

Research Article

The Equivalence between T -Stabilities of The Krasnoselskij and The Mann Iterations

Ştefan M. Şoltuz

Received 20 June 2007; Accepted 14 September 2007

Recommended by Hichem Ben-El-Mechaiekh

We prove the equivalence between the T -stabilities of the Krasnoselskij and the Mann iterations; a consequence is the equivalence with the T -stability of the Picard-Banach iteration.

Copyright © 2007 Ştefan M. Şoltuz. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Let X be a normed space and T a selfmap of X . Let x_0 be a point of X , and assume that $x_{n+1} = f(T, x_n)$ is an iteration procedure, involving T , which yields a sequence $\{x_n\}$ of points from X . Suppose $\{x_n\}$ converges to a fixed point x^* of T . Let $\{\xi_n\}$ be an arbitrary sequence in X , and set $\epsilon_n = \|\xi_{n+1} - f(T, \xi_n)\|$ for all $n \in \mathbb{N}$.

Definition 1.1 [1]. If $(\lim_{n \rightarrow \infty} \epsilon_n = 0) \Rightarrow (\lim_{n \rightarrow \infty} \xi_n = p)$, then the iteration procedure $x_{n+1} = f(T, x_n)$ is said to be T -stable with respect to T .

Remark 1.2 [1]. In practice, such a sequence $\{\xi_n\}$ could arise in the following way. Let x_0 be a point in X . Set $x_{n+1} = f(T, x_n)$. Let $\xi_0 = x_0$. Now $x_1 = f(T, x_0)$. Because of rounding or discretization in the function T , a new value ξ_1 approximately equal to x_1 might be obtained instead of the true value of $f(T, x_0)$. Then to approximate x_2 , the value $f(T, \xi_1)$ is computed to yield ξ_2 , an approximation of $f(T, \xi_1)$. This computation is continued to obtain $\{\xi_n\}$ an approximate sequence of $\{x_n\}$.

Let X be a normed space, D a nonempty, convex subset of X , and T a selfmap of D , let $p_0 = e_0 \in D$. The Mann iteration (see [2]) is defined by

$$e_{n+1} = (1 - \alpha_n)e_n + \alpha_n T e_n, \quad (1.1)$$

2 Fixed Point Theory and Applications

where $\{\alpha_n\} \subset (0, 1)$. The Ishikawa iteration is defined (see [3]) by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \end{aligned} \quad (1.2)$$

where $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1)$. The Krasnoselskij iteration (see [4]) is defined by

$$p_{n+1} = (1 - \lambda)p_n + \lambda T p_n, \quad (1.3)$$

where $\lambda \in (0, 1)$. Recently, the equivalence between the T -stabilities of Mann and Ishikawa iterations, respectively, for modified Mann-Ishikawa iterations was shown in [5]. In the present paper, we shall prove the equivalence between the T -stabilities of the Krasnoselskij and the Mann iterations. Next, $\{u_n\}, \{v_n\} \subset X$ are arbitrary.

Definition 1.3.

- (i) The Mann iteration (1.1) is said to be T -stable if and only if for all $\{\alpha_n\} \subset (0, 1)$ and for every sequence $\{u_n\} \subset X$,

$$\lim_{n \rightarrow \infty} \varepsilon_n = 0 \implies \lim_{n \rightarrow \infty} u_n = x^*, \quad (1.4)$$

where $\varepsilon_n := \|u_{n+1} - (1 - \alpha_n)u_n - \alpha_n T u_n\|$.

- (ii) The Krasnoselskij iteration (1.3) is said to be T -stable if and only if for all $\lambda \in (0, 1)$, and for every sequence $\{v_n\} \subset X$,

$$\lim_{n \rightarrow \infty} \delta_n = 0 \implies \lim_{n \rightarrow \infty} v_n = x^*, \quad (1.5)$$

where $\delta_n := \|v_{n+1} - (1 - \lambda)v_n - \lambda T v_n\|$.

2. Main results

THEOREM 2.1. *Let X be a normed space and $T : X \rightarrow X$ a map with bounded range and $\{\alpha_n\} \subset (0, 1)$ satisfy $\lim_{n \rightarrow \infty} \alpha_n = \lambda$, $\lambda \in (0, 1)$. Then the following are equivalent:*

- (i) *the Mann iteration is T -stable,*
- (ii) *the Krasnoselskij iteration is T -stable.*

Proof. We prove that (i) $\Rightarrow$ (ii). If $\lim_{n \rightarrow \infty} \delta_n = 0$, then $\{v_n\}$ is bounded. Set

$$M_1 := \max \left\{ \sup_{x \in X} \{ \|T(x)\| \}, \|v_0\|, \|u_0\| \right\}. \quad (2.1)$$

Observe that $\|v_1\| \leq \delta_0 + (1 - \lambda)\|v_0\| + \lambda\|T v_0\| \leq \delta_0 + M_1$. Set $M := M_1 + 1/\lambda$. Suppose that $\|v_n\| \leq M$ to prove that $\|v_{n+1}\| \leq M$. Remark that

$$\begin{aligned} \|v_{n+1}\| &\leq \delta_n + (1 - \lambda)\delta_{n-1} + \cdots + (1 - \lambda)^n \delta_0 + M_1 \\ &\leq 1 + (1 - \lambda) + \cdots + (1 - \lambda)^n + M_1 \\ &\leq \frac{1}{1 - (1 - \lambda)} + M_1 = M. \end{aligned} \quad (2.2)$$

Suppose that $\lim_{n \rightarrow \infty} \delta_n = 0$ to note that

$$\begin{aligned}
 \varepsilon_n &= \|v_{n+1} - (1 - \alpha_n)v_n - \alpha_n T v_n\| \\
 &= \|v_{n+1} - v_n + \lambda v_n - \lambda v_n + \alpha_n v_n - \lambda T v_n + \lambda T v_n - \alpha_n T v_n\| \\
 &\leq \|v_{n+1} - (1 - \lambda)v_n - \lambda T v_n\| + |\lambda - \alpha_n| \|v_n - T v_n\| \\
 &\leq \|v_{n+1} - (1 - \lambda)v_n - \lambda T v_n\| + 2M |\lambda - \alpha_n| \\
 &= \delta_n + 2M |\lambda - \alpha_n| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
 \end{aligned} \tag{2.3}$$

Condition (i) assures that if $\lim_{n \rightarrow \infty} \varepsilon_n = 0$, then $\lim_{n \rightarrow \infty} v_n = x^*$. Thus, for a $\{v_n\}$ satisfying

$$\lim_{n \rightarrow \infty} \delta_n = \lim_{n \rightarrow \infty} \|v_{n+1} - (1 - \lambda)v_n - \lambda T v_n\| = 0, \tag{2.4}$$

we have shown that $\lim_{n \rightarrow \infty} v_n = x^*$.

Conversely, we prove (ii) $\Rightarrow$ (i). First, we prove that $\{u_n\}$ is bounded. Since $\lim_{n \rightarrow \infty} \alpha_n = \lambda$, for $\beta \in (0, 1)$ given, there exists $n_0 \in N$, such that $1 - \alpha_n \leq \beta$, for all $n \geq n_0$. Set $M_1 := \max\{\sup_{x \in X} \|Tx\|, \|u_0\|\}$ and $M := n_0 + 1 + \beta/(1 - \beta) + M_1$ to obtain

$$\begin{aligned}
 \|u_{n+1}\| &\leq [\varepsilon_n + (1 - \alpha_1)\varepsilon_{n-1} + (1 - \alpha_1)(1 - \alpha_2)\varepsilon_{n-2} \\
 &\quad + \cdots + (1 - \alpha_1)(1 - \alpha_2) \cdots (1 - \alpha_{n_0})\varepsilon_{n-n_0}] \\
 &\quad + (1 - \alpha_1)(1 - \alpha_2) \cdots (1 - \alpha_{n_0})(1 - \alpha_{n_0+1})\varepsilon_{n-n_0-1} \\
 &\quad + \cdots + (1 - \alpha_1)(1 - \alpha_2) \cdots (1 - \alpha_n)\varepsilon_0 + M_1 \\
 &\leq (n_0 + 1) + (1 - \alpha_{n_0+1}) + (1 - \alpha_{n_0+1})(1 - \alpha_{n_0+2}) \cdots \\
 &\quad + (1 - \alpha_{n_0+1}) \cdots (1 - \alpha_n)\varepsilon_0 + M_1 \\
 &\leq n_0 + 1 + \beta + \beta^2 + \cdots + \beta^{n-n_0} + M_1 < M.
 \end{aligned} \tag{2.5}$$

Suppose $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. Observe that

$$\begin{aligned}
 \delta_n &= \|u_{n+1} - (1 - \lambda)u_n - \lambda T u_n\| \\
 &= \|u_{n+1} - u_n + \lambda u_n - \lambda T u_n + \alpha_n u_n - \alpha_n u_n - \alpha_n T u_n + \alpha_n T u_n\| \\
 &\leq \|u_{n+1} - (1 - \alpha_n)u_n - \alpha_n T u_n\| + |\lambda - \alpha_n| \|u_n - T u_n\| \\
 &\leq \|u_{n+1} - (1 - \alpha_n)u_n - \alpha_n T u_n\| + 2M |\lambda - \alpha_n| \\
 &= \varepsilon_n + 2M |\lambda - \alpha_n| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
 \end{aligned} \tag{2.6}$$

4 Fixed Point Theory and Applications

Condition (ii) assures that if $\lim_{n \rightarrow \infty} \delta_n = 0$, then $\lim_{n \rightarrow \infty} v_n = x^*$. Thus, for a $\{u_n\}$ satisfying

$$\lim_{n \rightarrow \infty} \varepsilon_n = \lim_{n \rightarrow \infty} \|u_{n+1} - (1 - \alpha_n)u_n - \alpha_n T u_n\| = 0, \quad (2.7)$$

we have shown that $\lim_{n \rightarrow \infty} u_n = x^*$. $\square$

Remark 2.2. Let X be a normed space and $T : X \rightarrow X$ a map with bounded range and $\{\alpha_n\} \subset (0, 1)$ satisfy $\lim_{n \rightarrow \infty} \alpha_n = \lambda$, $\lambda \in (0, 1)$. If the Mann iteration is not T -stable, then the Krasnoselskij iteration is not T -stable, and conversely.

Example 2.3. Let $T : [0, 1] \rightarrow [0, 1]$ be given by $Tx = x^2$, and $\lambda = 1/2$. Then the Krasnoselskij iteration converges to the unique fixed point $x^* = 0$, and it is not T -stable.

The Krasnoselskij iteration converges because, supposing $F := \sup_n p_n < 1$, the sequence $p_n \rightarrow 0$, as we can see from

$$\begin{aligned} p_{n+1} &= \left(1 - \frac{1}{2}\right)p_n + \frac{1}{2}p_n^2 = \frac{1}{2}p_n + \frac{1}{2}p_n^2 \\ &= \frac{1}{2}p_n(1 + p_n) \leq \frac{1+F}{2}p_n = \left(\frac{1+F}{2}\right)^n p_0 \rightarrow 0; \end{aligned} \quad (2.8)$$

set $v_n = n/(n+1)$ and note that v_n does not converge to zero, while δ_n does:

$$\delta_n = \left| \frac{n+1}{n+2} - \frac{1}{2} \frac{n}{n+1} - \frac{1}{2} \frac{n^2}{(n+1)^2} \right| = \frac{n^2 + 4n + 2}{2(n+1)^2(n+2)} \rightarrow 0. \quad (2.9)$$

The Mann iteration also converges because (supposing $E := \sup_n e_n < 1$) one has

$$\begin{aligned} e_{n+1} &= (1 - \alpha_n)e_n + \alpha_n e_n^2 = (1 - (1 - E)\alpha_n)e_n \\ &\leq \prod_{k=1}^n (1 - (1 - E)\alpha_k) e_0 \leq \exp\left(- (1 - E) \sum_{k=1}^n \alpha_k\right) e_0 \rightarrow 0; \end{aligned} \quad (2.10)$$

the last inequality is true because $1 - x \leq \exp(-x)$, $\forall x \geq 0$, and $\sum \alpha_n = +\infty$.

Take $u_n = n/(n+1) \rightarrow 1$, and note that $\varepsilon_n \rightarrow 0$ because

$$\varepsilon_n = \left| \frac{n+1}{n+2} - (1 - \alpha_n) \frac{n}{n+1} - \alpha_n \frac{n^2}{(n+1)^2} \right| = \frac{\alpha_n n^2 + (2\alpha_n + 1)n + 1}{(n+1)^2(n+2)}. \quad (2.11)$$

So the Mann iteration is not T -stable. Actually, by use of Theorem 2.1, one can easily obtain the non- T -stability of the other iteration, provided that the previous one is not stable.

The following result takes in consideration the case in which no condition on $\{\alpha_n\}$ are imposed.

THEOREM 2.4. *Let X be a normed space and $T : X \rightarrow X$ a map, and $\{\alpha_n\} \subset (0, 1)$. If*

$$\lim_{n \rightarrow \infty} \|v_n - T v_n\| = 0, \quad \lim_{n \rightarrow \infty} \|u_n - T u_n\| = 0, \quad (2.12)$$

then the following are equivalent:

- (i) the Mann iteration is T -stable,
- (ii) the Krasnoselskij iteration is T -stable.

Proof. We prove that (i) $\Rightarrow$ (ii). Suppose $\lim_{n \rightarrow \infty} \delta_n = 0$, to note that,

$$\begin{aligned} \varepsilon_n &= \|v_{n+1} - (1 - \alpha_n)v_n - \alpha_n T v_n\| \\ &= \|v_{n+1} - v_n + \lambda v_n - \lambda v_n + \alpha_n v_n - \lambda T v_n + \lambda T v_n - \alpha_n T v_n\| \\ &\leq \|v_{n+1} - (1 - \lambda)v_n - \lambda T v_n\| + |\lambda - \alpha_n| \|v_n - T v_n\| \\ &\leq \delta_n + 2\|v_n - T v_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{2.13}$$

Condition (i) assures that if $\lim_{n \rightarrow \infty} \varepsilon_n = 0$, then $\lim_{n \rightarrow \infty} v_n = x^*$. Thus, for a $\{v_n\}$ satisfying

$$\lim_{n \rightarrow \infty} \delta_n = \lim_{n \rightarrow \infty} \|v_{n+1} - (1 - \lambda)v_n - \lambda T v_n\| = 0, \tag{2.14}$$

we have shown that $\lim_{n \rightarrow \infty} v_n = x^*$.

Conversely, we prove (ii) $\Rightarrow$ (i). Suppose $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. Observe that

$$\begin{aligned} \delta_n &= \|u_{n+1} - (1 - \lambda)u_n - \lambda T u_n\| \\ &= \|u_{n+1} - u_n + \lambda u_n - \lambda T u_n + \alpha_n u_n - \alpha_n u_n - \alpha_n T u_n + \alpha_n T u_n\| \\ &\leq \|u_{n+1} - (1 - \alpha_n)u_n - \alpha_n T u_n\| + |\lambda - \alpha_n| \|u_n - T u_n\| \\ &\leq \varepsilon_n + 2\|u_n - T u_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{2.15}$$

Condition (ii) assures that if $\lim_{n \rightarrow \infty} \delta_n = 0$, then $\lim_{n \rightarrow \infty} v_n = x^*$. Thus, for a $\{u_n\}$ satisfying

$$\lim_{n \rightarrow \infty} \varepsilon_n = \lim_{n \rightarrow \infty} \|u_{n+1} - (1 - \alpha_n)u_n - \alpha_n T u_n\| = 0, \tag{2.16}$$

we have shown that $\lim_{n \rightarrow \infty} u_n = x^*$. □

Remark 2.5. Let X be a normed space and $T : X \rightarrow X$ a map, $\{\alpha_n\} \subset (0, 1)$ and $\lim_{n \rightarrow \infty} \|v_n - T v_n\| = 0$, $\lim_{n \rightarrow \infty} \|u_n - T u_n\| = 0$. If the Mann iteration is not T -stable, then the Krasnoselskij iteration is not T -stable, and conversely.

Note that one can consider the usual conditions $\lambda = 1/2$, $\lim \alpha_n = 0$, and $\sum \alpha_n = \infty$ in Theorem 2.4 and Remark 2.5.

Example 2.6. Again, let $T : [0, 1) \rightarrow [0, 1)$ be given by $Tx = x^2$, and $\lambda = 1/2$, $\alpha_n = 1/n$. Set $v_n = u_n = n/(n+1)$, to note that $\lim_{n \rightarrow \infty} u_n = 1$, and

$$\lim_{n \rightarrow \infty} \|v_n - T v_n\| = \lim_{n \rightarrow \infty} \frac{n}{(n+1)^2} = 0. \tag{2.17}$$

Hence, neither the Mann nor the Krasnoselskij iteration is T -stable, as we can see from Example 2.3.

3. Further results

Let $q_0 \in X$ be fixed, and let $q_{n+1} = Tq_n$ be the Picard-Banach iteration.

Definition 3.1. The Picard iteration is said to be T -stable if and only if for every sequence $\{q_n\} \subset X$ given,

$$\lim_{n \rightarrow \infty} \Delta_n = 0 \implies \lim_{n \rightarrow \infty} q_n = x^*, \quad (3.1)$$

where $\Delta_n := \|q_{n+1} - Tq_n\|$.

In [6], the equivalence between the T -stabilities of Picard-Banach iteration and Mann iteration is given, that is, the following holds.

THEOREM 3.2 [6]. *Let X be a normed space and $T : X \rightarrow X$ a map. If*

$$\lim_{n \rightarrow \infty} \|q_n - Tq_n\| = 0, \quad \lim_{n \rightarrow \infty} \|u_n - Tu_n\| = 0, \quad (3.2)$$

then the following are equivalent:

- (i) *for all $\{\alpha_n\} \subset (0, 1)$, the Mann iteration is T -stable,*
- (ii) *the Picard iteration is T -stable.*

Theorems 2.4 and 3.2 lead to the following conclusion.

COROLLARY 3.3. *Let X be a normed space and $T : X \rightarrow X$ a map. If*

$$\lim_{n \rightarrow \infty} \|q_n - Tq_n\| = 0, \quad \lim_{n \rightarrow \infty} \|v_n - Tv_n\| = 0, \quad \lim_{n \rightarrow \infty} \|u_n - Tu_n\| = 0, \quad (3.3)$$

then the following are equivalent:

- (i) *for all $\{\alpha_n\} \subset (0, 1)$, the Mann iteration is T -stable,*
- (ii) *the Picard-Banach iteration is T -stable,*
- (iii) *the Krasnoselskij iteration is T -stable.*

Remark 3.4. Let X be a normed space and $T : X \rightarrow X$ a map, $\{\alpha_n\} \subset (0, 1)$ and $\lim_{n \rightarrow \infty} \|q_n - Tq_n\| = 0$, $\lim_{n \rightarrow \infty} \|v_n - Tv_n\| = 0$, $\lim_{n \rightarrow \infty} \|u_n - Tu_n\| = 0$. If the Mann or Krasnoselskij iteration is not T -stable, then the Picard-Banach iteration is not T -stable, and conversely.

Example 3.5. To see that the Picard-Banach iteration is also not T -stable, consider $T : [0, 1) \rightarrow [0, 1)$, by $Tx = x^2$.

Indeed, setting $q_n = n/(n+1)$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} q_n &= \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1, \\ \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} - \left(\frac{n}{n+1} \right)^2 \right| &= \frac{n}{(n+1)^2} = 0. \end{aligned} \quad (3.4)$$

Acknowledgment

The author is indebted to referee for carefully reading the paper and for making useful suggestions.

References

- [1] A. M. Harder and T. L. Hicks, "Stability results for fixed point iteration procedures," *Mathematica Japonica*, vol. 33, no. 5, pp. 693–706, 1988.
- [2] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, no. 3, pp. 506–510, 1953.
- [3] S. Ishikawa, "Fixed points by a new iteration method," *Proceedings of the American Mathematical Society*, vol. 44, no. 1, pp. 147–150, 1974.
- [4] M. A. Krasnosel'skiĭ, "Two remarks on the method of successive approximations," *Uspekhi Matematicheskikh Nauk*, vol. 10, no. 1(63), pp. 123–127, 1955.
- [5] B. E. Rhoades and Ş. M. Şoltuz, "The equivalence between the T -stabilities of Mann and Ishikawa iterations," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 2, pp. 472–475, 2006.
- [6] Ş. M. Şoltuz, "The equivalence between the T -stabilities of Picard-Banach and Mann-Ishikawa iterations," to appear in *Applied Mathematics E—Notes*.

Ştefan M. Şoltuz: Departamento de Matematicas, Universidad de Los Andes, Carrera 1 no. 18A-10, Bogota, Colombia

Current address: Tiberiu Popoviciu Institute of Numerical Analysis, 400110 Cluj-Napoca, Romania

Email address: smsoltuz@gmail.com

Special Issue on Modeling Experimental Nonlinear Dynamics and Chaotic Scenarios

Call for Papers

Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the *Mathematical Problems in Engineering* aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at <http://www.hindawi.com/journals/mpe/>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/> according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

José Roberto Castilho Piqueira, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil ; elbert@lac.inpe.br

Celso Grebogi, Center for Applied Dynamics Research, King's College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk