DIAMETRICALLY CONTRACTIVE MAPS AND FIXED POINTS

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Contractive maps have nice properties concerning fixed points; a big amount of literature has been devoted to fixed points of nonexpansive maps. The class of shrinking (or strictly contractive) maps is slightly less popular: few specific results on them (not applicable to all nonexpansive maps) appear in the literature and some interesting problems remain open. As an attempt to fill this gap, a condition half way between shrinking and contractive maps has been studied recently. Here we continue the study of the latter notion, solving some open problems concerning these maps.

1. Introduction

Let X be a Banach space and M a nonempty convex closed bounded subset of X. In the theory of fixed points, two classes of maps $T : M \to M$ are well known and deeply studied: the class of contractive maps

$$\forall x, y \text{ in } M, \quad \|Tx - Ty\| \le \alpha\|x - y\|, \quad \alpha \in (0,1), \tag{1.1}$$

and the class of nonexpansive maps

$$\forall x, y \text{ in } M, \quad \|Tx - Ty\| \le \|x - y\|. \tag{1.2}$$

An intermediate class consists of the maps that satisfy the following condition:

$$\|Tx - Ty\| < \|x - y\| \quad \forall x \ne y, \text{ with } x, y \in M. \tag{S}$$

In the literature, these maps appear under different names, see for example [5] and the

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references therin; we will call them *shrinking*. We briefly recall some results and properties of maps in this class:

(1) the fixed point, if it exists, is unique;
(2) if M is a compact set (or more generally if $\overline{TM}$ is compact), then T has a fixed point x^*, and moreover for each $x \in M$, $T^n x \to x^*$;
(3) there is an example (see [5]) of a map on the unit ball of Hilbert spaces with fixed point x^* such that $T^n x$ does not converge to the fixed point for any $x \neq x^*$;
(4) there are examples of maps without fixed points [4, 6, 9].

Not so much attention has been paid to shrinking maps; indeed the following questions are open. Let M be a weakly compact convex of a Banach space and let $T : M \to M$ be a shrinking mapping. Must T have a fixed point? If T has a fixed point x^*, is it true that $T^n x \to x^*$ for every x?

Conditions stronger than (S) were considered, also in more general settings, see for example [3]. Another rather weak strengthening, which appeared probably for the first time in [2], is the one given by the following definition. T is *diametrically contractive* (DC) if $\delta(T(A)) < \delta(A)$ for every closed, convex, bounded nonsingleton subset A of M, where $\delta(A)$ is the diameter of A.

Such a notion was studied in details in [10]. We collect some relations between the previous classes of mappings:

(1) diametrically contractive maps are shrinking;
(2) if M is a compact set and T is shrinking, then it is diametrically contractive;
(3) there are examples of shrinking maps that are not diametrically contractive [4, 10].

A most important result is the following, see [10, Theorem 2.3].

Theorem 1.1. *Let M be a weakly compact subset of a Banach space X and let $T : M \to M$ be diametrically contractive, then T has a fixed point.*

The proof of this theorem appeared probably for the first time in [7, Theorem 2] and in the case of reflexive spaces can be found in [1, 8].

The following problems appear to be open (see [10]).

Problem 1.2. Can we substitute weakly compact subset with closed convex bounded one in Theorem 1.1?

Problem 1.3. If T is diametrically contractive and x^* is the fixed point of T, do we have $T^n x \to x^*$ for all (or at least for some) $x \in M$?

In this paper, we solve in the negative both problems: the first example (Section 2) solves Problem 1.2; the second example (Section 3) solves Problem 1.3.

2. First example

Now we give an example of a fixed point free DC self-map of a closed convex bounded set.

Consider the vector space of all continuous real functions on the closed unit interval, with the norm (equivalent to the classical one)

$$\|f\| = \|f\|_\infty + \|f\|_1 = \max_{0\le x\le 1} |f(x)| + \int_0^1 |f(x)|\,dx. \tag{2.1}$$

Let $M = \{f \in X : f(0) = 0;\ f(1) = 1;\ 0 \le f(x) \le x;\ f$ is monotone nondecreasing$\}$.

Define $T : M \to M$ in the following way:

$$Tf(x) = \begin{cases} 0 & 0 \le x \le \dfrac{1}{2}, \\ (2x-1)f(2x-1) & \dfrac{1}{2} \le x \le 1. \end{cases} \tag{2.2}$$

Claim 2.1. The map T is fixed point free.

Proof. Suppose that $f \in M$ is such that $Tf = f$. Clearly $f(x) = 0$ for every $x \in [0;1/2]$. If $x \in [1/2;1]$, then $(2x-1)f(2x-1) = f(x)$ implies that $f(x) = 0$ for every $x \in [0;3/4]$. By iterating the reasoning, we can easily prove that $f(x) = 0$ for all $x \in [0;1-1/2^n]$ and all $n \in \mathbb{N}$. Since f is continuous and $f(1) = 1$, this is a contradiction proving the claim. □

Claim 2.2. The map T is shrinking.

Proof. Let be $f,g \in M$ with $f \ne g$. Then

$$\begin{aligned} \|Tf - Tg\| &= \max_{0\le x\le 1} |Tf(x) - Tg(x)| + \int_0^1 |Tf(x) - Tg(x)|\,dx \\ &= \max_{1/2\le x\le 1} (2x-1)\,|\,(f(2x-1) - g(2x-1))\,| \\ &\quad + \int_{1/2}^1 (2x-1)\,|f(2x-1) - g(2x-1)|\,dx \\ &= \max_{0\le x\le 1} |x(f(x) - g(x))| + \frac{1}{2}\int_0^1 x\,|f(x) - g(x)|\,dx \\ &< \|f-g\|_\infty + \frac{1}{2}\|f-g\|_1 \le \|f-g\|. \end{aligned} \tag{2.3}$$

□

Claim 2.3. The map T is diametrically contractive.

Proof. Let A be a closed subset of M such that $\delta(A) > 0$. We have, for two suitable subsequences f_n, g_n,

$$\begin{aligned} \delta(T(A)) &= \lim_{n\to\infty} \|Tf_n - Tg_n\| = \lim_{n\to\infty}\left(\|Tf_n - Tg_n\|_\infty + \|Tf_n - Tg_n\|_1\right) \\ &\le \lim_{n\to\infty}\left(\|f_n - g_n\|_\infty + \frac{1}{2}\|f_n - g_n\|_1\right) \le \lim_{n\to\infty}\|f_n - g_n\| \le \delta(A). \end{aligned} \tag{2.4}$$

So, if we assume that $\delta(T(A)) = \delta(A)$, then (by passing again if necessary to a subsequence) we have

$$\begin{aligned} \lim_{n\to\infty}\|f_n - g_n\|_1 &= \lim_{n\to\infty}\|Tf_n - Tg_n\|_1 = 0, \\ \lim_{n\to\infty}\|f_n - g_n\|_\infty &= \lim_{n\to\infty}\|Tf_n - Tg_n\|_\infty = \delta(T(A)) = \delta(A). \end{aligned} \tag{2.5}$$

But we can choose a sequence (x_n) such that $\|Tf_n - Tg_n\|_\infty = x_n|f_n(x_n) - g_n(x_n)|$. By considering eventually a subsequence, we may assume that $x_n \to x_o \in [0;1]$. Then

$$\delta(A) = \lim_{n\to\infty} x_n \,|\, f_n(x_n) - g_n(x_n)\,| \le \lim_{n\to\infty} x_n \|f_n - g_n\|_\infty = x_o \delta(A), \tag{2.6}$$

thus $x_o = 1$.

By considering subsequences, and by exchanging eventually the sequences, we may assume that

$$f_n(x_n) \longrightarrow l, \qquad g_n(x_n) \longrightarrow L \tag{2.7}$$

with $L \le l \le 1$.

Therefore (2.6) implies that

$$l - L = \delta(A), \tag{2.8}$$

so

$$f_n(x_n) \longrightarrow l, \qquad g_n(x_n) \longrightarrow l - \delta(A). \tag{2.9}$$

Now take any $f \in A$; since $\lim_{n\to\infty} x_n = 1$, we have

$$\delta(A) \ge |\, f(x_n) - g_n(x_n)\,| \xrightarrow[n\to\infty]{} |\, 1 - l + \delta(A)\,| \ge \delta(A). \tag{2.10}$$

Thus we have $l = 1$; $\lim_{n\to\infty} |f(x_n) - g_n(x_n)| = \delta(A)$ for every $f \in A$, and then

$$\lim_{n\to\infty} \|f - g_n\|_\infty = \delta(A). \tag{2.11}$$

Now take $\epsilon \in (0, \delta(A))$, then there exists $\eta > 0$ such that for every $x \in [1-\eta, 1]$, we have $1 - \epsilon \le f(x) \le 1$. For n large, $x_n > 1 - \eta$; therefore, by using also the monotonicity assumption for the functions, we have (for suitable points c_n)

$$\begin{aligned} \int_0^1 |\, f(x) - g_n(x)\,|\, dx &\ge \int_{1-\eta}^{x_n} |\, f(x) - g_n(x)\,|\, dx = (x_n - 1 + \eta)\,|\, f(c_n) - g_n(c_n)\,| \\ &\ge (x_n - 1 + \eta)(1 - \epsilon - g_n(x_n)); \end{aligned} \tag{2.12}$$

also, since $\lim_{n\to\infty} g_n(x_n) = 1 - \delta(A)$,

$$\lim_{n\to\infty} (x_n - 1 + \eta)(1 - \epsilon - g_n(x_n)) = \eta(\delta(A) - \epsilon). \tag{2.13}$$

Thus we obtain

$$\liminf_{n\to\infty} \|f - g_n\|_1 \ge \eta(\delta(A) - \epsilon) \tag{2.14}$$

and this implies that

$$\liminf_{n\to\infty} \|f - g_n\| \ge \lim_{n\to\infty} \|f - g_n\|_\infty + \liminf_{n\to\infty} \|f - g_n\|_1 \ge \delta(A) + \eta(\delta(A) - \epsilon). \tag{2.15}$$

This is a contradiction, proving the claim and thus the result. □

3. Second example

The next example shows that for a DC self-map of a bounded closed convex set M, the existence of a fixed point does not imply the convergence of iterates $T^n x$ to the fixed point.

Consider the vector space c_o, endowed with the following norm (equivalent to the usual one):

$$\|x\| = \|x\|_\infty + \sum_{n=1}^{\infty} \frac{|x_n|}{2^n}. \tag{3.1}$$

We denote by B^+ the intersection of the positive cone with the unit closed ball. Define $T : B^+ \to B^+$ in this way:

$$(Tx)_1 = 0, \qquad \text{for } n \geq 2, \quad (Tx)_n = a_{n-1}x_{n-1}, \tag{3.2}$$

where (a_n), $n \geq 1$, is a strictly positive and strictly increasing sequence such that $\prod_{n=1}^{\infty} a_n = \alpha > 0$. Clearly T is linear and its unique fixed point is the null vector.

The map T is shrinking: in fact, for $x \neq y$,

$$\begin{aligned} \|Tx - Ty\| &= ||(0, a_1(x_1 - y_1), a_2(x_2 - y_2), \dots)|| \\ &< ||(0, (x_1 - y_1), (x_2 - y_2), \dots)|| < \|x - y\|. \end{aligned} \tag{3.3}$$

Consider now the orbit of non-null elements in B^+. Take x and let for example $x_k \neq 0$. We have

$$||T^n x|| \geq |(T^n x)_{k+n}| = a_k a_{k+1} \cdots a_{k+n-1} x_k \xrightarrow[n\to\infty]{} \left(\prod_{n=k}^{\infty} a_n\right) \quad x_k \neq 0. \tag{3.4}$$

Now we will prove that our map T is diametrically contractive.

Consider a bounded closed convex set A contained in B^+. Let us suppose that

$$\delta(A) = \delta(T(A)) > 0. \tag{3.5}$$

Consider two sequences $x^{(n)}$ and $y^{(n)}$ such that

$$\lim_{n\to\infty} ||Tx^{(n)} - Ty^{(n)}|| = \delta(T(A)). \tag{3.6}$$

Since T is shrinking, this implies that $\lim_{n\to\infty}\|x^{(n)}-y^{(n)}\|=\delta(A)$. We have

$$\begin{aligned}\delta(T(A)) &= \lim_{n\to\infty}\left(\|T(x^{(n)}-y^{(n)})\|_\infty+\sum_{k=1}^{\infty}\frac{|T((x^{(n)})-(y^{(n)}))_k|}{2^k}\right)\\ &= \lim_{n\to\infty}\left(\max_{k\geq 2}\left|a_{k-1}\left(x_{k-1}^{(n)}-y_{k-1}^{(n)}\right)\right|+\sum_{k=2}^{\infty}\frac{a_{k-1}\left|x_{k-1}^{(n)}-y_{k-1}^{(n)}\right|}{2^k}\right)\\ &= \lim_{n\to\infty}\left(\max_{k\geq 1}a_k\left|x_k^{(n)}-y_k^{(n)}\right|+\sum_{k=1}^{\infty}\frac{a_k\left|x_k^{(n)}-y_k^{(n)}\right|}{2^{k+1}}\right)\\ &\leq \limsup_{n\to\infty}\left(\|x^{(n)}-y^{(n)}\|_\infty+\frac{1}{2}\sum_{k=1}^{\infty}\frac{\left|x_k^{(n)}-y_k^{(n)}\right|}{2^k}\right)\leq\delta(A).\end{aligned}\tag{3.7}$$

From this, we obtain

$$\begin{gathered}\lim_{n\to\infty}\|x^{(n)}-y^{(n)}\|_\infty=\delta(A),\\ \lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{\left|x_k^{(n)}-y_k^{(n)}\right|}{2^k}=0.\end{gathered}\tag{3.8}$$

For every n, there exists $k(n)$ such that $\|x^{(n)}-y^{(n)}\|_\infty=|x_{k(n)}^{(n)}-y_{k(n)}^{(n)}|$, so

$$\lim_{n\to\infty}\left|x_{k(n)}^{(n)}-y_{k(n)}^{(n)}\right|=\delta(A).\tag{3.9}$$

Set $K=\{k(n);\ n\in\mathbb{N}\}$. If K is finite, then $k(n)=k_o$ for infinitely many n, so

$$\sum_{k=1}^{\infty}\frac{\left|x_k^{(n)}-y_k^{(n)}\right|}{2^k}\geq\frac{\left|x_{k_o}^{(n)}-y_{k_o}^{(n)}\right|}{2^{k_o}}\xrightarrow[n\to\infty]{}\frac{\delta(A)}{2^{k_o}}\neq 0,\tag{3.10}$$

which is an absurdity since we have proved that the left-hand side tends to 0. Thus K is infinite. Take a subsequence of $k(n)$ tending to infinity, that we still call $k(n)$, such that $x_{k(n)}^{(n)}\to\delta(A)+l$ and $y_{k(n)}^{(n)}\to l(\geq 0)$.

Now let $x\in A$; we have

$$\delta(A)+l=\lim_{n\to\infty}\left|x_{k(n)}-x_{k(n)}^{(n)}\right|\leq\lim_{n\to\infty}\|x-x^{(n)}\|_\infty\leq\lim_{n\to\infty}\|x-x^{(n)}\|\leq\delta(A).\tag{3.11}$$

This implies that $l=0$.

Therefore, for every $x\in A$, $\lim_{n\to\infty}\|x-x^{(n)}\|=\delta(A)$. So

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{\left|x_k-x_k^{(n)}\right|}{2^k}=0\tag{3.12}$$

which implies, for every k, that

$$\lim_{n\to\infty} x_k^{(n)} = x_k, \tag{3.13}$$

(remember that this should be true for every $x \in A$) so A cannot contain two or more elements. This would imply $\delta(A) = 0$, against the assumption. This contradiction proves the assertion.

4. Final remarks

After discussing Problems 1.2 and 1.3, another rather awkward condition, stronger than DC, was introduced in [10].

Given a set M, say that $T : M \to M$ is *asymptotically diametrically contractive* ADC if for all nested sequences (A_n) of closed bounded subsets of M with $lim_{n\to\infty}\delta(A_n) = \delta > 0$, we have $lim_{n\to\infty}\delta(T(A_n)) < \delta$.

We try to clarify its position among other simpler conditions.

Clearly, ADC maps are DC; as proved in [10, Theorem 2.6], the following result holds. If $T : M \to M$ is an ADC map and T has a bounded orbit for some $x_o \in M$, then T has a unique fixed point ξ, and for every $x \in M : T^n(x) \to \xi$. In particular, this fact is true whenever M is bounded.

If M is compact, then (S) implies DC and DC implies ADC. But there are (S) maps on compact sets which are not contractive; thus ADC does not imply contractiveness, also when the map is defined on a compact set. An example of a map, on an unbounded set, which is ADC but not contractive, was given in [10, Remark 2.7].

An example of a map satisfying (S), but which is not DC, was given in [10]; according to the previous result, our first and second examples (Sections 2 and 3) show that DC maps are not in general ADC.

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