NOTE ON KKM MAPS AND APPLICATIONS

Y. Q. CHEN, Y. J. CHO, J. K. KIM, AND B. S. LEE

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We apply the KKM technique to study fixed point theory, minimax inequality and coincidence theorem. Some new results on Fan-Browder fixed point theorem, Fan's minimax theorem and coincidence theorem are obtained.

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1. Introduction

In 1929, the KKM map was introduced by Knaster et al. [13] and it provides the foundation for many well-known existence results, such as Ky Fan's minimax inequality theorem, Ky Fan-Browder's fixed point theorem, Nash's equilibrium theorem, Hartman-Stampacchia's variational inequality theorem and many others (see [1, 2, 5–12, 14–17]). The central idea of applying KKM theory to prove that a family of sets has nonempty intersection is to find a suitable space and a mapping defined on that space such that this mapping is a KKM mapping and the original family of sets has finite intersection property provided the resulted family of sets by this mapping has finite intersection property. Based this idea, we first introduce a large class of mappings that can be interpreted as KKM mappings, then we apply the KKM technique to study fixed point theory, minimax inequality and coincidence theorem. A new concept on lower (upper) semi-continuous function is given and some new results on Fan-Browder's fixed point theorem, Fan's minimax theorem and coincidence theorem are obtained.

2. The KKM maps

In the sequel, let X be a set and 2^X be the collection of nonempty subsets of X. To begin our results, let us first recall the following definition.

Definition 2.1. Let E be a subset of topological vector space X. A map $G : E \to 2^X$ is called a *KKM map* if

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$$\operatorname{co}\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} G(x_i) \tag{2.1}$$

for $x_i \in E$, $i = 1, 2, \ldots, n$.

Definition 2.2. Let E be a set and X be a topological space. A map $G : E \to 2^X$ is called a *map with the KKM property* if there exists a topological vector space Y such that, for any $\{x_i : 1 \le i \le n\} \subseteq E$, there exist $F = \{y_i : 1 \le i \le n\} \subseteq Y$, a closed (or closed under appropriate topology) mapping $L : X \to Y$ or 2^Y, that is, maps closed set to closed set, and $G' : F \to 2^X$ with $G'(y_i) \subseteq G(x_i)$ for $i = 1, 2, \ldots, n$ such that the composition mapping $LG' : F \to 2^Y$ defined by $LG'(f) = \cup_{x \in G'(f)} L(x)$ for $f \in F$ is a KKM map and $\cap_{i=1}^{n} LG'(y_i) \neq \emptyset$ implies that $\cap_{i=1}^{n} G(x_i) \neq \emptyset$.

Remark 2.3. Definition 2.2 simply says that the map G has the KKM property if G or the part of G can be mapped onto another space such that the composite map is a KKM map. One can easily check that the generalized KKM map in [4, 18] is a map with the KKM property.

In the following, we give some examples of maps with the KKM property.

Example 2.4. Let $E = [0, 1]$ be the closed interval of R, $X = R$, and let $G : E \to 2^X$ be a map with $G(x) = (1, 2 + x)$ for $x \in E$. For any $(x_i) \subset [0, 1]$, $i = 1, 2, \ldots, n$, put $y_i = 3/2 + x_i$, $F = \{y_1, y_2, \ldots, y_n\}$, $Y = R$, and define $G' : F \to 2^Y$ by $G'(y_i) = [3/2, 7/4 + x_i]$. Take L as the identity mapping on R. Then the map $LG' = G'$ is a KKM map and so G is a map with the KKM property.

Example 2.5. Let $\phi : [0, \infty) \to R$ be the convex function defined by

$$\phi(x) = \begin{cases} 1 & \text{if } x = 0, \\ (x-1)^2 - 1 & \text{if } x > 0. \end{cases} \tag{2.2}$$

Define $G : [0, \infty) \to 2^R$ by $G(x) = \{y : \phi(y) \le \phi(x)\}$. It is easy to see that ϕ is not lower semi-continuous at 0 and so $G(2) = \{y : \phi(y) \le \phi(2)\}$ is not closed. For $\{x_i : 1 \le i \le n\} \subset [0, \infty)$, if $\phi(x_i) < 0$ or $\phi(x_i) \ge 1$, we set $y_i = x_i$, otherwise, set $y_i = x_i/2$. Put $F = \{y_1, y_2, \ldots, y_n\}$, $X = Y = R$, and define $G : F \to 2^X$ by $G'(y_i) = \{y : \phi(y) \le \phi(y_i)\}$. Take L as the identity mapping on R. Then $LG' = G'$ is a KKM map on $F = \{y_i : 1 \le i \le n\}$, thus G is a map with the KKM property.

The following results are direct consequences of the KKM theorem.

Theorem 2.6. *Let X be a topological space and E be a set. Suppose that $G : E \to 2^X$ is a closed valued map with the KKM property. Then $\{G(x)\}_{x \in E}$ has a finite intersection property.*

Theorem 2.7 (Ky Fan's theorem). *Let X be a topological space and E be a subset of X. If $G : E \to 2^X$ is a closed valued map with the KKM property and there is a set $G(x)$ such that $G(x)$ is compact. Then $\cap_{x \in E} G(x) \neq \emptyset$.*

3. Fan-Browder's fixed point theorem without compactness condition

The following result is a generalization of Fan-Browder's fixed point theorem without compactness condition.

Theorem 3.1 (Fan-Browder's fixed point theorem). *Let E be a convex subset of a vector space X and $G : E \to 2^E$ be a map satisfying the following conditions:*

(1) *there exists $\{y_i : 1 \le i \le n\} \subset E$ such that $\mathrm{co}\{y_i : 1 \le i \le n\} \subseteq \cup_{i=1}^{n} G^{-1}(y_i)$ and $G^{-1}(y_i) \cap \mathrm{co}\{y_i : 1 \le i \le n\}$ is open in $\mathrm{co}\{y_i : 1 \le i \le n\}$ with $\mathrm{co}\{y_i : 1 \le i \le n\}$ inherited with the Euclidean topology, where $G^{-1}(y) = \{x \in E : y \in G(x)\}$;*

(2) *$G(y)$ is convex for all $y \in E$.*

Then G has a fixed point.

Proof. Let $F = \{y_i : 1 \le i \le n\}$. Define a map $K : F \to 2^{\mathrm{co}F}$ by

$$K(y_i) = \mathrm{co}F \setminus G^{-1}(y_i) \bigcap \mathrm{co}F \tag{3.1}$$

for $i = 1,2,\ldots,n$. We may assume that $K(y_i) \neq \emptyset$ for $i = 1,2,\ldots$. (Otherwise, $K(y_i) = \emptyset$ for some i and so we have $\mathrm{co}F \subset G^{-1}(y_i)$. Thus y_i is a fixed point of G, and the conclusion holds.) One can easily see that

$$\bigcap_{i=1}^{n} K(y_i) = \mathrm{co}F \setminus \bigcup_{i=1}^{n} G^{-1}(y_i) \bigcap \mathrm{co}F. \tag{3.2}$$

By assumption (1), we have $\cap_{i=1}^{n} K(y_i) = \emptyset$. In view of Theorem 2.6, K cannot be a KKM map on $\{y_i : 1 \le i \le n\}$. Hence there exist $y_{i_1}, y_{i_2}, \ldots, y_{i_k}$ such that $\mathrm{co}\{y_{i_1}, y_{i_2}, \ldots, y_{i_k}\} \nsubseteq \cup_{j=1}^{k} K(y_{i_j})$, that is, there exists $y \in \mathrm{co}\{y_{i_1}, y_{i_2}, \ldots, y_{i_k}\}$ such that $y \notin K(y_{i_j})$ for $j = 1,2,\ldots,k$. Thus we have

$$y \in G^{-1}(y_{i_j}), \quad j = 1,2,\ldots,k, \tag{3.3}$$

that is, $y_{i_j} \in G(y)$ for $j = 1,2,\ldots,k$ and the convexity of $G(y)$ immediately implies that $y \in G(y)$. This completes the proof. □

Remark 3.2. Theorem 3.1 only requires the intersection $G^{-1}(y) \cap \mathrm{co}F$ for $y \in F$ is relatively open in the convex hull of some finite subset F of E and also E is not compact, which is different to the result in [3]. See also Theorem 1.2 on page 143 of Granas-Dugundji's book [11].

Example 3.3. Let $E = (0,1)$ and a map $T : E \to 2^E$ be defined by

$$Tx = \begin{cases} \left(x, x + \dfrac{1}{2}\right) & \text{if } x \in \left(0, \dfrac{1}{2}\right), \\ \left(\dfrac{1}{3}, x + \dfrac{1}{4}\right) & \text{if } x \in \left[\dfrac{1}{2}, \dfrac{3}{4}\right], \\ \left(x - \dfrac{1}{2}, x\right) & \text{otherwise.} \end{cases} \tag{3.4}$$

It is obvious that E is not compact and Tx is convex for all $x \in E$. Put $y_1 = 1/2$ and $y_2 = 3/4$. Then it follows that

$$\begin{aligned} \operatorname{co}\{y_1, y_2\} = \left[\frac{1}{2}, \frac{3}{4}\right] &\subset T^{-1}y_1 \bigcup T^{-1}y_2, \\ T^{-1}\frac{1}{2} \bigcap \left[\frac{1}{2}, \frac{3}{4}\right] &= \left[\frac{1}{2}, \frac{3}{4}\right], \\ T^{-1}\frac{3}{4} \bigcap \left[\frac{1}{2}, \frac{3}{4}\right] &= \left(\frac{1}{2}, \frac{3}{4}\right] \end{aligned} \tag{3.5}$$

are open in $[1/2, 3/4]$. Therefore, the map T satisfies the conditions of Theorem 2.6.

Corollary 3.4. *Let C be a nonempty convex subset of a topological vector space E and V be an open convex subset with $0 \in V$. Suppose that a map $T : C \to E$ is continuous and $T(C) \subset \cup_{i=1}^{n}\{y_i + V\}$, where $y_i \in C$ for $i = 1, 2, \ldots, n$. Then there exists $x_0 \in C$ such that $Tx_0 \in x_0 + V$.*

Proof. Let a map $G : C \to 2^C$ be defined by

$$G(x) = \{y \in C : Tx - y \in V\}. \tag{3.6}$$

Then $G(x)$ is convex for all $x \in C$ since V is convex. The continuity of T implies that $G^{-1}(y_i)$ is open. Moreover, $C = \cup_{i=1}^{n} G^{-1}(y_i)$ and thus

$$\operatorname{co}\{y_1, y_2, \ldots, y_n\} \subseteq \bigcup_{i=1}^{n} G^{-1}(y_i). \tag{3.7}$$

Therefore, by Theorem 3.1, we know that there exists $x_0 \in C$ such that $x_0 \in G(x_0)$. This implies that $Tx_0 \in x_0 + V$. □

Corollary 3.5. *Let C be a nonempty convex subset of a locally convex space E and K be a convex compact subset of E. Suppose that $T : C \to E$ is continuous and $T(C) \subset \cup_{i=1}^{n}\{y_i + K\}$, where $y_i \in C$ for $i = 1, 2, \ldots, n$. Then there is an $x_0 \in C$ such that $Tx_0 \in x_0 + K$.*

4. Coincidence theorem and minimax theorem

Theorem 4.1 (Ky Fan's coincidence theorem). *Let X and Y be nonempty convex subsets of topological vector spaces E and F, respectively. Let $A, B : X \to 2^Y$ be two maps satisfying the following conditions:*

(1) *there exists $x_i \in X$ such that Ax_i is open for $i = 1, 2, \ldots, n$, $Y = \cup_{i=1}^{n} Ax_i$ and $A^{-1}y$ is a convex set for each $y \in Y$;*
(2) *there exists $y_j \in y$ such that $B^{-1}y_j$ is open for $j = 1, 2, \ldots, m$, $X = \cup_{j=1}^{m} B^{-1}y_j$ and Bx is a convex set for each $x \in Y$.*

Then there exists $x_0 \in X$ such that $Ax_0 \cap Bx_0 \neq \emptyset$.

Proof. Let a map $K : X \times Y \to 2^{X \times Y}$ be defined by

$$K(x, y) = X \times Y \setminus (B^{-1}y \times Ax) \tag{4.1}$$

for all $(x, y) \in X \times Y$. By the assumptions, we have

$$X \times Y = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (B^{-1} y_j \times A x_i). \tag{4.2}$$

Therefore, we have

$$\bigcap_{i=1}^{n} \bigcap_{j=1}^{m} K(x_i, y_j) = \varnothing. \tag{4.3}$$

In view of Theorem 2.6, we know that K cannot be a KKM map on $\{x_i : 1 \le i \le n\} \times \{y_j : 1 \le j \le m\}$. So there exist $x_0, x_{i_1}, x_{i_2}, \ldots, x_{i_l}$ and $y_0, y_{j_1}, y_{j_2}, \ldots, y_{j_k}$ such that $x_0 \in \mathrm{co}\{x_{i_1}, x_{i_2}, \ldots, x_{i_l}\}$, $y_0 \in \mathrm{co}\{y_{j_1}, y_{j_2}, \ldots, y_{j_k}\}$ and

$$(x_0, y_0) \notin \bigcup_{s=1}^{l} \bigcup_{t=1}^{k} K(x_{i_s}, y_{j_t}), \tag{4.4}$$

which implies that

$$(x_0, y_0) \in (B^{-1} y_{j_t} \times A x_{i_s}) \tag{4.5}$$

for $s = 1, \ldots, l$ and $t = 1, 2, \ldots, k$. By the convexities of $A^{-1}x$ and By, we have $y_0 \in Ax_0$ and $y_0 \in Bx_0$. This completes the proof. □

Remark 4.2. The classical Ky Fan's coincidence theorem assume that both X and Y are compact. See Theorem 3.12 in Singh-Watson-Srivastava's book [15]. We do not require this condition in Theorem 4.1.

Definition 4.3. Let X be a topological space. A function $f : X \to R$ is said to be *lower semi-continuous from above* at x_0 if, for any net $(x_t)_{t \in T}$ with $x_t \to x_0$, $f(x_{t'}) \le f(x_t)$ for $t' \ge t$ implies that $f(x_0) \le \lim_t f(x_t)$. Similarly, f is said to *upper semi-continuous from below* at x_0 if, for any net $(x_t)_{t \in T}$ with $x_t \to x_0$, $f(x_t) \le f(x_{t'})$ for $t \le t'$ implies that $f(x_0) \le \lim_t f(x_t)$.

One can easily see that a lower (resp., upper) semi-continuous function is also a lower (resp., upper) semi-continuous from above (resp., below) function.

The following example shows that the converse is not true.

Example 4.4. Let a function $f : R \to R$ be defined by

$$f(x) = \begin{cases} x + 1 & \text{if } x \ge 0, \\ x & \text{if } x < 0. \end{cases} \tag{4.6}$$

Since R is a metric space, we consider a sequence $\{x_n\}$ such that $x_n \to 0$ with $f(x_1) \ge f(x_2) \ge \cdots \ge f(x_n) \ge \cdots$. Then, by the definition of $f(x)$, we know that $x_n \ge 0$ for all $n \ge 1$. Therefore, it follows that

$$\lim_{n \to \infty} f(x_n) = 1 = f(0) \tag{4.7}$$

and so f is lower semi-continuous from above at 0. If we take $x_n = -1/n$, then we have

$$\lim_{n\to\infty} f(x_n) = 0 < f(0) \tag{4.8}$$

and so f cannot be lower semi-continuous at 0.

Lemma 4.5. *Let X be a compact topological space and $f : X \to R$ be a real valued function. If f is lower semi-continuous from above (resp., upper semi-continuous from below), then there exists $x_0 \in X$ such that $f(x_0) = \min_{x\in X} f(x)$ (resp., $f(x_0) = \max_{x\in X} f(x)$).*

Proof. Assume that f is lower semi-continuous from above on X. There exists a net $(y_t) \subset C$ such that $f(y_{t'}) \le f(y_t)$ if $t' \ge t$ and $f(y_t) \to \inf_{y\in C} f(y)$. Since C is compact, without loss of generality, we may assume that $y_t \to y_0$. By the lower semi-continuity from above of $f(y)$, we have $f(y_0) \le \lim_t f(y_t)$ and so $f(y_0) = \inf_{y\in C} f(y)$. The proof of upper semi-continuous from below case is similar and hence we omit the detail. This completes the proof. □

Theorem 4.6 (von Neuman's minimax principle). *Let X and Y be two nonempty compact convex subsets of topological vector spaces E and F, respectively. Suppose that $f : X \times Y \to R$ is a real valued function satisfying the following conditions:*

(1) *$y \to f(x, y)$ is lower semi-continuous from above and quasi convex for each fixed $x \in X$, that is, $\{y : f(x, y) < r\}$ is convex for each $x \in X$;*
(2) *$x \to f(x, y)$ is upper semi-continuous from below and quasi concave for each fixed $y \in Y$, that is, $\{x : f(x, y) > r\}$ is convex for each $y \in Y$;*
(3) *for each $r \in R$, there exist x_i, $i = 1, 2, \ldots, n$, such that $A_i = \{y : f(x_i, y) > r\}$ is open and $Y = \cup_{i=1}^{n} A_i$;*
(4) *for each $r \in R$, there exist y_j, $j = 1, 2, \ldots, m$, such that $B_j = \{x : f(x, y_j) < r\}$ is open and $X = \cup_{j=1}^{m} B_j$.*

Then $\max_{x\in X} \min_{y\in Y} f(x, y) = \min_{y\in Y} \max_{x\in X} f(x, y)$.

Proof. By the assumptions (1), (2) and Lemma 4.5, we know that $\max_{x\in X} \min_{y\in Y} f(x, y)$ and $\min_{y\in Y} \max_{x\in X} f(x, y)$ both exist. It is obviously that

$$\max_{x\in X} \min_{y\in Y} f(x, y) \le \min_{y\in Y} \max_{x\in X} f(x, y). \tag{4.9}$$

Now we show that

$$\max_{x\in X} \min_{y\in Y} f(x, y) = \min_{y\in Y} \max_{x\in X} f(x, y). \tag{4.10}$$

If this is not true, then there would be a number $r \in R$ such that

$$\max_{x\in X} \min_{y\in Y} f(x, y) < r < \min_{y\in Y} \max_{x\in X} f(x, y). \tag{4.11}$$

Define two maps $A, B : X \to 2^Y$ by $Ax = \{y : f(x, y) > r\}$ and $Bx = \{y : f(x, y) < r\}$ for $x \in X$. It is obvious that

$$Y = \bigcup_{i=1}^{n} Ax_i, \qquad X = \bigcup_{j=1}^{m} B^{-1} y_j. \tag{4.12}$$

It is direct to check that $A^{-1}y$ is convex for $y \in Y$ and Bx is convex for each $x \in X$ and, by Theorem 4.1, there exists $x_0 \in X$ and $y_0 \in Y$ such that $y_0 \in Ax_0 \cap Bx_0 \neq \emptyset$. Hence we have $f(x_0, y_0) < r < f(x_0, y_0)$, which is a contradiction. This completes the proof. □

Theorem 4.7 (Ky Fan's minimax inequality). *Let C be a compact convex subset of a topological vector space X. Let $f : C \times C \to R$ be a real valued function satisfying the following conditions:*

(1) $\sup_{x\in C} f(x, y)$ *is lower semi-continuous from above on C;*

(2) $\{y : f(x, y) \leq \sup_{x\in C} f(x,x)\}$ *is closed for each $x \in C$;*

(3) $x \to f(x, y)$ *is quasi-concave on C for each $y \in C$.*

Then $\min_{y\in C} \sup_{x\in C} f(x, y) \leq \sup_{x\in C} f(x,x)$.

Proof. By Lemma 4.5, we know that $\sup_{x\in C} f(x, y)$ obtains its minimum on C.

Now, we may assume that $\sup_{x\in C} f(x,x) = \mu < \infty$. Define a map $G : C \to 2^C$ by

$$G(x) = \{y \in C : f(x, y) \leq \mu\} \tag{4.13}$$

for all $x \in C$. The quasi-concavity of $x \to f(x, y)$ on C for each $y \in C$ implies that G is a KKM map. By the assumption (2), we know that $G(x)$ is compact. Therefore, it follows from Theorem 2.7 that $\cap_{x\in C} G(x) \neq \emptyset$, thus there exists $y_0 \in C$ such that $y_0 \in G(x)$ for all $x \in C$, that is, $f(x, y_0) \leq \mu$ for all $x \in C$. This immediately implies that

$$\min_{y\in C} \sup_{x\in C} f(x, y) \leq \sup_{x\in C} f(x,x). \tag{4.14}$$

□

To end this paper, we give a function f which satisfies all the conditions of Theorem 4.6.

Example 4.8. Let a function $f : [0,1] \times [0,1] \to R$ be defined by

$$f(x, y) = \begin{cases} x + y & \text{if } y \in [0,1), \\ x + 2 & \text{if } y = 1. \end{cases} \tag{4.15}$$

Then we have

$$\sup_{x\in[0,1]} f(x, y) = \begin{cases} 1 + y & \text{if } y \in [0,1), \\ 3 & \text{if } y = 1. \end{cases} \tag{4.16}$$

Thus it follows that $\sup_{x\in[0,1]} f(x, y)$ is not lower semi-continuous, but lower semi-continuous from above. It is obvious that the set

$$\left\{y : f(x, y) \leq \sup_{x\in[0,1]} f(x,x) = 3\right\} = [0,1] \tag{4.17}$$

is closed and

$$\{x : f(x,1) > r\} = \{x : x > r-2\}, \qquad \{x : f(x,y) > r\} = \{x : x > r-y\} \tag{4.18}$$

for all $y \in [0,1)$ are convex sets, that is, $x \to f(x,y)$ is quasi-concave on C for each $y \in C$. Therefore, the function f satisfies all the conditions of Theorem 4.6.

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Y. Q. Chen: Department of Mathematics, Foshan University, Foshan, Guangdong 528000, China
E-mail address: yqchen@foshan.net

Y. J. Cho: Department of Mathematics Education and the RINS, College of Education, Gyeongsang National University, Chinju 660-701, Korea
E-mail address: yjcho@gsnu.ac.kr

J. K. Kim: Department of Mathematics Education, College of Education, Kyungnam University, Masan 631-701, Korea
E-mail address: jongkyuk@kyungnam.ac.kr

B. S. Lee: Department of Mathematics, Kyungsung University, Pusan 608-735, Korea
E-mail address: bslee@ks.ac.kr

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This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

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Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
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Guest Editors

Edson Denis Leonel, Departamento de Estatística, Matemática Aplicada e Computação, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil ; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob'evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru