VISCOSITY APPROXIMATION FIXED POINTS FOR NONEXPANSIVE AND m-ACCRETIVE OPERATORS

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Let X be a real reflexive Banach space, let C be a closed convex subset of X, and let A be an m-accretive operator with a zero. Consider the iterative method that generates the sequence $\{x_n\}$ by the algorithm $x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)J_{r_n}x_n$, where α_n and γ_n are two sequences satisfying certain conditions, J_r denotes the resolvent $(I+rA)^{-1}$ for $r>0$, and let $f : C \to C$ be a fixed contractive mapping. The strong convergence of the algorithm $\{x_n\}$ is proved assuming that X has a weakly continuous duality map.

1. Introduction

Let X be a real Banach space, let C be a nonempty closed convex subset of X, and let $T : C \to C$ be a nonexpansive mapping if for all $x, y \in C$, such that

$$\|Tx - Ty\| \le \|x - y\|. \tag{1.1}$$

We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in C : x = Tx\}$. And $\rightharpoonup$ denotes weak convergence, $\to$ denotes strong convergence. Recall that a self-mapping $f : C \to C$ is a contraction on C if there exists a constant $\beta \in (0,1)$ such that

$$\left\|f(x) - f(y)\right\| \le \beta\|x - y\|, \quad x, y \in C. \tag{1.2}$$

Browder [2] considered an iteration in a Hilbert space as follows. Fix $u \in C$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1-t)Tx, \quad x \in C, \tag{1.3}$$

where $t \in (0,1)$. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in C.

Xu [7] defined the following one viscosity iteration for nonexpansive mappings in uniformly smooth Banach space.

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THEOREM 1.1 [2, Theorem 4.1, page 287]. *Let X be a uniformly smooth Banach space, let C be a closed convex subset of X, $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \phi$, and $f \in \Pi_C$, where Π_C denotes the set of all contractions on C. Then $\{x_t\}$ defined by the following:*

$$x_t = tf(x_t) + (1-t)Tx_t, \quad x \in C, \tag{1.4}$$

converges strongly to a point in $F(T)$. Define $Q : \Pi_C \to F(T)$ by

$$Q(f) := \lim_{t \to 0} x_t, \quad f \in \prod_C, \tag{1.5}$$

then Q(f) solves the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p) \rangle \leq 0, \quad f \in \prod_C,\ p \in F(T). \tag{1.6}$$

Xu [8] proved the strong convergence of $\{x_t\}$ defined by (1.3) in a reflexive Banach space with a weakly continuous duality map J_φ with gauge φ.

Recall that an operator A with $D(A)$ and range $R(A)$ in X is said to be accretive, if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $(i = 1,2)$ such that

$$\langle y_2 - y_1, J(x_2 - x_1) \rangle \geq 0, \tag{1.7}$$

where J is the duality map from X to the dual space X^* given by

$$J(x) = \{f \in X^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \quad x \in X. \tag{1.8}$$

An accretive operator A is m-accretive if $R(I + \lambda A) = X$ for all $\lambda > 0$.

Denote by J_r the resolvent of A for $r > 0$,

$$J_r = (I + rA)^{-1}. \tag{1.9}$$

It is known that J_r is a nonexpansive mapping from X to $C := \overline{D(A)}$ which will be assumed convex.

Also in [8], Xu considered the following algorithm:

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J_{r_n} x_n, \quad n \geq 0, \tag{1.10}$$

where $u \in C$ is arbitrarily fixed, $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{r_n\}$ is a sequence of positive numbers. Xu proved that if X is a reflexive Banach space with weakly continuous duality mapping, then the sequence $\{x_n\}$ given by (1.10) converges strongly to a point in F provided the sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy certain conditions.

The main purpose of this paper is to consider the following two iterations both in a reflexive Banach space X which has a weakly continuous duality mapping:

$$x_t = tf(x_t) + (1-t)Tx_t, \quad t \in (0,1), \tag{1.11}$$

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J_{r_n} x_n, \quad n \geq 0. \tag{1.12}$$

2. Preliminaries

In order to prove our main results, we need the following lemmas. The proof of Lemma 2.1 can be found in [5, 6]. Lemma 2.2 is an immediate consequence of the subdifferential inequality of the function $(1/2)\|\cdot\|^2$.

LEMMA 2.1. *Let $\{a_n\}_n$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\beta_n, \quad n \geq 0, \tag{2.1}$$

where $\{\alpha_n\}_n \subset (0,1)$, and β_n satisfy the conditions:
(i) $\lim_{n\to\infty}\alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$,
(iii) $\limsup_{n\to\infty}\beta_n \leq 0$.
Then $\lim_{n\to\infty} a_n = 0$.

LEMMA 2.2. *Let X be an arbitrary real Banach space. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle, \quad x,y \in X. \tag{2.2}$$

Recall that a gauge is a continuous strictly increasing function $\varphi : [0,+\infty) \to [0,+\infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$. Associated to a gauge φ is the duality map $J_\varphi : X \to X^*$ defined by

$$J_\varphi(x) = \{f \in X^* : \langle x, f\rangle = \|x\|\varphi(\|x\|),\ \|f\| = \varphi(\|x\|)\}, \quad x \in X. \tag{2.3}$$

Following Browder [3], we say that a Banach space X has a weakly continuous duality map if there exists a gauge φ for which the duality map J_φ is single valued and weak-to-weak* sequentially continuous, that is, if $\{x_n\}$ is a sequence in X weakly convergent to a point x, then the sequence $\{J_\varphi(x_n)\}$ converges weakly* to $J_\varphi(x)$. It is known that ℓ^p has a weakly continuous duality map for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \quad \tau \geq 0. \tag{2.4}$$

Then

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad x \in X, \tag{2.5}$$

where ∂ denotes the subdifferential in the sense of convex analysis.

We also need the next lemma, and the first part of Lemma 2.3 is an immediate consequence of the subdifferential inequality and the proof of the second part can be found in [4].

LEMMA 2.3. *Assume that X has a weakly continuous duality map J_φ with gauge φ.*
(i) *For all $x, y \in X$, there holds the inequality*

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle. \tag{2.6}$$

(ii) *Assume a sequence $\{x_n\}$ in X is weakly convergent to a point x. Then there holds the identity*

$$\limsup_{n\to\infty}\Phi(||x_n - y||) = \limsup_{n\to\infty}\Phi(||x_n - x||) + \Phi(\|y - x\|), \quad x, y \in X. \tag{2.7}$$

Lemma 2.4 is the resolvent identity which can be found in [1].

Lemma 2.4. *For $\lambda, \mu > 0$, there holds the identity*

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda x\right), \quad x \in X. \tag{2.8}$$

3. Main results

Theorem 3.1. *Let X be a real reflexive Banach space and have a weakly continuous duality mapping J_φ with φ. Suppose C is a closed convex subset of X, and $T : C \to C$ is a nonexpansive mapping, let $f : C \to C$ be a fixed contractive mapping. For $t \in (0,1)$, $\{x_t\}$ is defined by (1.11). Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 0^+$, and in this case, $\{x_t\}$ converges strongly to a fixed point of T as $t \to 0^+$.*

Proof. Assume first that $F(T) \neq \phi$. Take $u \in F(T)$, it follows that

$$\begin{aligned} ||x_t - u|| &= ||tf(x_t) + (1-t)Tx_t - u|| \\ &\le t||f(x_t) - u|| + (1-t)||Tx_t - u|| \\ &\le t\beta||x_t - u|| + t||f(u) - u|| + (1-t)||x_t - u|| \\ &= (1 - (1-\beta)t)||x_t - u|| + t||f(u) - u||. \end{aligned} \tag{3.1}$$

Hence

$$||x_t - u|| \le \frac{1}{1-\beta}||f(u) - u||. \tag{3.2}$$

Therefore, $\{x_t\}$ is bounded, so are $\{Tx_t\}$ and $\{f(x_t)\}$.

Next assume that $\{x_t\}$ is bounded as $t \to 0^+$. Assume $t_n \to 0^+$ and $\{x_{t_n}\}$ is bounded. Since X is reflexive, we may assume that $x_{t_n} \rightharpoonup p$ for some $p \in C$. Since J_φ is weakly continuous, we have by Lemma 2.3,

$$\limsup_{n\to\infty}\Phi(||x_{t_n} - x||) = \limsup_{n\to\infty}\Phi(||x_{t_n} - p||) + \Phi(||x - p||), \quad \forall x \in X. \tag{3.3}$$

Put

$$g(x) = \limsup_{n\to\infty}\Phi(||x_{t_n} - x||), \quad x \in X. \tag{3.4}$$

It follows that

$$g(x) = g(p) + \Phi(\|x - p\|), \quad x \in X. \tag{3.5}$$

Since

$$||x_{t_n} - Tx_{t_n}|| = \frac{t_n}{1-t_n}||f(x_{t_n}) - x_{t_n}|| \longrightarrow 0, \tag{3.6}$$

we obtain

$$\begin{aligned} g(Tp) &= \limsup_{n\to\infty} \Phi(||x_{t_n} - Tp||) \le \limsup_{n\to\infty} \Phi(||Tx_{t_n} - Tp||) \\ &\le \limsup_{n\to\infty} \Phi(||x_{t_n} - p||) = g(p). \end{aligned} \tag{3.7}$$

On the other hand, however,

$$g(Tp) = g(p) + \Phi(\|Tp - p\|). \tag{3.8}$$

From (3.7) and (3.8), we get

$$\Phi(\|Tp - p\|) \le 0. \tag{3.9}$$

Hence $Tp = p$ and $p \in F(T)$.

Now we prove that $\{x_t\}$ converges strongly to a fixed point of T provided it remains bounded when $t \to 0$.

Let $\{t_n\}$ be a sequence in $(0,1)$ such that $t_n \to 0$ and $x_{t_n} \rightharpoonup p$ as $n \to \infty$. Then the argument above shows that $p \in F(T)$. We next show that $x_{t_n} \to p$. As a matter of fact, we have by Lemma 2.3,

$$\begin{aligned} \Phi(||x_{t_n} - p||) &= \Phi(||t_n(Tx_{t_n} - p) + (1-t_n)(f(x_{t_n}) - p)||) \\ &\le \Phi(t_n||Tx_{t_n} - p||) + (1-t_n)\langle f(x_{t_n}) - p, J_\varphi(x_{t_n} - p)\rangle \\ &\le t_n\Phi(||x_{t_n} - p||) + (1-t_n)\langle f(x_{t_n}) - f(p), J_\varphi(x_{t_n} - p)\rangle \\ &\quad + (1-t_n)\langle f(p) - p, J_\varphi(x_{t_n} - p)\rangle. \end{aligned} \tag{3.10}$$

This implies that

$$\begin{aligned} \Phi(||x_{t_n} - p||) &\le \langle f(x_{t_n}) - f(p), J_\varphi(x_{t_n} - p)\rangle + \langle f(p) - p, J_\varphi(x_{t_n} - p)\rangle \\ &\le \beta||x_{t_n} - p||\,||J_\varphi(x_{t_n} - p)|| + \langle f(p) - p, J_\varphi(x_{t_n} - p)\rangle \\ &= \beta\Phi(||x_{t_n} - p||) + \langle f(p) - p, J_\varphi(x_{t_n} - p)\rangle, \end{aligned} \tag{3.11}$$

that is,

$$\Phi(||x_{t_n} - p||) \le \frac{1}{1-\beta}\langle f(p) - p, J_\varphi(x_{t_n} - p)\rangle. \tag{3.12}$$

Now noting that $x_{t_n} \rightharpoonup p$ implies $J_\varphi(x_{t_n} - p) \rightharpoonup 0$, we get

$$\Phi(||x_{t_n} - p||) \longrightarrow 0. \tag{3.13}$$

Hence $x_{t_n} \to p$.

We have proved for any sequence $\{x_{t_n}\}$ in $\{x_t : t \in (0,1)\}$ that there exists a subsequence which is still denoted by $\{x_{t_n}\}$ that converges to some fixed point p of T. To prove that the entire net $\{x_t\}$ converges strongly to p, we assume there exists another sequence $\{s_n\} \in (0,1)$ such that $x_{s_n} \to q$, then $q \in F(T)$. We have to show $p = q$. Indeed, for $u \in F(T)$, it is easy to see that

$$\begin{aligned} \langle x_t - Tx_t, J_\varphi(x_t - u)\rangle &= \Phi(||x_t - u||) + \langle u - Tx_t, J_\varphi(x_t - u)\rangle \\ &\geq \Phi(||x_t - u||) - ||u - Tx_t|| \cdot ||J_\varphi(x_t - u)|| \\ &\geq \Phi(||x_t - u||) - \Phi(||x_t - u||) = 0. \end{aligned} \tag{3.14}$$

On the other hand, since

$$x_t - Tx_t = \frac{t}{1-t}(f(x_t) - x_t), \tag{3.15}$$

we get for $t \in (0,1)$ and $u \in F(T)$,

$$\langle x_t - f(x_t), J_\varphi(x_t - u)\rangle \leq 0. \tag{3.16}$$

Since the sets $\{x_t - u\}$ and $\{x_t\}$ are bounded and a Banach space X has a weakly continuous duality map J_φ, then J_φ is single valued and weak-to-weak* sequentially continuous, for any $u \in F(T)$, by $x_{s_n} \to q(s_n \to 0)$, we have

$$\begin{aligned} &||x_{s_n} - f(x_{s_n}) - (q - f(q))|| \longrightarrow 0 \quad (s_n \longrightarrow 0), \\ &|\langle x_{s_n} - f(x_{s_n}), J_\varphi(x_{s_n} - u)\rangle - \langle q - f(q), J_\varphi(q - u)\rangle| \\ &\quad = |\langle x_{s_n} - f(x_{s_n}) - (q - f(q)), J_\varphi(x_{s_n} - u)\rangle \\ &\qquad + \langle q - f(q), J_\varphi(x_{s_n} - u) - J_\varphi(q - u)\rangle| \\ &\quad \leq ||x_{s_n} - f(x_{s_n}) - (q - f(q))|| \, ||J_\varphi(x_{s_n} - u)|| \\ &\qquad + |\langle q - f(q), J_\varphi(x_{s_n} - u) - J_\varphi(q - u)\rangle| \quad \text{as } s_n \longrightarrow 0. \end{aligned} \tag{3.17}$$

Therefore, we get

$$\langle q - f(q), J_\varphi(q - u) = \lim_{s_n \to 0} \langle x_{s_n} - f(x_{s_n}), J_\varphi(x_{s_n} - u)\rangle \leq 0. \tag{3.18}$$

Interchange p and u to obtain

$$\langle q - f(q), J_\varphi(q - p)\rangle \leq 0. \tag{3.19}$$

Interchange q and u to obtain

$$\langle p - f(p), J_\varphi(p - q)\rangle \leq 0. \tag{3.20}$$

This implies that

$$\langle (p - q) - (f(p) - f(q)), J_\varphi(p - q)\rangle \leq 0. \tag{3.21}$$

That is,

$$\|p-q\|\varphi(\|p-q\|) \le \beta\|p-q\|\varphi(\|p-q\|). \tag{3.22}$$

This is a contradiction, so we have $p=q$.

The proof is complete. □

Remark 3.2. Theorem 3.1 is proved in a weaker condition than [7, Theorem 4.1], and the method of proof is different from [7], we introduce a continuous strict increasing function.

Next two main results are about accretive operators, we consider the problem of finding a zero of an m-accretive operator A in a reflexive Banach space X, $0 \in Ax$. Denote by $F(A)$ the zero set of A, that is,

$$F(A) := \{x \in D(A) : 0 \in Ax\} = A^{-1}(0). \tag{3.23}$$

Theorem 3.3. *Suppose that X is reflexive and has a weakly continuous duality map J_φ with gauge φ. Suppose that A is an m-accretive operator in X such that $C = \overline{D(A)}$ is convex with $F(A) \neq \phi$, and $f : C \to C$ is a fixed contractive map. Assume*

(i) $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\gamma_n \to \infty$.

Then the sequence $\{x_n\}$ defined by (1.12) converges strongly to a point in $F(A)$.

Proof. First we prove $\{x_t\}$ is bounded. Indeed, take $u \in F(A)$, then

$$\begin{aligned} ||x_{n+1}-u|| &\le \alpha_n||f(x_n)-u|| + (1-\alpha_n)||J_{r_n}x_n - u|| \\ &\le \alpha_n\beta||x_n-u|| + \alpha_n||f(u)-u|| + (1-\alpha_n)||x_n-u|| \\ &\le (1-(1-\beta)\alpha_n)||x_n-u|| + \alpha_n||f(u)-u||. \end{aligned} \tag{3.24}$$

By induction, we get

$$||x_n-u|| \le \max\left\{||x_0-u||, \frac{1}{1-\beta}||f(u)-u||\right\} \quad \forall n \ge 0. \tag{3.25}$$

This implies that $\{x_n\}$ is bounded, so are $\{f(x_n)\}$ and $\{J_{r_n}x_n\}$, and hence

$$||x_{n+1} - J_{r_n}x_n|| = \alpha_n||f(x_n) - J_{r_n}x_n|| \longrightarrow 0. \tag{3.26}$$

We next prove

$$\limsup_{n\to\infty} \langle f(p)-p, J_\varphi(x_n-p)\rangle \le 0, \quad p \in F(A). \tag{3.27}$$

By Theorem 3.1, put $p = \lim_{t\to 0} x_t$, we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(p)-p, J_\varphi(x_n-p)\rangle = \limsup_{n\to\infty} \langle f(p)-p, J_\varphi(x_{n_k}-p)\rangle. \tag{3.28}$$

Since X is reflexive, we may further assume that $x_{n_k} \rightharpoonup \widetilde{x}$. Moreover, since

$$\|x_{n+1} - J_{r_n}x_n\| \longrightarrow 0, \tag{3.29}$$

we obtain

$$J_{r_{n_k}-1}x_{r_{n_k}-1} \rightharpoonup \overline{x}. \tag{3.30}$$

Taking the limit as $k \to \infty$ in the relation

$$[J_{r_{n_k}-1}x_{r_{n_k}-1}, A_{r_{n_k}-1}x_{r_{n_k}-1}] \in A, \tag{3.31}$$

we get $[\overline{x}, 0] \in A$, that is, $\overline{x} \in F(A)$. Hence by (3.28) and (3.18), we have

$$\limsup_{n\to\infty} \langle f(p) - p, J_\varphi(x_n - p)\rangle = \langle f(p) - p, J_\varphi(\overline{x} - p)\rangle \le 0. \tag{3.32}$$

That is, (3.27) holds.

Finally, we prove that $x_n \to p$.

We apply Lemma 2.3 to get

$$\begin{aligned}
\Phi(\|x_{n+1} - p\|) &= \Phi(\|(1-\alpha_n)(J_{r_n}x_n - p) + \alpha_n(f(x_n) - p)\|) \\
&= \Phi(\|(1-\alpha_n)(J_{r_n}x_n - p) + \alpha_n(f(x_n) - f(p)) + \alpha_n(f(p) - p)\|) \\
&\le \Phi(\|(1-\alpha_n)(J_{r_n}x_n - p) + \alpha_n(f(x_n) - f(p))\|) \\
&\quad + \alpha_n\langle f(p) - p, J_\varphi(x_{n+1} - p)\rangle \\
&\le (1 - (1-\beta)\alpha_n)\Phi(\|x_n - p\|) + \alpha_n\langle f(p) - p, J_\varphi(x_{n+1} - p)\rangle.
\end{aligned} \tag{3.33}$$

Applying Lemma 2.1, we get

$$\Phi(\|x_n - p\|) \longrightarrow 0. \tag{3.34}$$

That is, $\|x_n - p\| \to 0$, that is, $x_n \to p$.

The proof is complete. □

Theorem 3.4. *Suppose that X is reflexive and has a weakly continuous duality map J_φ with gauge φ. Suppose that A is an m-accretive operator in X such that $C = \overline{D(A)}$ is convex with $F(A) \ne \phi$, and $f : C \to C$ is a fixed contractive map. Assume*

$$\begin{aligned}
&\text{(i) } \alpha_n \longrightarrow 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \\
&\text{(ii) } \gamma_n \ge \varepsilon \quad \forall n, \quad \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty.
\end{aligned} \tag{3.35}$$

Then $\{x_n\}$ defined by (1.12) converges strongly to a point in $F(A)$.

Proof. We only include the differences. We have

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)J_{\gamma_n}x_n, \quad x_n = \alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1})J_{\gamma_{n-1}}x_{n-1}. \tag{3.36}$$

Thus,

$$\begin{aligned} x_{n+1} - x_n &= \alpha_n f(x_n) + (1-\alpha_n)J_{\gamma_n}x_n - (\alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1})J_{\gamma_{n-1}}x_{n-1}) \\ &= \alpha_n (f(x_n) - f(x_{n-1})) + (\alpha_n - \alpha_{n-1})(f(x_{n-1}) - J_{\gamma_{n-1}}x_{n-1}) \\ &\quad + (1-\alpha_n)(J_{\gamma_n}x_n - J_{\gamma_{n-1}}x_{n-1}). \end{aligned} \tag{3.37}$$

If $\gamma_{r_{n-1}} \leq \gamma_n$, by Lemma 2.4, we get

$$J_{\gamma_n}x_n = J_{\gamma_{n-1}}\left(\frac{\gamma_{n-1}}{\gamma_n}x_n + \left(1 - \frac{\gamma_{n-1}}{\gamma_n}\right)J_{\gamma_n}x_n\right), \tag{3.38}$$

we have

$$\begin{aligned} ||J_{\gamma_n}x_n - J_{\gamma_{n-1}}x_{n-1}|| &\leq \frac{\gamma_{n-1}}{\gamma_n}||x_n - x_{n-1}|| + \left(1 - \frac{\gamma_{n-1}}{\gamma_n}\right)||J_{\gamma_n}x_n - x_{n-1}|| \\ &\leq ||x_n - x_{n-1}|| + \left(\frac{\gamma_n - \gamma_{n-1}}{\gamma_n}\right)||J_{\gamma_n}x_n - x_{n-1}|| \\ &\leq ||x_n - x_{n-1}|| + \frac{1}{\varepsilon}\,|\,\gamma_{n-1} - \gamma_n\,|\,||J_{\gamma_n}x_n - x_{n-1}||. \end{aligned} \tag{3.39}$$

It follows from the above results that

$$\begin{aligned} ||x_{n+1} - x_n|| &\leq |\,\alpha_n - \alpha_{n-1}\,|\,||f(x_{n-1}) - J_{\gamma_{n-1}}x_{n-1}|| + (1-\alpha_n)||x_n - x_{n-1}|| \\ &\quad + \frac{1}{\varepsilon}(1-\alpha_n)\,|\,\gamma_{n-1} - \gamma_n\,|\,||J_{\gamma_n}x_n - x_{n-1}|| + \alpha_n\beta||x_n - x_{n-1}|| \\ &\leq M(\,|\,\alpha_n - \alpha_{n-1}\,| + |\,\gamma_{n-1} - \gamma_n\,|\,) + (1 - (1-\beta)\alpha_n)||x_n - x_{n-1}||, \end{aligned} \tag{3.40}$$

where $M > 0$ is some appropriate constant. Similarly, we can prove the last inequality if $\gamma_{n-1} \geq \gamma_n$. By assumptions (i) and (ii) and Lemma 2.1, we have

$$||x_{n+1} - x_n|| \longrightarrow 0. \tag{3.41}$$

This implies that

$$||x_n - J_{\gamma_n}x_n|| \leq ||x_{n+1} - x_n|| + ||x_{n+1} - J_{\gamma_n}x_n||. \tag{3.42}$$

Since $\|x_{n+1} - J_{\gamma_n}x_n\| = \alpha_n\|f(x_n) - J_{\gamma_n}x_n\| \to 0$. It follows from (3.42) that

$$||A_{\gamma_n}x_n|| = \frac{1}{\gamma_n}||x_n - J_{\gamma_n}x_n|| \leq \frac{1}{\varepsilon}||x_n - J_{\gamma_n}x_n|| \longrightarrow 0. \tag{3.43}$$

Now if $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ converging weakly to a point $\overline{x}$, then taking the limit as $k \to \infty$ in the relation

$$[J_{\gamma_{n_k}}x_{n_k}, A_{\gamma_{n_k}}x_{n_k}] \in A, \tag{3.44}$$

we get $[\overline{x},0] \in A$, that is, $\overline{x} \in F(A)$. We therefore conclude that all weak limit points of $\{x_n\}$ are zeros of A.

The rest of the proof follows from Theorem 3.3.

The proof is complete. □

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