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Research Article

Convergence Theorem for Equilibrium Problems and Fixed Point Problems of Infinite Family of Nonexpansive Mappings

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We introduce an iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of infinite nonexpansive mappings in a Hilbert space. We prove a strong-convergence theorem under mild assumptions on parameters.

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1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $h : C \times C \to R$ be an equilibrium bifunction, that is, $h(u,u) = 0$ for every $u \in C$. Then one can define the equilibrium problem that is to find an element $u \in C$ such that

$$\mathrm{EP}(h) : h(u,v) \geq 0 \quad \forall v \in C. \tag{1.1}$$

Denote the set of solutions of EP(h) by SEP(h). This problem contains fixed point problems, optimization problems, variational inequality problems, and Nash equilibrium problems as special cases, see [1]. Some methods have been proposed to solve the equilibrium problem, please consult [2–4].

Recently, Combettes and Hirstoaga [2] introduced an iterative scheme of finding the best approximation to the initial data when SEP(h)$\neq\emptyset$ and proved a strong convergence theorem. Motivated by the idea of Combettes and Hirstoaga, very recently, Takahashi and Takahashi [4] introduced a new iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. Their results extend and

improve the corresponding results announced by Combettes and Hirstoaga [2], Moudafi [5], Wittmann [6], and Tada and Takahashi [7].

In this paper, motivated and inspired by Combettes and Hirstoaga [2] and Takahashi and Takahashi [4], we introduce an iterative scheme for finding a common element of the set of solutions of EP(h) and the set of fixed points of infinite nonexpansive mappings in a Hilbert space. We obtain a strong convergence theorem which improves and extends the corresponding results of [2, 4].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Then for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection of H onto C. We know that P_C is nonexpansive. Further, for $x \in H$ and $x^* \in C$,

$$x^* = P_C(x) \Longleftrightarrow \langle x - x^*, x^* - y \rangle \geq 0 \quad \forall y \in C. \tag{2.1}$$

Recall that a mapping $T : C \to H$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Denote the set of fixed points of T by $F(T)$. It is well known that if C is a bounded closed convex and $T : C \to C$ is nonexpansive, then $F(T) \neq \emptyset$; see, for instance, [8]. We call a mapping $f : H \to H$ contractive if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha \|x - y\|$ for all $x, y \in H$.

For an equilibrium bifunction $h : C \times C \to R$, we call h satisfying condition (A) if h satisfies the following three conditions:

(i) h is monotone, that is, $h(x, y) + h(y, x) \leq 0$ for all $x, y \in C$;

(ii) for each $x, y, z \in C$, $\lim_{t \downarrow 0} h(tz + (1 - t)x, y) \leq h(x, y)$;

(iii) for each $x \in C$, $y \mapsto h(x, y)$ is convex and lower semicontinuous.

If an equilibrium bifunction $h : C \times C \to R$ satisfies condition (A), then we have the following two important results. You can find the first lemma in [1] and the second one in [2].

Lemma 2.1. *Let C be a nonempty closed convex subset of H and let h be an equilibrium bifunction of $C \times C$ into R, satisfying condition (A). Let $r > 0$ and $x \in H$. Then there exists $y \in C$ such that*

$$h(y, z) + \frac{1}{r} \langle z - y, y - x \rangle \geq 0 \quad \forall z \in C. \tag{2.2}$$

Lemma 2.2. *Assume that h satisfies the same assumptions as Lemma 2.1. For $r > 0$ and $x \in H$, define a mapping $S_r : H \to C$ as follows:*

$$S_r(x) = \left\{ y \in C : h(y, z) + \frac{1}{r} \langle z - y, y - x \rangle \geq 0, \ \forall z \in C \right\} \tag{2.3}$$

for all $y \in H$. Then the following holds:

(1) S_r *is single-valued and* S_r *is firmly nonexpansive, that is, for any* $x, y \in H$,

$$||S_r x - S_r y||^2 \leq \langle S_r x - S_r y, x - y \rangle; \tag{2.4}$$

(2) $F(S_r) = \mathrm{SEP}(h)$ *and* $\mathrm{SEP}(h)$ *is closed and convex.*

We also need the following lemmas for proving our main results.

Lemma 2.3 (see [9]). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. *Suppose* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then* $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 2.4 (see [10]). *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that* $a_{n+1} \leq (1-\gamma_n)a_n + \delta_n$, *where* $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that* $\sum_{n=1}^{\infty}\gamma_n = \infty$ *and* $\limsup_{n\to\infty}\delta_n/\gamma_n \leq 0$. *Then* $\lim_{n\to\infty}a_n = 0$.

3. Iterative scheme and strong convergence theorems

In this section, we first introduce our iterative scheme. Consequently, we will establish strong convergence theorems for this iteration scheme. To be more specific, let $T_1, T_2, \ldots$ be infinite mappings of C into C and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i \in N$. For any $n \in N$, define a mapping W_n of C into C as follows:

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I,\\
&\vdots\\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
&\vdots\\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
W_n = U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned} \tag{3.1}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$; see [11].

Now we introduce the following iteration scheme: Let f be a contraction of H into itself with coefficient $\alpha \in (0,1)$ and given $x_0 \in H$ arbitrarily. Suppose the sequences $\{x_n\}_{n=1}^{\infty}$

and $\{y_n\}_{n=1}^{\infty}$ are generated iteratively by

$$\begin{aligned} h(y_n,x)+\frac{1}{r_n}\langle x-y_n, y_n-x_n\rangle \geq 0, \quad \forall x\in C,\\ x_{n+1}=\alpha_n f(x_n)+\beta_n x_n+\gamma_n W_n y_n, \end{aligned} \tag{3.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n=1$, $\{r_n\}$ is a real sequence in $(0,\infty)$, h is an equilibrium bifunction, and W_n is the W-mapping defined by (3.1).

We have the following crucial conclusions concerning W_n. You can find them in [12, 13]. Now we only need the following similar version in Hilbert spaces.

Lemma 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T_1, T_2,… be nonexpansive mappings of C into C such that $\bigcap_{i=1}^{\infty}F(T_i)$ is nonempty, and let $\lambda_1,\lambda_2,\ldots$ be real numbers such that $0<\lambda_i\leq b<1$ for any $i\in N$. Then for every $x\in C$ and $k\in N$, the limit $\lim_{n\to\infty}U_{n,k}x$ exists.*

Remark 3.2. From Lemma 3.1, we have that if C is bounded, then for all $\varepsilon>0$, there exists a common positive integer number N_0 such that for $n>N_0$, $\|U_{n,k}x-U_k(x)\|<\varepsilon$ for all $x\in C$. Indeed, by the similar argument to Lemma 3.2 in [13], let $w\in\bigcap_{n=1}^{\infty}F(T_n)$. Since C is bounded, there exists a constant $M>0$ such that $\|x-w\|\leq M$ for all $x\in C$. Fix $k\in N$. Then for all $x\in C$ and any $n\in N$ with $n\geq k$, we have $\|U_{n+1,k}x-U_{n,k}x\|\leq 2(\prod_{i=k}^{n+1}\lambda_i)\|x-w\|\leq 2M(\prod_{i=k}^{n+1}\lambda_i)$.

Let $\varepsilon>0$. Then there exists $n_0\in N$ with $n_0\geq k$ such that for all $x\in C$, $b^{n_0-k+2}<\varepsilon(1-b)/2M$. So for all $x\in C$ and every m, n with $m>n>n_0$, we have

$$\begin{aligned} &\left\|U_{m,k}x-U_{n,k}x\right\|\\ &\quad\leq\sum_{j=n}^{m-1}\left\|U_{j+1,k}x-U_{j,k}x\right\|\leq\sum_{j=n}^{m-1}\left\{2\left(\prod_{i=k}^{j+1}\lambda_i\right)\|x-w\|\right\}\\ &\quad\leq 2M\sum_{j=n}^{m-1}b^{j-k+2}\leq\frac{2Mb^{n-k+2}}{1-b}<\varepsilon. \end{aligned} \tag{3.3}$$

Remark 3.3. Using Lemma 3.1, one can define a mapping W of C into C as $Wx=\lim_{n\to\infty}W_nx=\lim_{n\to\infty}U_{n,1}x$ for every $x\in C$. Such a W is called the W-mapping generated by $T_1,T_2,\ldots$ and $\lambda_1,\lambda_2,\ldots$. We observe that if $\{x_n\}$ is a bounded sequence in C, then we have

$$\lim_{n\to\infty}\left\|Wx_n-W_nx_n\right\|=0. \tag{3.4}$$

Indeed, from Remark 3.1, we have: for any $\varepsilon>0$, there is n_0 such that $\|Wx-W_nx\|\leq\varepsilon$ for all $x\in\{x_n\}$ and for all $n\geq n_0$. In particular, $\|Wx_n-W_nx_n\|\leq\varepsilon$ for all $n\geq n_0$. Consequently, $\lim_{n\to\infty}\|Wx_n-W_nx_n\|=0$, as claimed.

Throughout this paper, we will assume that $0<\lambda_i\leq b<1$ for every $i\in N$.

Lemma 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T_1, T_2,... be nonexpansive mappings of C into C such that $\bigcap_{i=1}^{\infty}F(T_i)$ is nonempty, and let $\lambda_1,\lambda_2,\ldots$ be real numbers such that $0<\lambda_i\le b<1$ for any $i\in N$. Then $F(W)=\bigcap_{i=1}^{\infty}F(T_i)$.*

Now we state and prove our main results.

Theorem 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $h: C\times C\to R$ be an equilibrium bifunction satisfying condition (A) and let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into C such that $\bigcap_{i=1}^{\infty}F(T_i)\cap \mathrm{SEP}(h)\neq\varnothing$. Suppose $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n=1$ and $\{r_n\}\subset(0,\infty)$. Suppose the following conditions are satisfied:*

(i) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=0}^{\infty}\alpha_n=\infty$;

(ii) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$;

(iii) $\liminf_{n\to\infty}r_n>0$ *and* $\lim_{n\to\infty}(r_{n+1}-r_n)=0$.

Let f be a contraction of H into itself and given $x_0\in H$ arbitrarily. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated iteratively by (3.2) converge strongly to $x^*\in\bigcap_{i=1}^{\infty}F(T_i)\cap\mathrm{SEP}(h)$, where $x^*=P_{\bigcap_{i=1}^{\infty}F(T_i)\cap\mathrm{SEP}(h)}f(x^*)$.

Proof. Let $Q=P_{\bigcap_{i=1}^{\infty}F(T_i)\cap\mathrm{SEP}(h)}$. Note that f is a contraction mapping with coefficient $\alpha\in(0,1)$. Then $\|Qf(x)-Qf(y)\|\le\|f(x)-f(y)\|\le\alpha\|x-y\|$ for all $x,y\in H$. Therefore, Qf is a contraction of H into itself, which implies that there exists a unique element $x^*\in H$ such that $x^*=Qf(x^*)$. At the same time, we note that $x^*\in C$.

Let $p\in\bigcap_{i=1}^{\infty}F(T_i)\cap\mathrm{SEP}(h)$. From the definition of S_r, we note that $y_n=S_{r_n}x_n$. It follows that $\|y_n-p\|=\|S_{r_n}x_n-S_{r_n}p\|\le\|x_n-p\|$. Next, we prove that $\{x_n\}$ and $\{y_n\}$ are bounded. From (3.1) and (3.2), we obtain

$$\begin{aligned}
\|x_{n+1}-p\| &\le \alpha_n\|f(x_n)-p\|+\beta_n\|x_n-p\|+\gamma_n\|W_ny_n-p\| \\
&\le \alpha_n(\|f(x_n)-f(p)\|+\|f(p)-p\|)+\beta_n\|x_n-p\|+\gamma_n\|y_n-p\| \\
&\le \alpha_n(\alpha\|x_n-p\|+\|f(p)-p\|)+(1-\alpha_n)\|x_n-p\| \\
&\le \max\left\{\|x_0-p\|,\frac{1}{1-\alpha}\|f(p)-p\|\right\}.
\end{aligned}\tag{3.5}$$

Therefore, $\{x_n\}$ is bounded. We also obtain that $\{y_n\}$, $\{W_nx_n\}$, and $\{f(x_n)\}$ are all bounded. We shall use M to denote the possible different constants appearing in the following reasoning.

Setting $x_{n+1}=\beta_nx_n+(1-\beta_n)z_n$ for all $n\ge0$, we have that

$$\begin{aligned}
z_{n+1}-z_n &= \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_nx_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1})-f(x_n))+\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)f(x_n) \\
&\quad+\frac{\gamma_{n+1}}{1-\beta_{n+1}}(W_{n+1}y_{n+1}-W_ny_n)+\left(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right)W_ny_n.
\end{aligned}\tag{3.6}$$

So we have

$$\begin{aligned}||z_{n+1}-z_n|| \le{}& \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}||x_{n+1}-x_n|| \\ &+ \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|(||f(x_n)||+||W_n y_n||) \\ &+ \frac{\gamma_{n+1}}{1-\beta_{n+1}}||W_{n+1}y_{n+1}-W_n y_n||.\end{aligned} \tag{3.7}$$

Since T_i and $U_{n,i}$ are nonexpansive from (3.1), we have

$$\begin{aligned}||W_{n+1}y_n - W_n y_n|| &= ||\lambda_1 T_1 U_{n+1,2}y_n - \lambda_1 T_1 U_{n,2}y_n|| \\ &\le \lambda_1||U_{n+1,2}y_n - U_{n,2}y_n|| \\ &\le \lambda_1\lambda_2||U_{n+1,3}y_n - U_{n,3}y_n|| \\ &\le \cdots \\ &\le \lambda_1\lambda_2\cdots\lambda_n||U_{n+1,n+1}y_n - U_{n,n+1}y_n|| \\ &\le M\prod_{i=1}^{n}\lambda_i,\end{aligned} \tag{3.8}$$

and hence

$$\begin{aligned}||W_{n+1}y_{n+1}-W_n y_n|| &\le ||W_{n+1}y_{n+1}-W_{n+1}y_n|| + ||W_{n+1}y_n - W_n y_n|| \\ &\le ||y_{n+1}-y_n|| + M\prod_{i=1}^{n}\lambda_i.\end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.7), we have

$$\begin{aligned}||z_{n+1}-z_n|| \le{}& \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}||x_{n+1}-x_n|| + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|(||f(x_n)||+||W_n y_n||) \\ &+ \frac{\gamma_{n+1}}{1-\beta_{n+1}}||y_{n+1}-y_n|| + \frac{My_{n+1}}{1-\beta_{n+1}}\prod_{i=1}^{n}\lambda_i.\end{aligned} \tag{3.10}$$

On the other hand, from $y_n = S_{r_n}x_n$ and $y_{n+1} = S_{r_{n+1}}x_{n+1}$, we have

$$h(y_n,x) + \frac{1}{r_n}\langle x-y_n, y_n-x_n\rangle \ge 0 \quad \forall x\in C, \tag{3.11}$$

$$h(y_{n+1},x) + \frac{1}{r_{n+1}}\langle x-y_{n+1}, y_{n+1}-x_{n+1}\rangle \ge 0 \quad \forall x\in C. \tag{3.12}$$

Putting $x = y_{n+1}$ in (3.11) and $x = y_n$ in (3.12), we have

$$h(y_n, y_{n+1}) + \frac{1}{r_n}\langle y_{n+1} - y_n, y_n - x_n\rangle \geq 0, \tag{3.13}$$

$$h(y_{n+1}, y_n) + \frac{1}{r_{n+1}}\langle y_n - y_{n+1}, y_{n+1} - x_{n+1}\rangle \geq 0. \tag{3.14}$$

From the monotonicity of h, we have

$$h(y_n, y_{n+1}) + h(y_{n+1}, y_n) \leq 0. \tag{3.15}$$

So from (3.13), we can conclude that

$$\left\langle y_{n+1} - y_n, \frac{y_n - x_n}{r_n} - \frac{y_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geq 0, \tag{3.16}$$

and hence

$$\left\langle y_{n+1} - y_n, y_n - y_{n+1} + y_{n+1} - x_n - \frac{r_n}{r_{n+1}}(y_{n+1} - x_{n+1})\right\rangle \geq 0. \tag{3.17}$$

Since $\liminf_{n\to\infty} r_n > 0$, without loss of generality, we may assume that there exists a real number τ such that $r_n > \tau > 0$ for all $n \in N$. Then we have

$$\begin{aligned} ||y_{n+1} - y_n||^2 &\leq \left\langle y_{n+1} - y_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(y_{n+1} - x_{n+1})\right\rangle \\ &\leq ||y_{n+1} - y_n||\left\{||x_{n+1} - x_n|| + \left|1 - \frac{r_n}{r_{n+1}}\right| ||y_{n+1} - x_{n+1}||\right\}, \end{aligned} \tag{3.18}$$

and hence

$$||y_{n+1} - y_n|| \leq ||x_{n+1} - x_n|| + \frac{M}{\tau}|r_{n+1} - r_n|. \tag{3.19}$$

Substituting (3.19) into (3.10), we have

$$\begin{aligned} ||z_{n+1} - z_n|| &\leq \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}||x_{n+1} - x_n|| + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|(||f(x_n)|| + ||W_n y_n||) \\ &\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}}||x_{n+1} - x_n|| + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \times \frac{M}{\tau}|r_{n+1} - r_n| + \frac{M\gamma_{n+1}}{1-\beta_{n+1}}\prod_{i=1}^{n}\lambda_i \\ &\leq ||x_{n+1} - x_n|| + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|(||f(x_n) + ||W_n y_n||) \\ &\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \times \frac{M}{\tau}|r_{n+1} - r_n| + \frac{M\gamma_{n+1}}{1-\beta_{n+1}}\prod_{i=1}^{n}\lambda_i. \end{aligned} \tag{3.20}$$

This together with $\alpha_n \to 0$ and $r_{n+1} - r_n \to 0$ imply that $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Hence by Lemma 2.3, we obtain $\|z_n - x_n\| \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty}||x_{n+1} - x_n|| = \lim_{n\to\infty}(1-\beta_n)||z_n - x_n|| = 0. \tag{3.21}$$

From (3.19) and $\lim_{n\to\infty}(r_{n+1}-r_n)=0$, we have $\lim_{n\to\infty}\|y_{n+1}-y_n\|=0$. Since $x_{n+1}=\alpha_n f(x_n)+\beta_n x_n+\gamma_n W_n y_n$, we have

$$\begin{aligned}||x_n-W_ny_n||&\le||x_n-x_{n+1}||+||x_{n+1}-W_ny_n||\\&\le||x_n-x_{n+1}||+\alpha_n||f(x_n)-W_ny_n||+\beta_n||x_n-W_ny_n||,\end{aligned}\tag{3.22}$$

that is,

$$||x_n-W_ny_n||\le\frac{1}{1-\beta_n}||x_n-x_{n+1}||+\frac{\alpha_n}{1-\beta_n}||f(x_n)-W_ny_n||.\tag{3.23}$$

Hence we have $\lim_{n\to\infty}\|x_n-W_ny_n\|=0$. For $p\in\bigcap_{i=1}^{\infty}F(T_i)\cap\mathrm{SEP}(h)$, note that S_r is firmly nonexpansive. Then we have

$$\begin{aligned}||y_n-p||^2&=||S_{r_n}x_n-S_{r_n}p||^2\\&\le\langle S_{r_n}x_n-S_{r_n}p,x_n-p\rangle\\&=\langle y_n-p,x_n-p\rangle\\&=\frac{1}{2}\left(||y_n-p||^2+||x_n-p||^2-||x_n-y_n||^2\right),\end{aligned}\tag{3.24}$$

and hence

$$||y_n-p||^2\le||x_n-p||^2-||x_n-y_n||^2.\tag{3.25}$$

Therefore, we have

$$\begin{aligned}||x_{n+1}-p||^2&\le\alpha_n||f(x_n)-p||^2+\beta_n||x_n-p||^2+\gamma_n||W_ny_n-p||^2\\&\le\alpha_n||f(x_n)-p||^2+\beta_n||x_n-p||^2+\gamma_n||y_n-p||^2\\&\le\alpha_n||f(x_n)-p||^2+\beta_n||x_n-p||^2\\&\quad+\gamma_n\left(||x_n-p||^2-||x_n-y_n||^2\right)\\&\le\alpha_n||f(x_n)-p||^2+||x_n-p||^2-\gamma_n||x_n-y_n||^2.\end{aligned}\tag{3.26}$$

Then we have

$$\begin{aligned}\gamma_n||x_n-y_n||^2&\le\alpha_n||f(x_n)-p||^2+(||x_n-p||+||x_{n+1}-p||)\\&\quad\times(||x_n-p||-||x_{n+1}-p||)\\&\le\alpha_n||f(x_n)-p||^2+||x_n-x_{n+1}||(||x_n-p||+||x_{n+1}-p||).\end{aligned}\tag{3.27}$$

It is easily seen that $\liminf_{n\to\infty}\gamma_n>0$. So we have

$$\lim_{n\to\infty}||x_n-y_n||=0.\tag{3.28}$$

From $\|W_ny_n-y_n\|\le\|W_ny_n-x_n\|+\|x_n-y_n\|$, we also have $\|W_ny_n-y_n\|\to0$. At that same time, we note that

$$||Wy_n-y_n||\le||Wy_n-W_ny_n||+||W_ny_n-y_n||.\tag{3.29}$$

It follows from (3.29) and Remark 3.2 that $\lim_{n\to\infty}\|Wy_n - y_n\| = 0$. Next, we show that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle \le 0, \tag{3.30}$$

where $x^* = P_{F(W)\cap \mathrm{SEP}(h)} f(x^*)$. First, we can choose a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\lim_{j\to\infty}\langle f(x^*) - x^*, y_{n_j} - x^*\rangle = \limsup_{n\to\infty}\langle f(x^*) - x^*, y_n - x^*\rangle. \tag{3.31}$$

Since $\{y_{n_j}\}$ is bounded, there exists a subsequence $\{y_{n_{ji}}\}$ of $\{y_{n_j}\}$, which converges weakly to w. Without loss of generality, we can assume that $y_{n_j}\to w$ weakly. From $\|Wy_n - y_n\|\to 0$, we obtain $Wy_{n_j}\to w$ weakly. Now we show $w\in \mathrm{SEP}(h)$.

By $y_n = S_{r_n}x_n$, we have $h(y_n,x) + (1/r_n)\langle x - y_n, y_n - x_n\rangle \ge 0$ for all $x\in C$. From the monotonicity of h, we have $(1/r_n)\langle x - y_n, y_n - x_n\rangle \ge -h(y_n,x) \ge h(x,y_n)$, and hence

$$\left\langle x - y_{n_j}, \frac{y_{n_j} - x_{n_j}}{r_{n_j}}\right\rangle \ge h(x,y_{n_j}). \tag{3.32}$$

Since $(y_{n_j} - x_{n_j})/r_{n_j}\to 0$ and $y_{n_j}\to w$ weakly, from the lower semicontinuity of $h(x,y)$ on the second variable y, we have $h(x,w)\le 0$ for all $x\in C$. For t with $0 < t \le 1$ and $x\in C$, let $x_t = tx + (1-t)w$. Since $x\in C$ and $w\in C$, we have $x_t\in C$, and hence $h(x_t,w)\le 0$. So from the convexity of equilibrium bifunction $h(x,y)$ on the second variable y, we have

$$0 = h(x_t,x_t) \le th(x_t,x) + (1-t)h(x_t,w) \le th(x_t,x). \tag{3.33}$$

Hence $h(x_t,x)\ge 0$. Therefore, we have $h(w,x)\ge 0$ for all $x\in C$, and hence $w\in \mathrm{SEP}(h)$.

We will show $w\in F(W)$. Assume $w\notin F(W)$. Since $y_{n_j}\to w$ weakly and $w\ne Ww$, from Opial's condition, we have

$$\begin{aligned}\liminf_{j\to\infty}||y_{n_j} - w|| &< \liminf_{j\to\infty}||y_{n_j} - Ww|| \\ &\le \liminf_{j\to\infty}(||y_{n_j} - Wy_{n_j}|| + ||Wy_{n_j} - Ww||) \\ &\le \liminf_{j\to\infty}||y_{n_j} - w||.\end{aligned} \tag{3.34}$$

This is a contradiction. So we get $w\in F(W) = \bigcap_{i=1}^{\infty}F(T_i)$. Therefore, $w\in\bigcap_{i=1}^{\infty}F(T_i)\cap \mathrm{SEP}(h)$. Since $x^* = P_{\bigcap_{i=1}^{\infty}F(T_i)\cap \mathrm{SEP}(h)} f(x^*)$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle &= \lim_{j\to\infty}\langle f(x^*) - x^*, x_{n_j} - x^*\rangle \\ &= \lim_{j\to\infty}\langle f(x^*) - x^*, y_{n_j} - x^*\rangle \\ &= \langle f(x^*) - x^*, w - x^*\rangle \le 0.\end{aligned} \tag{3.35}$$

First, we prove that $\{x_n\}$ converges strongly to $x^* \in \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{SEP}(h)$. From (3.2), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \|\beta_n(x_n - x^*) + \gamma_n(W_n y_n - x^*)\|^2 + 2\alpha_n \langle f(x_n) - x^*, x_{n+1} - x^* \rangle \\
&\le \{\beta_n \|x_n - x^*\| + \gamma_n \|W_n y_n - x^*\|\}^2 + 2\alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&\quad + 2\alpha_n \langle f(x_n) - f(x^*), x_{n+1} - x^* \rangle \\
&\le \{\beta_n \|x_n - x^*\| + \gamma_n \|y_n - x^*\|\}^2 + 2\alpha\alpha_n \|x_n - x^*\| \|x_{n+1} - x^*\| \\
&\quad + 2\alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&\le (1 - \alpha_n)^2 \|x_n - x^*\|^2 + \alpha\alpha_n (\|x_{n+1} - x^*\|^2 + \|x_n - x^*\|^2) \\
&\quad + 2\alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle,
\end{aligned} \tag{3.36}$$

which implies that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \frac{(1 - \alpha_n)^2 + \alpha\alpha_n}{1 - \alpha\alpha_n} \|x_n - x^*\|^2 + \frac{2\alpha_n}{1 - \alpha\alpha_n} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&= \frac{1 - 2\alpha_n + \alpha\alpha_n}{1 - \alpha\alpha_n} \|x_n - x^*\|^2 + \frac{\alpha_n^2}{1 - \alpha\alpha_n} \|x_n - x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1 - \alpha\alpha_n} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&\le \left\{1 - \frac{2(1 - \alpha)\alpha_n}{1 - \alpha\alpha_n}\right\} \|x_n - x^*\|^2 + \frac{2(1 - \alpha)\alpha_n}{1 - \alpha\alpha_n} \\
&\quad \times \left\{\frac{M\alpha_n}{2(1 - \alpha)} + \frac{1}{1 - \alpha} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle\right\} \\
&= (1 - \varphi_n) \|x_n - x^*\|^2 + \phi_n \varphi_n,
\end{aligned} \tag{3.37}$$

where $\varphi_n = 2(1 - \alpha)\alpha_n / (1 - \alpha\alpha_n)$ and $\phi_n = M\alpha_n / 2(1 - \alpha) + 1/(1 - \alpha)\langle f(x^*) - x^*, x_{n+1} - x^* \rangle$. It is easily seen that $\sum_{n=0}^{\infty} \varphi_n = \infty$ and $\limsup_{n \to \infty} \phi_n \le 0$. Now applying Lemma 2.4 and (3.35) to (3.37), we conclude that $x_n \to x^*$ $(n \to \infty)$. Consequently, from (3.28), we have $y_n \to x^*$ $(n \to \infty)$. This completes the proof. □

Corollary 3.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $h : C \times C \to R$ be an equilibrium bifunction satisfying condition (A) such that $\mathrm{SEP}(h) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ and $\{r_n\} \subset (0, \infty)$ is a real sequence. Suppose the following conditions are satisfied:*

(i) $\lim_{n \to \infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n \to \infty} \beta_n \le \limsup_{n \to \infty} \beta_n < 1$;

(iii) $\liminf_{n \to \infty} r_n > 0$ *and* $\lim_{n \to \infty} (r_{n+1} - r_n) = 0$.

Let f be a contraction of H into itself and given $x_0 \in H$ arbitrarily. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated iteratively by

$$\begin{aligned} h(y_n, x) + \frac{1}{r_n}\langle x - y_n, y_n - x_n\rangle \geq 0, \quad \forall x \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n. \end{aligned} \tag{3.38}$$

Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (3.38) converge strongly to $x^* \in \mathrm{SEP}(h)$, where $x^* = P_{\mathrm{SEP}(h)} f(x^*)$.

Proof. Take $T_i x = x$ for all $i = 1, 2, \ldots$ and for all $x \in C$ in (3.1). Then $W_n x = x$ for all $x \in C$. The conclusion follows from Theorem 3.1. This completes the proof. □

Corollary 3.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into C such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Suppose the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Let f be a contraction of H into itself and given $x_0 \in H$ arbitrarily. Let $\{x_n\}$ be a sequence generated iteratively by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n P_C x_n. \tag{3.39}$$

Then the sequence $\{x_n\}$ converges strongly to $x^* = P_{\bigcap_{i=1}^{\infty} F(T_i)} f(x^*)$.

Proof. Set $h(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in N$. Then we have $y_n = P_C x_n$. From (3.2), we have

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n P_C x_n. \tag{3.40}$$

Then the conclusion follows from Theorem 3.5. This completes the proof. □

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