\| = 0$, by the continuity of T_1, T_2, and T_3, we have $p \in F$, that is, p is a common fixed point of T_1, T_2, and T_3. This completes the proof. □

Corollary 3.4. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. Let $T_1, T_2, T_3 : K \to E$ be nonself asymptotically nonexpansive mappings with sequences $\{r_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$, for all $i = 1, 2, 3$, satisfying condition (B). Suppose that $\{x_n\}$ is defined by (1.6) with $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$, and $\sum_{n=1}^{\infty} \gamma_n'' < \infty$, where α_n, α_n', and α_n'' are three sequences in $[\varepsilon, 1-\varepsilon]$, for some $\varepsilon > 0$. If $F = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2, and T_3.*

Proof. Since every nonself asymptotically nonexpansive mapping is uniformly L-Lipschitzian and nonself asymptotically quasi-non-expansive, the result can be deduced immediately from Theorem 3.3. This completes the proof. □

Theorem 3.5. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. Let $T_1, T_2, T_3 : K \to E$ be uniformly L-Lipschitzian and nonself asymptotically quasi-non-expansive mappings with sequences $\{r_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$, for all $i = 1, 2, 3$. Suppose that $\{x_n\}$ is defined by (1.6) with $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$, and $\sum_{n=1}^{\infty} \gamma_n'' < \infty$, where α_n, α_n', and α_n'' are three sequences in $[\varepsilon, 1-\varepsilon]$, for some $\varepsilon > 0$. If $F = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$ and one of T_1, T_2, and T_3 is demicompact, then $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2, and T_3.*

Proof. Without loss of generality, we may assume that T_1 is demicompact. Since $\lim_{n\to\infty}\|x_n - T_1 x_n\| = 0$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to x^* \in K$. Hence, from (3.39), we have

$$\left\|x^* - T_i x^*\right\| = \lim_{n\to\infty}\left\|x_{n_j} - T_i x_{n_j}\right\| = 0, \quad i = 1, 2, 3. \tag{3.40}$$

This implies that $x^* \in F$. By the arbitrariness of $p \in F$, from Lemma 3.1, and taking $p = x^*$, similarly we can prove that

$$\lim_{n\to\infty}\left\|x_n - x^*\right\| = d, \tag{3.41}$$

where $d \geq 0$ is some nonnegative number. From $x_{n_j} \to x^*$, we know that $d = 0$, that is, $x_n \to x^*$. This completes the proof. □

Corollary 3.6. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. Let $T_1, T_2, T_3 : K \to E$ be nonself asymptotically nonexpansive mappings with sequences $\{r_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$, for all $i = 1, 2, 3$. Suppose that $\{x_n\}$ is defined by (1.6) with $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$, and $\sum_{n=1}^{\infty} \gamma_n'' < \infty$, where α_n, α_n', and α_n'' are three sequences in $[\varepsilon, 1-\varepsilon]$, for some $\varepsilon > 0$. If $F = F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$ and one of T_1, T_2, and T_3 is demicompact, then $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2, and T_3.*

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