FIXED POINTS AND CONTROLLABILITY IN DELAY SYSTEMS

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Schaefer's fixed point theorem is used to study the controllability in an infinite delay system $x'(t) = G(t,x_t) + (Bu)(t)$. A compact map or homotopy is constructed enabling us to show that if there is an *a priori* bound on all possible solutions of the companion control system $x'(t) = \lambda[G(t,x_t) + (Bu)(t)]$, $0 < \lambda < 1$, then there exists a solution for $\lambda = 1$. The *a priori* bound is established by means of a Liapunov functional or applying an integral inequality. Applications to integral control systems are given to illustrate the approach.

1. Introduction

This paper is concerned with the problem of controllability in an infinity delay system

$$x'(t) = G(t,x_t) + (Bu)(t), \quad t \in J = [0,b], \tag{1.1}$$

where $x(t) \in R^n$, $x_t(\theta) = x(t+\theta)$ for $-\infty < \theta \le 0$, $u(t)$, the control, is a real m-vector valued function on J, B is a bounded linear operator acting on u, and G is defined on $J \times C$ with C being the Banach space of bounded continuous functions $\phi : (-\infty,0] \to R^n$ with the supremum norm $|\cdot|_C$.

The problem of controllability in delay systems has been the subject of extensive investigations by many scientists and researchers for over half of a century. A large number of applications have appeared in biology, medicine, economics, engineering, and information technology. Many actual systems have the property of "after-effect," that is, the future state depends not only on the present, but also on the past history. It is well-known that such a delay factor, when properly controlled, can essentially improve system's qualitative and quantitative characteristics in many aspects. For historical background and discussion of applications, we refer the reader to the work of Balachandran and Sakthivel [1], Chukwu [4], Górniewicz and Nistri [8], and references therein.

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Equation (1.1) describes the state of a system (physical, chemical, economic, etc.) whose evolution in time t is governed by $G(t,x_t)+(Bu)(t)$. In general, we view a solution of (1.1) as a function of $u(t)$ so that the behavior of the system depends on (or is controlled by) the choice of u within a set U given in advance. Assume that two states of the system are given, one to be considered as an initial state ϕ, and the other as a final state γ. The problem of controllability is to determine whether there are available controls which can transfer the state x from ϕ to γ along a solution (1.1), that is, whether there exists a $u_0 \in U$ such that $\dot{x}(t) = G(t,x_t)+Bu_0(t)$ has a solution joining ϕ and γ. When this is possible for arbitrary choice of ϕ and γ, we can say, roughly speaking, that system (1.1) is controllable by means of U.

The main method of proving controllability has been to write (1.1) as an integral equation

$$x(t) = x_0 + \int_0^t [G(s,x_s)+Bu(s)]ds \tag{1.2}$$

viewing the right-hand side as a mapping Px on an appropriate space, when u is properly chosen in terms of x. Then, apply a fixed point theorem, say Schauder's, to the mapping P when $P : K \to K$ is compact for a closed, convex subset K of a Banach space. However, P, in general, does not satisfy this condition unless the growth of $G(t,x_t)$ in x is restricted. This presents a significant challenge to investigators. A modern approach to such a problem is to use topological degree or transversality method (see Górniewicz and Nistri [8]). Here we will use a fixed point theorem of Schaefer [12] which is a variant of the nonlinear alternative of Leray-Schauder degree theory, but much easier to use.

The paper is organized as follows. In Section 2, we prove our main result on controllability of (1.1). Applications to specific systems are given in Section 3 which also contains some general results and remarks concerning the approach.

2. Controllability

We start this section with some descriptions of spaces associated with our discussion. Let $R = (-\infty,\infty)$, $R^- = (-\infty,0]$, and $R^+ = [0,\infty)$, respectively, and $|\cdot|$ denote the Euclidean norm in R^n. For an $n\times n$ matrix $A = (a_{ij})_{n\times n}$, define $\|A\| = \sup_{|x|\le 1}|Ax|$. We denote by $C(X,Y)$ the set of bounded continuous functions $\phi : X \to Y$ for normed spaces X and Y. For J given in (1.1), we define $C_0 = \{\phi \in C(J,R^n) : \phi(0) = 0\}$ with the supremum norm $|\cdot|_{C_0}$, and set $C_0(\mu) = \{\phi \in C_0 : |\phi|_{C_0} \le \mu\}$. Let $L^2(J)$ denote the Banach space of square Lebesgue integrable functions $x : J \to R^n$ with the norm $|x|_2 = (\int_J |x(s)|^2 ds)^{1/2}$ with the understanding $x = 0 \in L^2(J)$ if and only if $x(t) = 0$ a.e. on J. We denote by U the space of admissible controls and choose U as a complete subspace of $L^2(J)$ with a norm $|\cdot|_U$. The symbol $\|\cdot\|$ is reserved for the norm of a linear operator.

We assume that $B : U \to L^2(J)$ is a bounded linear operator. For a given $u \in U$, we say that $x : (-\infty,b] \to R^n$ is a solution $x = x(t,\phi)$ of (1.1) on J with initial function ϕ if x is absolutely continuous on J and satisfies

$$x'(t) = G(t,x_t)+(Bu)(t), \quad \text{a.e. } t \in J = [0,b] \tag{2.1}$$

with $x_0 = \phi$, that is, $x(s) = \phi(s)$ for all $s \in R^-$.

Our result rests on a fixed point theorem of Schaefer [12]. Its relation to Leray-Schauder degree theorem is explained in Smart [13, page 29]. Schaefer's theorem has been used in a variety of areas in differential equations and control theory (see Burton [2], Burton and Zhang [3], Balachandran and Sakthivel [1]).

Theorem 2.1 (Schaefer). *Let V be a normed space, F a continuous mapping of V into V which is compact on each bounded subset of V. Then either*

(i) *the equation $x = \lambda F(x)$ has a solution for $\lambda = 1$, or*

(ii) *the set of all such solutions x, for $0 < \lambda < 1$, is unbounded.*

If we view $U_\lambda(x) = \lambda F(x)$ in Schaefer's theorem as a homotopy, then it can be restated in the form of Leray-Schauder Principle (cf. Zeidler [14, page 245]), which is often used in application.

Definition 2.2. System (1.1) is said to be controllable on the interval J if for each $\phi \in C$ and $y \in R^n$, there exists a control $u \in U$ such that the solution $x(t) = x(t,\phi)$ of (1.1) satisfies $x(b) = y$.

Throughout this paper, we let $\phi \in C$ and $y \in R^n$ be arbitrary, but fixed. For each $y \in C_0$, we define

$$\bar{y}(t) = \begin{cases} y(t) + \phi(0) & t \in J, \\ \phi(t) & t \in R^-. \end{cases} \tag{2.2}$$

This implies that $\bar{y}_0 = \phi$; that is, $\bar{y}(s) = \phi(s)$ for all $s \in R^-$. For each continuous function $z : (-\infty, b] \to R$ with $z_0 = \phi$, we define

$$[z] = y - \phi(0) - \int_J G(s, z_s)\, ds. \tag{2.3}$$

We now introduce a companion to (1.1)

$$x'(t) = \lambda\big[G(t, x_t) + (Bu)(t)\big], \quad \text{a.e. } t \in J = [0, b] \tag{1.1_λ}$$

for $\lambda \in [0,1]$ and make the following assumptions.

(H_1) The linear operator $T : U \to R^n$ defined by

$$T(u) = \int_J Bu(s)\, ds \tag{2.4}$$

is invertible; that is, for each $\alpha \in R^n$, there exists a unique $u_\alpha \in U$ such that $\alpha = \int_J (Bu_\alpha)(s)\, ds$.

(H_2) For each $y \in C_0$, $G(s, \bar{y}_s)$ is Lebesgue measurable in s on J, and for each $\mu > 0$, there exists an integrable function $M_\mu : J \to R^+$ such that $|G(s, \bar{y}_s)| \le M_\mu(s)$ for all $s \in J$ whenever $y \in C_0(\mu)$.

(H_3) For any $\varepsilon > 0$ and $y \in C_0$, there exists a $\delta > 0$ such that $[x \in C_0,\ |x - y|_{C_0} < \delta]$, imply

$$\int_J \big| G(s, \bar{x}_s) - G(s, \bar{y}_s) \big|\, ds < \varepsilon. \tag{2.5}$$

(H_4) For each $\phi \in C$ and $y \in R^n$, there exists a constant $L = L(\phi, y) > 0$ such that $|x_\lambda(t)| \le L$ for all $t \in J$ whenever $x_\lambda(t) = x(t,\phi)$ is a solution of (1.1_λ) with $u(t) = u_\alpha(t)$ for $\alpha = [x_\lambda]$ and $\lambda \in (0,1)$.

Theorem 2.3. *If (H_1)–(H_4) are satisfied, then (1.1) is controllable on J.*

Proof. Let $\phi \in C$ and $y \in R^n$ be fixed. We define a function $F : C_0 \to C_0$ by

$$(Fy)(t) = \int_0^t \left[G(s,\bar{y}_s) + Bu_{[\bar{y}]}(s)\right]ds \tag{2.6}$$

for each $y \in C_0$ and $t \in J$. It is clear that F is well-defined. For the linear operator T defined in (H_1), we have

$$|T(u)| = \left|\int_J Bu(s)ds\right| \le \int_J |Bu(s)|ds \le b^{1/2}|Bu|_2 \le b^{1/2}\|B\||u|_U. \tag{2.7}$$

Thus, $T : U \to R$ is bounded, and hence T^{-1} is also bounded, say $\|T^{-1}\| \le M_1$ (see Friedman [6, page 143]).

We now show that F is continuous on C_0. For each $\varepsilon > 0$ and $y \in C_0$, by (H_3) there exists a $\delta > 0$ such that $[x \in C_0,\ |x-y|_{C_0} < \delta]$ imply

$$\int_J |G(s,\bar{x}_s) - G(s,\bar{y}_s)|ds < \frac{\varepsilon}{(1 + b^{1/2}\|B\|\|T^{-1}\|)}. \tag{2.8}$$

Observe

$$\begin{aligned}
&\int_J |Bu_{[\bar{x}]}(s) - Bu_{[\bar{y}]}(s)|ds \\
&\quad \le b^{1/2}\left|Bu_{[\bar{x}]} - Bu_{[\bar{y}]}\right|_2 \le b^{1/2}\|B\|\left|u_{[\bar{x}]} - u_{[\bar{y}]}\right|_U = b^{1/2}\|B\|\left|T^{-1}[\bar{x}] - T^{-1}[\bar{y}]\right|_U \\
&\quad \le b^{1/2}\|B\|\|T^{-1}\||[\bar{x}] - [\bar{y}]| \le b^{1/2}\|B\|\|T^{-1}\|\left|\int_J [G(s,\bar{x}_s) - G(s,\bar{y}_s)]ds\right|,
\end{aligned} \tag{2.9}$$

where $[\bar{x}] = y - \phi(0) - \int_J G(s,\bar{x}_s)ds$ and $[\bar{y}] = y - \phi(0) - \int_J G(s,\bar{y}_s)ds$. Thus, we obtain

$$\begin{aligned}
|F(x)(t) - F(y)(t)| &\le \int_J |G(s,\bar{x}_s) - G(s,\bar{y}_s)|ds + \int_J |Bu_{[\bar{x}]}(s) - Bu_{[\bar{y}]}(s)|ds \\
&\le (1 + b^{1/2}\|B\|\|T^{-1}\|)\int_J |G(s,\bar{x}_s) - G(s,\bar{y}_s)|ds < \varepsilon.
\end{aligned} \tag{2.10}$$

This implies that $|F(x) - F(y)|_{C_0} < \varepsilon$ whenever $|x - y|_{C_0} < \delta$, and hence F is continuous on C_0.

Next, we show that for each $\mu > 0$, the set $\{F(y) : y \in C_0(\mu)\}$ is uniformly bounded and equicontinuous. By the definition of F, we have

$$|F(y)(t)| = \left|\int_0^t G(s,\bar{y}_s)ds + \int_0^t (Bu_{[\bar{y}]})(s)ds\right| \le \int_J M_\mu(s)ds + b^{1/2}\|B\|\left|u_{[\bar{y}]}\right|_U. \tag{2.11}$$

Notice that $[\bar{y}] = \int_J Bu_{[\bar{y}]}(s)ds = T(u_{[\bar{y}]})$ so that $u_{[\bar{y}]} = T^{-1}[\bar{y}]$. Therefore,

$$\begin{aligned} |u_{[\bar{y}]}|_U &\le \|T^{-1}\| |[\bar{y}]| \le M_1\left[|\gamma| + |\phi(0)| + \int_J |G(s,\bar{y}_s)| ds\right] \\ &\le M_1\left[|\gamma| + |\phi(0)| + \int_J M_\mu(s)ds\right] =: M_2. \end{aligned} \tag{2.12}$$

Combine (2.11) and (2.12) to obtain $|F(y)|_{C_0} \le M_3$ for some $M_3 > 0$ and for all $y \in C_0(\mu)$. Thus, $\{F(y) : y \in C_0(\mu)\}$ is uniformly bounded. Now let $t_1, t_2 \in J$ with $t_1 < t_2$. Then for $y \in C_0(\mu)$, we have

$$\begin{aligned} |F(y)(t_2) - F(y)(t_1)| &\le \int_{t_1}^{t_2} M_\mu(s)ds + \int_{t_1}^{t_2} |Bu_{[\bar{y}]}(s)| ds \\ &\le \int_{t_1}^{t_2} M_\mu(s)ds + (t_2 - t_1)^{1/2}\left(\int_{t_1}^{t_2} |Bu_{[\bar{y}]}(s)|^2 ds\right)^{1/2} \\ &\le \int_{t_1}^{t_2} M_\mu(s)ds + (t_2 - t_1)^{1/2}\|B\| |u_{[\bar{y}]}|_U \\ &\le \int_{t_1}^{t_2} M_\mu(s)ds + (t_2 - t_1)^{1/2}\|B\| M_2 \end{aligned} \tag{2.13}$$

by (2.12). Thus, $\{F(y) : y \in C_0(\mu)\}$ is uniformly bounded and equicontinuous on J, and hence F is compact by Ascoli-Arzela's theorem.

Finally, suppose that $x = \lambda F(x)$ for some $x \in C_0$ and $0 < \lambda < 1$; that is,

$$x(t) = \lambda\left[\int_0^t G(s,\bar{x}_s) ds + \int_0^t Bu_{[\bar{x}]}(s)ds\right] \tag{2.14}$$

for $t \in J$. Differentiate (2.14) with respect to t to obtain

$$\bar{x}'(t) = x'(t) = \lambda[G(t,\bar{x}_t) + Bu(t)], \quad \text{a.e. } t \in J \tag{2.15}$$

with $u = u_{[\bar{x}]}$. From (2.15), we see that $\bar{x}$ is a solution of (1.1_λ) with $\bar{x}_0 = \phi$. Moreover, $|\bar{x}|_{C_0} \le L$ by (H_4). This implies that $|x|_{C_0} \le L + |\phi(0)|$. Therefore, alternative (i) of Theorem 2.1 must hold, and there exists $y \in C_0$ such that $y = F(y)$. Following the argument in (2.14) and (2.15), we see that $\bar{y}$ is a solution of (1.1) with $\bar{y}_0 = \phi$ and

$$\begin{aligned} \bar{y}(b) - \bar{y}(0) &= \int_0^b G(s,\bar{y}_s) ds + \int_0^b Bu_{[\bar{y}]}(s)ds = \int_0^b G(s,\bar{y}_s) ds + [\bar{y}] \\ &= \int_0^b G(s,\bar{y}_s) ds + \gamma - \phi(0) - \int_0^b G(s,\bar{y}_s) ds. \end{aligned} \tag{2.16}$$

Since $\bar{y}(0) = \phi(0)$, we obtain $\bar{y}(b) = \gamma$, and so (1.1) is controllable on the interval J with $u \in U$. This completes the proof. □

3. Examples and remarks

In this section, we give several examples to illustrate how to apply Theorem 2.3 to some specific equations and systems. Our emphasis will be on obtaining a priori bounds. The examples are shown in simple forms for illustrative purposes, and they can easily be generalized.

Example 3.1. Consider the control problem

$$x'(t) = Ax(t) + \int_{-\infty}^{t} E(t,s,x(s))\,ds + u(t), \quad \text{a.e. } t \in J, \tag{3.1}$$

where $A = (a_{ij})_{n\times n}$ is an $n \times n$ matrix, $E : \Omega \times R^n \to R^n$ is measurable with $\Omega = \{(t,s) \in R^2 : t \geq s\}$, and $u(t)$ is an arbitrary control (to be determined later).

It is well known that

$$e^{-At} = \omega_1(t)I + \omega_2(t)A + \cdots + \omega_n(t)A^{n-1}, \tag{3.2}$$

where $\omega_i : R \to R$ $(i = 1,2,\dots,n)$ are continuous and linearly independent (see Godunov [7, page 32]). Denote the space of linear span of functions $w_1, w_2, \dots, w_n$ by

$$\operatorname{span}\{\omega_1,\omega_2,\dots,\omega_n\}. \tag{3.3}$$

Let $c \in R^n$ be fixed. We define U as

$$U = \{u = u^* c \mid u^* \in \operatorname{span}\{\omega_1,\omega_2,\dots,\omega_n\}\}, \tag{3.4}$$

and view $(U, |\cdot|_U)$, $|\cdot|_U = |\cdot|_2$, as a complete subspace of $(L^2(J), |\cdot|_2)$.

Introduce a transformation $y(t) = e^{-At}x(t)$ to write (3.1) as

$$y'(t) = G(t, y_t) + e^{-At}u(t), \quad \text{a.e. } t \in J, \tag{3.5}$$

where

$$G(t, y_t) = \int_0^t e^{-At}E(t,s,e^{As}y(s))\,ds + \int_{-\infty}^{0} e^{-At}E(t,s,\phi(s))\,ds. \tag{3.6}$$

We assume that

$$\int_{-\infty}^{0} E(t,s,\phi(s))\,ds \quad \text{is integrable on } J \text{ with respect to } t \tag{3.7}$$

for each $\phi \in C$. It is clear that (3.1) is controllable if and only if (3.5) is controllable.

We now write

$$(Bu)(t) = e^{-At}u(t) \tag{3.8}$$

and show that the linear operator $T : U \to R^n$ defined by

$$T(u) = \int_J (Bu)(s)\,ds = \int_J e^{-As}u(s)\,ds \tag{3.9}$$

is invertible. To this end, we assume

$$\operatorname{span}\{c, Ac, A^2c, \dots, A^{n-1}c\} = R^n. \tag{3.10}$$

Let $\alpha \in R^n$. We will find a unique $u \in U$ such that $T(u) = \alpha$. By (3.10), there exists a unique $a = (a_1, a_2, \dots, a_n)^T \in R^n$ satisfying

$$\alpha = a_1c + a_2Ac + \cdots + a_nA^{n-1}c. \tag{3.11}$$

Since $\omega_1, \omega_2, \dots, \omega_n$ are linearly independent, we have

$$\triangle =: \det\left[\left(\langle \omega_i, \omega_j \rangle\right)_{n\times n}\right] \neq 0, \tag{3.12}$$

where $\langle \omega_i, \omega_j \rangle = \int_J w_i(s)w_j(s)ds$ is the inner product in $L^2(J)$. Thus, the system of equations

$$\langle \omega_1, \omega_j \rangle k_1 + \langle \omega_2, \omega_j \rangle k_2 + \cdots + \langle \omega_n, \omega_j \rangle k_n = a_j \tag{3.13}$$

$(j = 1, 2, \dots, n)$ has a unique solution $(k_1, k_2, \dots, k_n)$. We now define

$$u^*(t) = \omega_1(t)k_1 + \omega_2(t)k_2 + \cdots + \omega_n(t)k_n. \tag{3.14}$$

Multiply (3.14) by $e^{-As}c$ and integrate on J to obtain

$$\begin{aligned}
\int_J e^{-As}u(s)ds = \int_J e^{-As}cu^*(s)ds &= \int_J \left[\sum_{j=1}^n \omega_j(s)A^{j-1}c\right]\left[\sum_{i=1}^n \omega_i(s)k_i\right]ds \\
&= \sum_{i=1}^n \langle \omega_i, \omega_1 \rangle k_i c + \cdots + \sum_{i=1}^n \langle \omega_i, \omega_n \rangle k_i A^{n-1}c \\
&= a_1c + a_2Ac + a_3A^2c + \cdots + a_nA^{n-1}c = \alpha.
\end{aligned} \tag{3.15}$$

This implies that T is invertible.

By Cramer's rule, we write k_j in (3.13) as $k_j = \triangle_j(a)/\triangle$ where $\triangle_j(a)$ is the $n \times n$ determinant obtained by replacing the jth column of $\triangle$ by $a = (a_1, a_2, \dots, a_n)^T$ defined in (3.11). Thus

$$|u^*(t)| \leq \sum_{j=1}^n \frac{|\triangle_j(a)|}{|\triangle|}|w_j(t)|. \tag{3.16}$$

Since

$$\triangle_j(a) = a_1C_{1j} + a_2C_{2j} + \cdots + a_nC_{nj}, \tag{3.17}$$

where C_{ij} is the cofactor of $\langle w_i, w_j\rangle$ in the matrix $(\langle w_i, w_j\rangle)_{n\times n}$, and $\ell_1|\alpha| \le |a| \le \ell_2|\alpha|$ for some positive constants ℓ_1 and ℓ_2, we have

$$|\triangle_j(a)| \le |a|\sqrt{\sum_{i=1}^{n}|C_{ij}|^2} \le \ell_2|\alpha|\sqrt{\sum_{i=1}^{n}|C_{ij}|^2}, \tag{3.18}$$

$$\int_0^b |e^{-At}u(t)|\,dt \le |c|\sum_{j=1}^{n}\frac{\ell_2}{|\triangle|}\sqrt{\sum_{i=1}^{n}|C_{ij}|^2}\int_0^b e^{\|A\|t}|w_j(t)|\,dt|\alpha| =: K|\alpha|. \tag{3.19}$$

This implies that

$$\int_J |e^{-At}u_\alpha(t)|\,dt \le K|\alpha|. \tag{3.20}$$

Theorem 3.2. *Suppose that (3.7), (3.10), and the following conditions hold.*

(i) There exist a positive constant M and a measurable function $q:\Omega_0 \to R^+$, $\Omega_0 = \{(t,s)\in R^n : 0 \le s \le t \le b\}$ such that

$$|E(t,s,x)| \le q(t,s)(|x|+M). \tag{3.21}$$

(ii) For any $\mu > 0$, there exists $k_\mu : \Omega_0 \to R^+$ with $\int_J\int_0^t k_\mu(t,s)\,ds\,dt < \infty$ such that

$$|E(t,s,x)-E(t,s,y)| \le k_\mu(t,s)|x-y| \tag{3.22}$$

for all $t,s\in\Omega_0$, $|x|\le\mu$, and $|y|\le\mu$.

(iii)

$$(K+1)\int_0^b\int_0^t e^{\|A\|(t+s)}q(t,s)\,ds\,dt =: r < 1, \tag{3.23}$$

where K is defined in (3.19).

Then (3.1) is controllable.

Proof. It suffices to show that (3.5) is controllable; that is, for any $\phi\in C$ and $y_1\in R^n$, there exists a control $u\in U$ such that the solution $y(t)=y(t,\phi)$ of (3.5) satisfies $y(b)=y_1$. We have shown that (H_1) holds. For $\phi\in C$ and $y\in C_0(\mu)$, it is clear that $G(t,\bar{y}_t)$ is measurable in t. For $\phi\in C$ and $y\in C_0(\mu)$, we also have

$$\begin{aligned}
|G(t,\bar{y}_t)| &= \left|\int_0^t e^{-At}E(t,s,e^{As}(y(s)+\phi(0)))\,ds + \int_{-\infty}^0 e^{-At}E(t,s,\phi(s))\,ds\right| \\
&\le \int_0^t e^{\|A\|t}q(t,s)[e^{\|A\|s}(|y(s)|+|\phi(0)|)+M]\,ds + e^{\|A\|b}\left|\int_{-\infty}^0 E(t,s,\phi(s))\,ds\right| \\
&\le (\mu+|\phi(0)|+M)\int_0^t e^{\|A\|(t+s)}q(t,s)\,ds + e^{\|A\|b}\left|\int_{-\infty}^0 E(t,s,\phi(s))\,ds\right| \\
&=: M_\mu(t).
\end{aligned} \tag{3.24}$$

Thus, (H_2) holds. Next, let $\mu_1 > 0$ and $x, y \in C_0(\mu_1)$. By (ii), there exists $k_\mu(t,s)$, $\mu = (\mu_1 + |\phi(0)|)e^{\|A\|b}$, such that

$$|E(t,s,e^{As}(x(s)+\phi(0))) - E(t,s,e^{As}(y(s)+\phi(0)))| \le e^{\|A\|b} k_\mu(t,s)|x(s)-y(s)| \tag{3.25}$$

for all $(t,s) \in \Omega_0$. This yields

$$\begin{aligned}
&\int_0^b |G(t,\bar{x}_t) - G(t,\bar{y}_t)|\,dt \\
&\quad = \int_0^b \left| \int_0^t e^{-At}E(t,s,e^{As}(x(s)+\phi(0)))\,ds - \int_0^t e^{-At}E(t,s,e^{As}(y(s)+\phi(0)))\,ds \right| dt \\
&\quad \le e^{2\|A\|b} \int_J \int_0^t k_\mu(t,s)\,ds\,dt\,|x-y|_{C_0}
\end{aligned} \tag{3.26}$$

for all $x, y \in C_0(\mu_1)$, and hence (H_3) is satisfied.

We now show that (H_4) holds. Let $y = y(t,\phi)$ be a solution of

$$y'(t) = \lambda[G(t,y_t) + e^{-At}u(t)], \quad \text{a.e. } t \in J \tag{3.2_λ}$$

with $\lambda \in (0,1)$, $y_0 = \phi$, and

$$\int_J e^{-As}u(s)\,ds = y_1 - \phi(0) - \int_J G(s,y_s)\,ds. \tag{3.27}$$

Integrate (3.2_λ) from 0 to t to obtain

$$y(t) = \phi(0) + \lambda \int_0^t G(\tau,y_\tau)\,d\tau + \lambda \int_0^t e^{-A\tau}u(\tau)\,d\tau. \tag{3.28}$$

Thus,

$$|y(t)| \le |\phi(0)| + \int_0^b |G(\tau,y_\tau)|\,ds + \int_0^b |e^{-A\tau}u(\tau)|\,d\tau. \tag{3.29}$$

It follows from (3.20) that

$$\int_0^b |e^{-A\tau}u(\tau)|\,d\tau \le K\left[|y_1| + |\phi(0)| + \int_0^b |G(\tau,y_\tau)|\,d\tau\right]. \tag{3.30}$$

Substituting (3.30) into (3.29), we arrive at

$$\begin{aligned}
|y(t)| &\le K|y_1| + (1+K)|\phi(0)| + (1+K)\int_0^b |G(\tau,y_\tau)|\,d\tau \\
&\le K|y_1| + (1+K)|\phi(0)| + (1+K)\int_0^b \left| \int_{-\infty}^0 e^{-A\tau}E(\tau,s,\phi(s))\,ds \right| d\tau \\
&\quad + (1+K)\int_0^b \int_0^\tau |e^{-A\tau}E(\tau,s,e^{As}y(s))|\,ds\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&\leq K|y_1| + (1+K)|\phi(0)| + (1+K)\int_0^b \left| \int_{-\infty}^0 e^{-A\tau}E(\tau,s,\phi(s))ds \right| d\tau \\
&\quad + (1+K)\int_0^b \int_0^\tau e^{\|A\|t}q(\tau,s)[e^{\|A\|s}|y(s)| + M]ds\,d\tau \\
&=: (1+K)\int_0^b \int_0^\tau e^{\|A\|(t+s)}q(\tau,s)|y(s)|ds\,d\tau + M^* \\
&\leq r|y|_{C_0} + M^*.
\end{aligned}
\tag{3.31}
$$

This yields $|y|_{C_0} \leq M^*/(1-r)$, and hence (H_4) holds. By Theorem 2.3, system (3.5) is controllable. The proof is complete. □

Remark 3.3. A more general expression can be introduced in the control term in (3.1) such as $B(t)u(t)$ where $B(t)$ is an $n \times m$ matrix function and $u(t) \in R^m$. When $E(t,s,x) \equiv 0$, (3.1) is reduced to

$$
x'(t) = Ax + u^*(t)c. \tag{3.32}
$$

A classical result states that (3.32) is controllable if and only if (3.10) holds (see Godunov [7, page 211] and Conti [5, page 98]).

Example 3.4. Consider the scalar Volterra equation

$$
x'(t) = -\int_{-\infty}^t a(t-s)q(x(s))ds + u(t), \quad \text{a.e. } t \in J, \tag{3.33}
$$

where $a : R \to R$, $q : R \times R$ are continuous, and $u \in U$. For a fixed $\xi : J \to R^+$ with $|\xi|_2 = 1$, we define

$$
U = \{u \in L^2(J) : u = k\xi,\ k \in R\} \tag{3.34}
$$

with $|u|_U = |u|_2 = |k||\xi|_2 = |k|$. Therefore, U is a Banach space (dimension 1) with $|\cdot|_U$.
Observe that (3.33) can be written in the form of (1.1) with

$$
G(t,x_t) = -\int_{-\infty}^t a(t-s)q(x(s))ds \tag{3.35}
$$

and $B : U \to L^2(J)$ being the identity operator ($\|B\| = 1$). Define $T : U \to R$ by

$$
T(u) = \int_J Bu(s)ds = \int_J u(s)ds. \tag{3.36}
$$

Notice that $\int_J \xi(s)ds \neq 0$ since $\xi(s) \geq 0$ and $|\xi|_2 = 1$. For each $\alpha \in R$, if there are $u_1, u_2 \in U$ with $u_1 = k_1\xi$, $u_2 = k_2\xi$ such that $T(u_1) = T(u_2) = \alpha$, then $k_1 \int_J \xi(s)ds = k_2 \int_J \xi(s)ds$, which yields $k_1 = k_2$. Thus, T is invertible.

Theorem 3.5. *Suppose the following conditions hold.*

($\tilde{i}$) $\int_{-\infty}^{0} a(t-s)q(\phi(s))ds$ is integrable on J with respect to t for each $\phi \in C$.

($\tilde{ii}$) $a \in C(R,R)$ with $a(t) \geq 0$, $a'(t) \leq 0$, $a''(t) \geq 0$ for all $t \geq 0$.

($\tilde{iii}$) There are positive constants α and K such that $Q(x)+K > 0$ and

$$|q(x)| \leq \alpha\sqrt{Q(x)+K} \quad \forall x \in R \text{ with } \lim_{|x|\to\infty} Q(x) = \infty, \tag{3.37}$$

where $Q(x) = \int_0^x q(s)ds$.

($\tilde{iv}$)

$$\left(\frac{\alpha^2}{2}\right)\int_J \int_0^t a(s)ds\,dt =: \eta < 1. \tag{3.38}$$

Then (3.33) is controllable.

Proof. We verify that conditions (H_2)–(H_4) hold. For each $\phi \in C$ and $x \in C_0(\mu)$, we have

$$\begin{aligned}
|G(t,\bar{x}_t)| &= \left|\int_{-\infty}^{0} a(t-s)q(\phi(s))ds + \int_0^t a(t-s)q(x(s)+\phi(0))ds\right| \\
&\leq \left|\int_{-\infty}^{0} a(t-s)q(\phi(s))ds\right| + \int_0^t a(t-s)|q(x(s)+\phi(0))|ds \\
&\leq \left|\int_{-\infty}^{0} a(t-s)q(\phi(s))ds\right| + \int_0^t a(s)ds\, q_\mu =: M_\mu(t),
\end{aligned} \tag{3.39}$$

where $q_\mu = \sup\{|q(z)| : |z| \leq \mu + |\phi(0)|\}$. Thus, ($H_2$) is satisfied.

Now let $\mu > 0$ and $x, y \in C_0(\mu)$. Since q is uniformly continuous on $[-(\mu+|\phi(0)|), \mu+|\phi(0)|]$, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|x-y|_{C_0} < \delta$ implies

$$|q(x(s)+\phi(0)) - q(y(s)+\phi(0))| < \varepsilon \tag{3.40}$$

for all $s \in J$, and so

$$\begin{aligned}
\int_J |G(s,\bar{x}_s) - G(s,\bar{y}_s)|ds &= \int_J \left|\int_0^t a(t-s)[q(x(s)+\phi(0)) - q(y(s)+\phi(0))]ds\right| dt \\
&\leq \int_J \int_0^t a(s)ds\,dt\,\varepsilon \leq \frac{2\varepsilon}{\alpha^2}.
\end{aligned} \tag{3.41}$$

This shows (H_3) holds.

Let $x(t) = x(t,\phi)$ be a solution of

$$x'(t) = \lambda\left[-\int_{-\infty}^{t} a(t-s)q(x(s))ds + u(t)\right], \quad \text{a.e. } t \in J, \tag{3.42}$$

where $\lambda \in (0,1)$, $x_0 = \phi$, and

$$\int_J u(s)ds = \gamma - \phi(0) + \int_J \int_{-\infty}^{\tau} a(\tau-s)q(x(s))ds\,d\tau. \tag{3.43}$$

We apply Liapunov's direct method to derive a priori bounds on x. Define

$$E(t) = Q(x(t)) + K + \frac{1}{2}\lambda a(t)\left[\int_0^t q(x(s))\,ds\right]^2 - \frac{1}{2}\lambda\int_0^t a'(t-s)\left[\int_s^t q(x(v))\,dv\right]^2 ds \tag{3.44}$$

for all $t \geq 0$ and set $V(t) = \sqrt{E(t)}$. Then

$$\begin{aligned} V'(t) = \frac{1}{2\sqrt{E(t)}}\Big\{&\Big[q(x)\lambda\Big(-\int_{-\infty}^t a(t-s)q(x(s))\,ds + u(t)\Big)\Big] \\ &+ \frac{1}{2}\lambda a'(t)\left[\int_0^t q(x(s))\,ds\right]^2 + \lambda a(t)\left[\int_0^t q(x(s))\,ds\right]q(x(t)) \\ &- \frac{1}{2}\lambda\int_0^t a''(t-s)\left[\int_s^t q(x(v))\,dv\right]^2 ds \\ &- \lambda\int_0^t a'(t-s)\left[\int_s^t q(x(v))\,dv\right]ds\, q(x(t))\Big\}. \end{aligned} \tag{3.45}$$

Integrating by parts in the last term, we get

$$\begin{aligned} &-\lambda\int_0^t a'(t-s)\left[\int_s^t q(x(v))\,dv\right]ds \\ &\quad = \lambda a(t-s)\int_s^t q(x(v))\,dv\,\Big|_0^t + \lambda\int_0^t a(t-s)q(x(s))\,ds \\ &\quad = -\lambda a(t)\int_0^t q(x(v))\,dv + \lambda\int_0^t a(t-s)q(x(s))\,ds. \end{aligned} \tag{3.46}$$

Substitute (3.46) into (3.45) and apply condition ($\tilde{\text{iii}}$) to obtain

$$\begin{aligned} V'(t) &= \frac{1}{2\sqrt{E(t)}}\Big\{\lambda q(x(t))u(t) - \lambda q(x(t))\int_{-\infty}^0 a(t-s)q(\phi(s))\,ds \\ &\qquad + \frac{1}{2}\lambda a'(t)\left[\int_0^t q(x(s))\,ds\right]^2 - \frac{1}{2}\lambda\int_0^t a''(t-s)\left[\int_s^t q(x(v))\,dv\right]^2 ds\Big\} \\ &\leq \frac{1}{2\sqrt{Q(x)+K}}\Big\{|q(x(t))|\Big(|u(t)| + \int_0^\infty a(v)\,dv\sup_{t\in R^-}|q(\phi(s))|\Big)\Big\} \\ &\leq \frac{\alpha}{2}\Big(|u(t)| + \int_0^\infty a(v)\,dv\sup_{s\in R^-}|q(\phi(s))|\Big) =: \frac{\alpha}{2}|u(t)| + \Gamma_1. \end{aligned} \tag{3.47}$$

Integrating $V'(t)$ from 0 to t and using (3.43), we find

$$\begin{aligned}
V(t) &\leq V(0)+\frac{\alpha}{2}\int_0^b |u(s)|\,ds+b\Gamma_1=\sqrt{Q(x(0))+K}+\frac{\alpha}{2}\left|\int_0^b u(s)ds\right|+b\Gamma_1\\
&=\sqrt{Q(\phi(0))+K}+b\Gamma_1+\frac{\alpha}{2}\left|\gamma-\phi(0)+\int_J\int_{-\infty}^{\tau}a(\tau-s)q(x(s))\,ds\,d\tau\right|\\
&\leq\frac{\alpha}{2}\int_J\int_0^{\tau}a(\tau-s)|q(x(s))|\,ds\,d\tau+\sqrt{Q(\phi(0))+K}+b\Gamma_1\\
&\quad+\frac{\alpha}{2}\left(|\gamma|+|\phi(0)|+\int_J\left|\int_{-\infty}^{0}a(\tau-s)q(\phi(s))ds\right|dt\right)\\
&=:\frac{\alpha}{2}\int_J\int_0^{\tau}a(\tau-s)|q(x(s))|\,ds\,d\tau+\Gamma_2.
\end{aligned} \tag{3.48}$$

Observe

$$\begin{aligned}
\sqrt{Q(x(t))+K}\leq V(t)&\leq\frac{\alpha}{2}\int_J\int_0^{\tau}a(\tau-s)|q(x(s))|\,ds\,d\tau+\Gamma_2\\
&\leq\left(\sup_{s\in J}\sqrt{Q(x(s))+K}\right)\frac{\alpha^2}{2}\int_J\int_0^{\tau}a(\nu)d\nu+\Gamma_2\\
&\leq\eta\sup_{s\in J}\sqrt{Q(x(s))+K}+\Gamma_2.
\end{aligned} \tag{3.49}$$

This implies

$$\sup_{s\in J}\sqrt{Q(x(s))+K}\leq\frac{\Gamma_2}{(1-\eta)}. \tag{3.50}$$

Since $Q(x)\to\infty$ as $|x|\to\infty$, there exists a positive constant $L=L(\Gamma_2,\eta)$ such that $|x(t)|<L$ for all $t\in J$. Therefore, (H_4) holds. By Theorem 2.3, we conclude that (3.33) is controllable. □

Remark 3.6. Equations such as (3.33) have been the center of much interest for a long time in connection with a problem of reactor dynamics (Levin and Nohel [11]). The Liapunov function here, having its root in the work of Levin [10], continues to play an important role in the investigation of Volterra equations. It is also well-known that under $xq(x)>0$ for $x\neq 0$ with $Q(x)\to\infty$ as $|x|\to\infty$, condition ($\tilde{\text{ii}}$) with small modifications guarantees that the zero solution of the unperturbed equation

$$x'(t)=\int_{-\infty}^{t}a(t-s)q(x(s))ds \tag{3.51}$$

is asymptotically stable (Hale [9]).

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