INVARIANT APPROXIMATIONS, GENERALIZED I-CONTRACTIONS, AND R-SUBWEAKLY COMMUTING MAPS

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We present common fixed point theory for generalized contractive R-subweakly commuting maps and obtain some results on invariant approximation.

1. Introduction and preliminaries

Let S be a subset of a normed space $X = (X, \|\cdot\|)$ and T and I self-mappings of X. Then T is called (1) nonexpansive on S if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in S$; (2) I-nonexpansive on S if $\|Tx - Ty\| \leq \|Ix - Iy\|$ for all $x, y \in S$; (3) I-contraction on S if there exists $k \in [0,1)$ such that $\|Tx - Ty\| \leq k\|Ix - Iy\|$ for all $x, y \in S$. The set of fixed points of T (resp., I) is denoted by $F(T)$ (resp., $F(I)$). The set S is called (4) p-starshaped with $p \in S$ if for all $x \in S$, the segment $[x, p]$ joining x to p is contained in S (i.e., $kx + (1-k)p \in S$ for all $x \in S$ and all real k with $0 \leq k \leq 1$); (5) convex if S is p-starshaped for all $p \in S$. The convex hull $\operatorname{co}(S)$ of S is the smallest convex set in X that contains S, and the closed convex hull $\operatorname{cl}\operatorname{co}(S)$ of S is the closure of its convex hull. The mapping T is called (6) compact if $\operatorname{cl} T(D)$ is compact for every bounded subset D of S. The mappings T and I are said to be (7) commuting on S if $ITx = TIx$ for all $x \in S$; (8) R-weakly commuting on S [7] if there exists $R \in (0, \infty)$ such that $\|TIx - ITx\| \leq R\|Tx - Ix\|$ for all $x \in S$. Suppose $S \subset X$ is p-starshaped with $p \in F(I)$ and is both T- and I-invariant. Then T and I are called (8) R-subweakly commuting on S [11] if there exists $R \in (0, \infty)$ such that $\|TIx - ITx\| \leq R\operatorname{dist}(Ix, [Tx, p])$ for all $x \in S$, where $\operatorname{dist}(Ix, [Tx, p]) = \inf\{\|Ix - z\| : z \in [Tx, p]\}$. Clearly commutativity implies R-subweak commutativity, but the converse may not be true (see [11]).

The set $P_S(\hat{x}) = \{y \in S : \|y - \hat{x}\| = \operatorname{dist}(\hat{x}, S)\}$ is called the set of best approximants to $\hat{x} \in X$ out of S, where $\operatorname{dist}(\hat{x}, S) = \inf\{\|y - \hat{x}\| : y \in S\}$. We define $C_S^I(\hat{x}) = \{x \in S : Ix \in P_S(\hat{x})\}$ and denote by $\mathfrak{I}_0$ the class of closed convex subsets of X containing 0. For $S \in \mathfrak{I}_0$, we define $S_{\hat{x}} = \{x \in S : \|x\| \leq 2\|\hat{x}\|\}$. It is clear that $P_S(\hat{x}) \subset S_{\hat{x}} \in \mathfrak{I}_0$.

In 1963, Meinardus [6] employed the Schauder fixed point theorem to establish the existence of invariant approximations. Afterwards, Brosowski [2] obtained the following extension of the Meinardus result.

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Theorem 1.1. *Let T be a linear and nonexpansive self-mapping of a normed space X, $S \subset X$ such that $T(S) \subset S$, and $\hat{x} \in F(T)$. If $P_S(\hat{x})$ is nonempty, compact, and convex, then $P_S(\hat{x}) \cap F(T) \neq \emptyset$.*

Singh [15] observed that Theorem 1.1 is still true if the linearity of T is dropped and $P_S(\hat{x})$ is only starshaped. He further remarked, in [16], that Brosowski's theorem remains valid if T is nonexpansive only on $P_S(\hat{x}) \cup \{\hat{x}\}$. Then Hicks and Humphries [5] improved Singh's result by weakening the assumption $T(S) \subset S$ to $T(\partial S) \subset S$; here ∂S denotes the boundary of S.

On the other hand, Subrahmanyam [18] generalized the Meinardus result as follows.

Theorem 1.2. *Let T be a nonexpansive self-mapping of X, S a finite-dimensional T-invariant subspace of X, and $\hat{x} \in F(T)$. Then $P_S(\hat{x}) \cap F(T) \neq \emptyset$.*

In 1981, Smoluk [17] noted that the finite dimensionality of S in Theorem 1.2 can be replaced by the linearity and compactness of T. Subsequently, Habiniak [4] observed that the linearity of T in Smoluk's result is superfluous.

In 1988, Sahab et al. [8] established the following result which contains Singh's result as a special case.

Theorem 1.3. *Let T and I be self-mappings of a normed space X, $S \subset X$ such that $T(\partial S) \subset S$, and $\hat{x} \in F(T) \cap F(I)$. Suppose T is I-nonexpansive on $P_S(\hat{x}) \cup \{\hat{x}\}$, I is linear and continuous on $P_S(\hat{x})$, and T and I are commuting on $P_S(\hat{x})$. If $P_S(\hat{x})$ is nonempty, compact, and p-starshaped with $p \in F(I)$, and if $I(P_S(\hat{x})) = P_S(\hat{x})$, then $P_S(\hat{x}) \cap F(T) \cap F(I) \neq \emptyset$.*

Recently, Al-Thagafi [1] generalized Theorem 1.3 and proved some results on invariant approximations for commuting mappings. More recently, with the introduction of noncommuting maps to this area, Shahzad [9, 10, 11, 12, 13, 14] further extended Al-Thagafi's results and obtained a number of results regarding best approximations. The purpose of this paper is to present common fixed point theory for generalized I-contraction and R-subweakly commuting maps. As applications, some invariant approximation results are also obtained. Our results extend, generalize, and complement those of Al-Thagafi [1], Brosowski [2], Dotson Jr. [3], Habiniak [4], Hicks and Humphries [5], Meinardus [6], Sahab et al. [8], Shahzad [9, 10, 11, 12], Singh [15, 16], Smoluk [17], and Subrahmanyam [18].

2. Main results

Theorem 2.1. *Let S be a closed subset of a metric space (X, d), and T and I R-weakly commuting self-mappings of S such that $T(S) \subset I(S)$. Suppose there exists $k \in [0,1)$ such that*

$$d(Tx, Ty) \leq k \max\left\{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), \frac{1}{2}\left[d(Ix, Ty) + d(Iy, Tx)\right]\right\} \tag{2.1}$$

for all $x, y \in S$. If $\mathrm{cl}(T(S))$ is complete and T is continuous, then $S \cap F(T) \cap F(I)$ is singleton.

Proof. Let $x_0 \in S$ and let $x_1 \in S$ be such that $Ix_1 = Tx_0$. Inductively, choose x_n so that $Ix_n = Tx_{n-1}$. This is possible since $T(S) \subset I(S)$. Notice

$$\begin{aligned} d(Ix_{n+1}, Ix_n) &= d(Tx_n, Tx_{n-1}) \\ &\le k\max\Big\{d(Ix_n, Ix_{n-1}), d(Ix_n, Tx_n), d(Ix_{n-1}, Tx_{n-1}), \\ &\qquad \frac{1}{2}\big[d(Ix_n, Tx_{n-1}) + d(Ix_{n-1}, Tx_n)\big]\Big\} \\ &= k\max\Big\{d(Ix_n, Ix_{n-1}), d(Ix_n, Tx_n), \\ &\qquad d(Ix_{n-1}, Tx_{n-1}), \frac{1}{2}d(Ix_{n-1}, Tx_n)\Big\} \\ &\le k\max\Big\{d(Ix_n, Ix_{n-1}), d(Ix_n, Tx_n), \\ &\qquad \frac{1}{2}\big[d(Ix_{n-1}, Ix_n) + d(Ix_n, Tx_n)\big]\Big\} \\ &\le kd(Ix_n, Ix_{n-1}) \end{aligned} \tag{2.2}$$

for all n. This shows that $\{Ix_n\}$ is a Cauchy sequence in S. Consequently, $\{Tx_n\}$ is a Cauchy sequence. The completeness of $\mathrm{cl}(T(S))$ further implies that $Tx_n \to y \in S$ and so $Ix_n \to y$ as $n \to \infty$. Since T and I are R-weakly commuting, we have

$$d(TIx_n, ITx_n) \le Rd(Tx_n, Ix_n). \tag{2.3}$$

This implies that $ITx_n \to Ty$ as $n \to \infty$. Now

$$\begin{aligned} d(Tx_n, TTx_n) \le k\max\Big\{&d(Ix_n, ITx_n), d(Ix_n, Tx_n), d(ITx_n, TTx_n), \\ &\frac{1}{2}\big[d(Ix_n, TTx_n) + d(ITx_n, Tx_n)\big]\Big\}. \end{aligned} \tag{2.4}$$

Taking the limit as $n \to \infty$, we obtain

$$\begin{aligned} d(y, Ty) &\le k\max\Big\{d(y, Ty), d(y, y), d(Ty, Ty), \\ &\qquad \frac{1}{2}\big[d(y, Ty) + d(Ty, y)\big]\Big\} \\ &= kd(y, Ty), \end{aligned} \tag{2.5}$$

which implies $y = Ty$. Since $T(S) \subset I(S)$, we can choose $z \in S$ such that $y = Ty = Iz$. Since

$$\begin{aligned} d(TTx_n, Tz) \le k\max\Big\{&d(ITx_n, Iz), d(ITx_n, TTx_n), d(Iz, Tz), \\ &\frac{1}{2}\big[d(ITx_n, Tz) + d(Iz, TTx_n)\big]\Big\}, \end{aligned} \tag{2.6}$$

taking the limit as $n \to \infty$ yields

$$d(Ty,Tz) \le kd(Ty,Tz). \tag{2.7}$$

This implies that $Ty = Tz$. Therefore, $y = Ty = Tz = Iz$. Using the R-weak commutativity of T and I, we obtain

$$d(Ty,Iy) = d(TIz,ITz) \le Rd(Tz,Iz) = 0. \tag{2.8}$$

Thus $y = Ty = Iy$. Clearly y is a unique common fixed point of T and I. Hence $S \cap F(T) \cap F(I)$ is singleton. □

Theorem 2.2. *Let S be a closed subset of a normed space X, and T and I continuous self-mappings of S such that $T(S) \subset I(S)$. Suppose I is linear, $p \in F(I)$, S is p-starshaped, and $\operatorname{cl}(T(S))$ is compact. If T and I are R-subweakly commuting and satisfy*

$$\begin{aligned} \|Tx - Ty\| \le \max\Big\{ &\|Ix - Iy\|, \operatorname{dist}(Ix,[Tx,p]), \operatorname{dist}(Iy,[Ty,p]), \\ &\frac{1}{2}\big[\operatorname{dist}(Ix,[Ty,p]) + \operatorname{dist}(Iy,[Tx,p])\big]\Big\} \end{aligned} \tag{2.9}$$

for all $x, y \in S$, then $S \cap F(T) \cap F(I) \neq \emptyset$.

Proof. Choose a sequence $\{k_n\} \subset [0,1)$ such that $k_n \to 1$ as $n \to \infty$. Define, for each n, a map T_n by $T_n(x) = k_nTx + (1-k_n)p$ for each $x \in S$. Then each T_n is a self-mapping of S. Furthermore, $T_n(S) \subset I(S)$ for each n since I is linear and $T(S) \subset I(S)$. Now the linearity of I and the R-subweak commutativity of T and I imply that

$$\begin{aligned} \|T_nIx - IT_nx\| = k_n\|TIx - ITx\| &\le k_nR\operatorname{dist}(Ix,[Tx,p]) \\ &\le k_nR\|T_nx - Ix\| \end{aligned} \tag{2.10}$$

for all $x \in S$. This shows that T_n and I are k_nR-weakly commuting for each n. Also

$$\begin{aligned} \|T_nx - T_ny\| &= k_n\|Tx - Ty\| \\ &\le k_n\max\Big\{\|Ix - Iy\|, \operatorname{dist}(Ix,[Tx,p]), \operatorname{dist}(Iy,[Ty,p]), \\ &\qquad \frac{1}{2}\big[\operatorname{dist}(Ix,[Ty,p]) + \operatorname{dist}(Iy,[Tx,p])\big]\Big\} \\ &\le k_n\max\Big\{\|Ix - Iy\|, \|Ix - T_nx\|, \|Iy - T_ny\|, \\ &\qquad \frac{1}{2}\big[\|Ix - T_ny\| + \|Iy - T_nx\|\big]\Big\} \end{aligned} \tag{2.11}$$

for all $x, y \in S$. Now Theorem 2.1 guarantees that $F(T_n) \cap F(I) = \{x_n\}$ for some $x_n \in S$. The compactness of $\operatorname{cl}(T(S))$ implies that there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $x_m \to y \in S$ as $m \to \infty$. By the continuity of T and I, we have $y \in F(T) \cap F(I)$. Hence $S \cap F(T) \cap F(I) \neq \emptyset$. □

The following corollaries extend and generalize [3, Theorem 1] and [4, Theorem 4].

Corollary 2.3. *Let S be a closed subset of a normed space X, and T and I continuous self-mappings of S such that $T(S) \subset I(S)$. Suppose I is linear, $p \in F(I)$, S is p-starshaped, and $\mathrm{cl}(T(S))$ is compact. If T and I are R-subweakly commuting and T is I-nonexpansive on S, then $S \cap F(T) \cap F(I) \neq \emptyset$.*

Corollary 2.4. *Let S be a closed subset of a normed space X, and T and I continuous self-mappings of S such that $T(S) \subset I(S)$. Suppose I is linear, $p \in F(I)$, S is p-starshaped, and $\mathrm{cl}(T(S))$ is compact. If T and I are commuting and satisfy (2.9) for all $x, y \in S$, then $S \cap F(T) \cap F(I) \neq \emptyset$.*

Let $D_S^{R,I}(\hat{x}) = P_S(\hat{x}) \cap G_S^{R,I}(\hat{x})$, where

$$G_S^{R,I}(\hat{x}) = \{x \in S : \|Ix - \hat{x}\| \leq (2R+1)\,\mathrm{dist}(\hat{x}, S)\}. \tag{2.12}$$

Theorem 2.5. *Let T and I be self-mappings of a normed space X with $\hat{x} \in F(T) \cap F(I)$ and $S \subset X$ such that $T(\partial S \cap S) \subset S$. Suppose I is linear on $D_S^{R,I}(\hat{x})$, $p \in F(I)$, $D_S^{R,I}(\hat{x})$ is closed and p-starshaped, $\mathrm{cl}\,T(D_S^{R,I}(\hat{x}))$ is compact, and $I(D_S^{R,I}(\hat{x})) = D_S^{R,I}(\hat{x})$. If T and I are R-subweakly commuting and continuous on $D_S^{R,I}(\hat{x})$ and satisfy, for all $x \in D_S^{R,I}(\hat{x}) \cup \{\hat{x}\}$,*

$$\|Tx - Ty\| \leq \begin{cases} \|Ix - I\hat{x}\| & \text{if } y = \hat{x}, \\ \max\Big\{\|Ix - Iy\|, \mathrm{dist}\,(Ix, [Tx, p]), \mathrm{dist}\,(Iy, [Ty, p]), \\ \qquad \dfrac{1}{2}\big[\mathrm{dist}\,(Ix, [Ty, p]) + \mathrm{dist}\,(Iy, [Tx, p])\big]\Big\} & \text{if } y \in D_S^{R,I}(\hat{x}), \end{cases} \tag{2.13}$$

then $P_S(\hat{x}) \cap F(T) \cap F(I) \neq \emptyset$.

Proof. Let $x \in D_S^{R,I}(\hat{x})$. Then $x \in \partial S \cap S$ (see [1]) and so $Tx \in S$ since $T(\partial S \cap S) \subset S$. Now

$$\|Tx - \hat{x}\| = \|Tx - T\hat{x}\| \leq \|Ix - I\hat{x}\| = \|Ix - \hat{x}\| = \mathrm{dist}(\hat{x}, S). \tag{2.14}$$

This shows that $Tx \in P_S(\hat{x})$. From the R-subweak commutativity of T and I, it follows that

$$\|ITx - \hat{x}\| = \|ITx - T\hat{x}\| \leq R\|Tx - Ix\| + \big\|I^2x - I\hat{x}\big\| \leq (2R+1)\,\mathrm{dist}(\hat{x}, S). \tag{2.15}$$

This implies that $Tx \in G_S^{R,I}(\hat{x})$. Consequently, $Tx \in D_S^{R,I}(\hat{x})$ and so $T(D_S^{R,I}(\hat{x})) \subset D_S^{R,I}(\hat{x}) = I(D_S^{R,I}(\hat{x}))$. Now Theorem 2.2 guarantees that $P_S(\hat{x}) \cap F(T) \cap F(I) \neq \emptyset$. □

Theorem 2.6. *Let T and I be self-mappings of a normed space X with $\hat{x} \in F(T) \cap F(I)$ and $S \subset X$ such that $T(\partial S \cap S) \subset I(S) \subset S$. Suppose I is linear on $D_S^{R,I}(\hat{x})$, $p \in F(I)$, $D_S^{R,I}(\hat{x})$ is closed and p-starshaped, $\mathrm{cl}\,T(D_S^{R,I}(\hat{x}))$ is compact, and $I(G_S^{R,I}(\hat{x})) \cap D_S^{R,I}(\hat{x}) \subset I(D_S^{R,I}(\hat{x})) \subset D_S^{R,I}(\hat{x})$. If T and I are R-subweakly commuting and continuous on $D_S^{R,I}(\hat{x})$ and satisfy, for all $x \in D_S^{R,I}(\hat{x}) \cup \{\hat{x}\}$, (2.13), then $P_S(\hat{x}) \cap F(T) \cap F(I) \neq \emptyset$.*

Proof. Let $x \in D_S^{R,I}(\hat{x})$. Then, as in Theorem 2.5, $Tx \in D_S^{R,I}(\hat{x})$, that is, $T(D_S^{R,I}(\hat{x})) \subset D_S^{R,I}(\hat{x})$. Also $\|(1-k)x + k\hat{x} - \hat{x}\| < \operatorname{dist}(\hat{x}, S)$ for all $k \in (0,1)$. This implies that $x \in \partial S \cap S$ (see [1]) and so $T(D_S^{R,I}(\hat{x})) \subset T(\partial S \cap S) \subset I(S)$. Thus we can choose $y \in S$ such that $Tx = Iy$. Since $Iy = Tx \in P_S(\hat{x})$, it follows that $y \in G_S^{R,I}(\hat{x})$. Consequently, $T(D_S^{R,I}(\hat{x})) \subset I(G_S^{R,I}(\hat{x})) \subset P_S(\hat{x})$. Therefore, $T(D_S^{R,I}(\hat{x})) \subset I(G_S^{R,I}(\hat{x})) \cap D_S^{R,I}(\hat{x}) \subset I(D_S^{R,I}(\hat{x})) \subset D_S^{R,I}(\hat{x})$. Now Theorem 2.2 guarantees that $P_S(\hat{x}) \cap F(T) \cap F(I) \neq \emptyset$. □

Remark 2.7. Theorems 2.5 and 2.6 remain valid when $D_S^{R,I}(\hat{x}) = P_S(\hat{x})$. If $I(P_S(\hat{x})) \subset P_S(\hat{x})$, then $P_S(\hat{x}) \subset C_S^I(\hat{x}) \subset G_S^{R,I}(\hat{x})$ (see [1]) and so $D_S^{R,I}(\hat{x}) = P_S(\hat{x})$. Consequently, Theorem 2.5 contains Theorem 1.3 as a special case.

The following result includes [1, Theorem 4.1] and [4, Theorem 8]. It also contains the well-known results due to Smoluk [17] and Subrahmanyam [18].

Theorem 2.8. *Let T be a self-mapping of a normed space X with $\hat{x} \in F(T)$ and $S \in \mathfrak{I}_0$ such that $T(S_{\hat{x}}) \subset S$. If $\operatorname{cl} T(S_{\hat{x}})$ is compact and T is continuous on $S_{\hat{x}}$ and satisfies for all $x \in S_{\hat{x}} \cup \{\hat{x}\}$*

$$\|Tx - Ty\| \leq \begin{cases} \|x - \hat{x}\| & \text{if } y = \hat{x}, \\ \max\Big\{\|x-y\|, \operatorname{dist}(x,[Tx,0]), \operatorname{dist}(y,[Ty,0]), & \\ \quad \frac{1}{2}\big[\operatorname{dist}(x,[Ty,0]) + \operatorname{dist}(y,[Tx,0])\big]\Big\} & \text{if } y \in S_{\hat{x}}, \end{cases} \tag{2.16}$$

then

(i) *$P_S(\hat{x})$ is nonempty, closed, and convex,*
(ii) *$T(P_S(\hat{x})) \subset P_S(\hat{x})$,*
(iii) *$P_S(\hat{x}) \cap F(T) \neq \emptyset$.*

Proof. (i) We may assume that $\hat{x} \notin S$. If $x \in S \setminus S_{\hat{x}}$, then $\|x\| > 2\|\hat{x}\|$. Notice that

$$\|x - \hat{x}\| \geq \|x\| - \|\hat{x}\| > \|\hat{x}\| \geq \operatorname{dist}(\hat{x}, S_{\hat{x}}). \tag{2.17}$$

Consequently, $\operatorname{dist}(\hat{x}, S_{\hat{x}}) = \operatorname{dist}(\hat{x}, S) \leq \|\hat{x}\|$. Also $\|z - \hat{x}\| = \operatorname{dist}(\hat{x}, \operatorname{cl} T(S_{\hat{x}}))$ for some $z \in \operatorname{cl} T(S_{\hat{x}})$. Thus

$$\begin{aligned} \operatorname{dist}(\hat{x}, S_{\hat{x}}) &\leq \operatorname{dist}(\hat{x}, \operatorname{cl} T(S_{\hat{x}})) \leq \operatorname{dist}(\hat{x}, T(S_{\hat{x}})) \\ &\leq \|Tx - \hat{x}\| = \|Tx - T\hat{x}\| \\ &\leq \|x - \hat{x}\| \end{aligned} \tag{2.18}$$

for all $x \in S_{\hat{x}}$. This implies that $\|z - \hat{x}\| = \operatorname{dist}(\hat{x}, S)$ and so $P_S(\hat{x})$ is nonempty. Furthermore, it is closed and convex.

(ii) Let $y \in P_S(\hat{x})$. Then

$$\|Ty - \hat{x}\| = \|Ty - T\hat{x}\| \leq \|y - \hat{x}\| = \operatorname{dist}(\hat{x}, S). \tag{2.19}$$

This implies that $Ty \in P_S(\hat{x})$ and so $T(P_S(\hat{x})) \subset P_S(\hat{x})$.

(iii) Theorem 2.2 guarantees that $P_S(\hat{x}) \cap F(T) \neq \emptyset$ since $\operatorname{cl} T(P_S(\hat{x})) \subset \operatorname{cl} T(S_{\hat{x}})$ and $\operatorname{cl} T(S_{\hat{x}})$ is compact. □

THEOREM 2.9. *Let I and T be self-mappings of a normed space X with $\hat{x} \in F(I) \cap F(T)$ and $S \in \Im_0$ such that $T(S_{\hat{x}}) \subset I(S) \subset S$. Suppose that I is linear, $\|Ix - \hat{x}\| = \|x - \hat{x}\|$ for all $x \in S$, $\operatorname{cl} I(S_{\hat{x}})$ is compact and I satisfies, for all $x, y \in S_{\hat{x}}$,*

$$\begin{aligned} \|Ix - Iy\| \leq \max\Big\{ &\|x - y\|, \operatorname{dist}(x, [Ix, 0]), \operatorname{dist}(y, [Iy, 0]), \\ &\frac{1}{2}\big[\operatorname{dist}(x, [Iy, 0]) + \operatorname{dist}(y, [Ix, 0])\big]\Big\}. \end{aligned} \tag{2.20}$$

If I and T are R-subweakly commuting and continuous on $S_{\hat{x}}$ and satisfy, for all $x \in S_{\hat{x}} \cup \{\hat{x}\}$, and $p \in F(I)$,

$$\|Tx - Ty\| \leq \begin{cases} \|Ix - I\hat{x}\| & \text{if } y = \hat{x}, \\ \max\Big\{\|Ix - Iy\|, \operatorname{dist}(Ix, [Tx, p]), \operatorname{dist}(Iy, [Ty, p]), & \\ \quad \frac{1}{2}\big[\operatorname{dist}(Ix, [Ty, p]) + \operatorname{dist}(Iy, [Tx, p])\big]\Big\} & \text{if } y \in S_{\hat{x}}, \end{cases} \tag{2.21}$$

then

(i) *$P_S(\hat{x})$ is nonempty, closed, and convex,*
(ii) *$T(P_S(\hat{x})) \subset I(P_S(\hat{x})) \subset P_S(\hat{x})$,*
(iii) *$P_S(\hat{x}) \cap F(I) \cap F(T) \neq \emptyset$.*

Proof. From Theorem 2.8, (i) follows immediately. Also, we have $I(P_S(\hat{x})) \subset P_S(\hat{x})$. Let $y \in T(P_S(\hat{x}))$. Since $T(S_{\hat{x}}) \subset I(S)$ and $P_S(\hat{x}) \subset S_{\hat{x}}$, there exist $z \in P_S(\hat{x})$ and $x_1 \in S$ such that $y = Tz = Ix_1$. Furthermore, we have

$$\big\|Ix_1 - \hat{x}\big\| = \|Tz - T\hat{x}\| \leq \|Iz - I\hat{x}\| \leq \|z - \hat{x}\| = d(\hat{x}, S). \tag{2.22}$$

Thus $x_1 \in C_S^I(\hat{x}) = P_S(\hat{x})$ and so (ii) holds.

Since, by Theorem 2.8, $P_S(\hat{x}) \cap F(I) \neq \emptyset$, it follows that there exists $p \in P_S(\hat{x})$ such that $p \in F(I)$. Hence (iii) follows from Theorem 2.2. □

The following corollary extends [1, Theorem 4.2(a)] to a class of noncommuting maps.

COROLLARY 2.10. *Let I and T be self-mappings of a normed space X with $\hat{x} \in F(I) \cap F(T)$ and $S \in \Im_0$ such that $T(S_{\hat{x}}) \subset I(S) \subset S$. Suppose that I is linear, $\|Ix - \hat{x}\| = \|x - \hat{x}\|$ for all $x \in S$, $\operatorname{cl} I(S_{\hat{x}})$ is compact, and I is nonexpansive on $S_{\hat{x}}$. If I and T are R-subweakly commuting on $S_{\hat{x}}$ and T is I-nonexpansive on $S_{\hat{x}} \cup \{\hat{x}\}$, then*

(i) *$P_S(\hat{x})$ is nonempty, closed and convex,*
(ii) *$T(P_S(\hat{x})) \subset I(P_S(\hat{x})) \subset P_S(\hat{x})$, and*
(iii) *$P_S(\hat{x}) \cap F(I) \cap F(T) \neq \emptyset$.*

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References

[1] M. A. Al-Thagafi, *Common fixed points and best approximation*, J. Approx. Theory **85** (1996), no. 3, 318–323.
[2] B. Brosowski, *Fixpunktsätze in der Approximationstheorie*, Mathematica (Cluj) **11 (34)** (1969), 195–220 (German).
[3] W. G. Dotson Jr., *Fixed point theorems for non-expansive mappings on star-shaped subsets of Banach spaces*, J. London Math. Soc. (2) **4** (1972), 408–410.
[4] L. Habiniak, *Fixed point theorems and invariant approximations*, J. Approx. Theory **56** (1989), no. 3, 241–244.
[5] T. L. Hicks and M. D. Humphries, *A note on fixed-point theorems*, J. Approx. Theory **34** (1982), no. 3, 221–225.
[6] G. Meinardus, *Invarianz bei linearen Approximationen*, Arch. Rational Mech. Anal. **14** (1963), 301–303 (German).
[7] R. P. Pant, *Common fixed points of noncommuting mappings*, J. Math. Anal. Appl. **188** (1994), no. 2, 436–440.
[8] S. A. Sahab, M. S. Khan, and S. Sessa, *A result in best approximation theory*, J. Approx. Theory **55** (1988), no. 3, 349–351.
[9] N. Shahzad, *A result on best approximation*, Tamkang J. Math. **29** (1998), no. 3, 223–226.
[10] ———, *Correction to: "A result on best approximation"*, Tamkang J. Math. **30** (1999), no. 2, 165.
[11] ———, *Invariant approximations and R-subweakly commuting maps*, J. Math. Anal. Appl. **257** (2001), no. 1, 39–45.
[12] ———, *Noncommuting maps and best approximations*, Rad. Mat. **10** (2001), no. 1, 77–83.
[13] ———, *On R-subcommuting maps and best approximations in Banach spaces*, Tamkang J. Math. **32** (2001), no. 1, 51–53.
[14] ———, *Remarks on invariant approximations*, Int. J. Math. Game Theory Algebra **13** (2003), no. 2, 157–159.
[15] S. P. Singh, *An application of a fixed-point theorem to approximation theory*, J. Approx. Theory **25** (1979), no. 1, 89–90.
[16] ———, *Application of fixed point theorems in approximation theory*, Applied Nonlinear Analysis (Proc. Third Internat. Conf., Univ. Texas, Arlington, Tex, 1978) (V. Lakshmikantham, ed.), Academic Press, New York, 1979, pp. 389–394.
[17] A. Smoluk, *Invariant approximations*, Mat. Stos. (3) **17** (1981), 17–22 (Polish).
[18] P. V. Subrahmanyam, *An application of a fixed point theorem to best approximation*, J. Approximation Theory **20** (1977), no. 2, 165–172.

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