COMPARISON OF FASTNESS OF THE CONVERGENCE AMONG KRASNOSELSKIJ, MANN, AND ISHIKAWA ITERATIONS IN ARBITRARY REAL BANACH SPACES

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Let E be an arbitrary real Banach space and K a nonempty, closed, convex (not necessarily bounded) subset of E. If T is a member of the class of Lipschitz, strongly pseudocontractive maps with Lipschitz constant $L \geq 1$, then it is shown that to each Mann iteration there is a Krasnosleskij iteration which converges faster than the Mann iteration. It is also shown that the Mann iteration converges faster than the Ishikawa iteration to the fixed point of T.

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1. Introduction

By approximation of fixed points of certain classes of operators which satisfy weak contractive-type conditions that do not guarantee the convergence of Picard iteration [2, Example 2.1, page 76], certain mean value fixed point iterations, namely, Krasnoselskij, Mann, and Ishikawa iteration methods are useful to approximate fixed points. For more details on these iterations and further literature, see Berinde [3].

When, for a certain class of mappings, two or more fixed point iteration procedures can be used to approximate their fixed points, it is of theoretical and practical importance to compare the rate of convergence of these iterations, and to find out, if possible, which one of them converges faster. Recent works in this direction are [1, 4, 5].

Verma [9] approximated fixed points of Lipschitzian and generalized pseudocontractive operators in Hilbert spaces by both Krasnoselskij and Mann iteration, and Berinde [4] established that, for any Mann iteration, there is a Krasnoselskij iteration which converges faster to the fixed point of such an operator.

Chidume and Osilike [7] approximated fixed points of Lipschitzian strongly pseudocontractive maps in Banach spaces, using both Mann and Ishikawa iterations.

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Now, the interest of this paper is to compare the fastness of the convergence to the fixed point among the Krasnoselskij, Mann, and Ishikawa iterations for the class of Lipschitz, strongly pseudocontractive operators in arbitrary real Banach spaces.

2. Preliminaries and known results

Suppose that E is a real Banach space with dual E^*, we denote by J, the normalized duality map from E to 2^{E^*} defined by

$$J(x) = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\}, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

A mapping T with domain $D(T)$ and range $R(T)$ in E is called *Lipschitz*, if there exists $L > 0$ such that for each $x, y \in D(T)$,

$$\|Tx - Ty\| \leq L\|x - y\|. \tag{2.2}$$

A mapping T with domain $D(T)$ and range $R(T)$ in E is called *strongly pseudocontractive* if and only if for any $x, y \in D(T)$, there exists $t > 1$ such that

$$\|x - y\| \leq \|(1+r)(x-y) - rt(Tx - Ty)\| \tag{2.3}$$

for any $r > 0$. If $t = 1$ in (2.3), then T is called *pseudocontractive*.

It follows from [8, Lemma 1.1] that T is strongly pseudocontractive if and only if the following condition holds: there exists $j(x-y) \in J(x-y)$ such that

$$\langle (I-T)(x) - (I-T)(y), j(x-y) \rangle \geq k\|x-y\|^2 \tag{2.4}$$

for each x, y in E, where $k = (t-1)/t \in (0,1)$.

Again by using [8, Lemma 1.1] and inequality (2.4) (Bogin [6]) it follows that T is strongly pseudocontractive if and only if the following inequality holds:

$$\|x - y\| \leq \|x - y + s[(I - T - kI)(x) - (I - T - kI)(y)]\| \tag{2.5}$$

for all $x, y \in D(T)$ and $s > 0$.

Notation 2.1. Throughout this paper, E denotes a real Banach space, K a closed convex (not necessarily bounded) subset of E, and $\mathrm{LS}(K)$ the class of all Lipschitz, strongly pseudocontractive maps on K. For any $T \in \mathrm{LS}(K)$, we assume that the Lipschitz constant $L \geq 1$ and pseudocontractive constant $k \in (0,1)$.

Let $x_0 \in E$ be arbitrary.

(i) For any $\lambda \in (0,1)$, the sequence $\{x_n\}_{n=0}^{\infty} \subseteq E$ defined by

$$x_{n+1} = T_\lambda x_n = (1-\lambda)x_n + \lambda T x_n, \quad n = 0, 1, 2, \ldots, \tag{2.6}$$

is called the *Krasnoselskij iteration*. We denote it by $K(x_0, \lambda, T)$.

(ii) The sequence $\{x_n\}_{n=0}^{\infty} \subseteq E$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n = 0, 1, 2, \ldots, \tag{2.7}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence satisfying $0 \le \alpha_n < 1$, $n = 0,1,2,\ldots$, is called the *Mann iteration*, and is denoted by $M(x_0,\alpha_n,T)$.

(iii) The sequence $\{x_n\}_{n=0}^{\infty} \subseteq E$ defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \quad n = 0,1,2,\ldots, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \quad n = 0,1,2,\ldots, \end{aligned} \tag{2.8}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are sequences of reals satisfying $0 \le \alpha_n$, $\beta_n < 1$, is called the *Ishikawa iteration*, and is denote by $I(x_0,\alpha_n,\beta_n,T)$.

Chidume and Osilike [7] established the strong convergence of Mann and Ishikawa iterations to the fixed point of $T \in \mathrm{LS}(K)$. Now, the following question arises: *for a member T of* $\mathrm{LS}(K)$ *which one of the following, namely, Krasnoselskij, Mann, and Ishikawa iterations converges faster to the fixed point of T?*

To answer this question, we use the following definitions introduced by Berinde [5].

Definition 2.2 [5]. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers that converge to a and b, respectively. Assume that there exists a real number l such that

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = l. \tag{2.9}$$

(i) If $l = 0$, then $\{a_n\}_{n=0}^{\infty}$ is said to converge faster to a than $\{b_n\}_{n=0}^{\infty}$ to b.

(ii) If $0 < l < \infty$, then $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are said to have the same rate of convergence.

Definition 2.3 [5]. Suppose that for two fixed point iteration procedures $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ both converging to the same fixed point p (say) with error estimates

$$\begin{aligned} ||u_n - p|| &\le a_n, \quad n = 0,1,2,\ldots, \\ ||v_n - p|| &\le b_n, \quad n = 0,1,2,\ldots, \end{aligned} \tag{2.10}$$

where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive numbers converging to zero. If $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$, then $\{u_n\}_{n=0}^{\infty}$ is said to *converge faster than* $\{v_n\}_{n=0}^{\infty}$ *to p*.

For more details on definitions, we refer, Berinde [4].

3. Results on the comparison of fastness of the convergence

Theorem 3.1. *If $T \in \mathrm{LS}(K)$, then the following hold:*

(a) *for any $x_0 \in K$ and $\epsilon \in (0, k/2M] \cap (0,1)$, the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$ defined by $K(x_0,\epsilon,T)$ converges strongly to the fixed point x^* of T, where $M = 1 + (2 - k + L)(L+1)$;*

(b) *for any $x_0 \in K$, the Mann iteration $\{x_n\}_{n=0}^{\infty}$ defined by $M(x_0,\alpha_n,T)$ with $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1)$ satisfying* (i) $\lim_{n\to\infty} \alpha_n = 0$ *and* (ii) $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ *converges strongly to the fixed point x^* of T;*

(c) *for any* $x_0 \in K$ *and for any Mann iteration* $\{x_n\}_{n=0}^{\infty}$ *defined by* $M(x_0, \alpha_n, T)$ *with* $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1)$ *satisfying (i) and (ii) of (b), converging to the fixed point* x^* *of* T, *there is an* $\epsilon_0 \in (0,1)$ *such that the Krasnoselskij iteration* $K(x_0, \epsilon_0, T)$ *converges faster to the fixed point* x^* *of* T. *Moreover,* x^* *is unique.*

Proof. From [7, Corollary 1], (b) follows. In order to establish (c), we need the following estimates, through which (a) follows.

Using Mann iteration $M(x_0, \alpha_n, T)$, from (2.7), we have

$$\begin{aligned} x_n = x_{n+1} + \alpha_n x_n - \alpha_n T x_n &= (1+\alpha_n)x_{n+1} + \alpha_n(I - T - kI)x_{n+1} \\ &\quad - (2-k)\alpha_n x_{n+1} + \alpha_n x_n + \alpha_n(Tx_{n+1} - Tx_n) \end{aligned} \tag{3.1}$$

so that

$$\begin{aligned} x_n - x^* &= (1+\alpha_n)(x_{n+1} - x^*) + \alpha_n(I - T - kI)(x_{n+1} - x^*) \\ &\quad - (1-k)\alpha_n(x_n - x^*) + (2-k)\alpha_n^2(x_n - Tx_n) + \alpha_n(Tx_{n+1} - Tx_n). \end{aligned} \tag{3.2}$$

Thus from (2.5), we get

$$\begin{aligned} ||x_n - x^*|| \geq (1+\alpha_n)||x_{n+1} - x^*|| - (1-k)\alpha_n||x_n - x^*|| \\ - (2-k)\alpha_n^2||x_n - Tx_n|| - \alpha_n||Tx_{n+1} - Tx_n||. \end{aligned} \tag{3.3}$$

Thus

$$\begin{aligned} &(1+\alpha_n)||x_{n+1} - x^*|| \\ &\quad \leq [1 + (1-k)\alpha_n]||x_n - x^*|| + (2-k)\alpha_n^2||x_n - Tx_n|| + \alpha_n||Tx_{n+1} - Tx_n||. \end{aligned} \tag{3.4}$$

We have

$$\begin{aligned} ||x_n - Tx_n|| &\leq ||x_n - x^*|| + ||x^* - Tx_n|| \leq (1+L)||x_n - x^*||, \\ ||Tx_{n+1} - Tx_n|| &\leq L||x_{n+1} - x_n|| = L||(1-\alpha_n)x_n + \alpha_n Tx_n - x_n|| \leq L(1+L)\alpha_n||x_n - x^*||. \end{aligned} \tag{3.5}$$

Thus from (3.4), (3.5), we have

$$(1+\alpha_n)||x_{n+1} - x^*|| \leq [1 + (1-k)\alpha_n + (2-k)\alpha_n^2(1+L) + \alpha_n^2 L(1+L)] \cdot ||x_n - x^*||. \tag{3.6}$$

Now

$$\begin{aligned} ||x_{n+1} - x^*|| &\leq \left[\frac{1 + (1-k)\alpha_n}{1+\alpha_n} + (2-k)\alpha_n^2(1+L) + \alpha_n^2 L(1+L)\right] \cdot ||x_n - x^*|| \\ &\leq [1 - k\alpha_n + \alpha_n^2 + \alpha_n^2(1+L)(2-k+L)] \cdot ||x_n - x^*|| \\ &= [1 - k\alpha_n + \alpha_n^2(1 + (2-k+L)(1+L))] \cdot ||x_n - x^*||. \end{aligned} \tag{3.7}$$

Therefore,

$$||x_{n+1} - x^*|| \leq [1 - k\alpha_n + \alpha_n^2 M] \cdot ||x_n - x^*||, \tag{3.8}$$

where $M = 1 + (2 - k + L)(1 + L)$.

On replacing α_n by ϵ in (3.8), we get the following estimate for the Krasnoselskij iteration $K(x_0, \epsilon, T)$:

$$||x_{n+1} - x^*|| \leq [1 - k\epsilon + \epsilon^2 M] \cdot ||x_n - x^*||. \tag{3.9}$$

Here we observe that $1 - k\epsilon + \epsilon^2 M < 1$ for any $\epsilon < k/M$. Thus (a) follows.

From the elementary calculus, the function f defined on $[0,1]$ by $f(\epsilon) = [1 - k\epsilon + \epsilon^2 M]$ has the minimum value at $\epsilon = \epsilon_0$, where $\epsilon_0 = k/2M$.

In particular, for this $\epsilon_0 > 0$, from (3.9), we have the following estimate for the Krasnoselskij iteration:

$$||x_{n+1} - x^*|| \leq \theta_0 \cdot ||x_n - x^*||, \tag{3.10}$$

where $\theta_0 = 1 - (k\epsilon_0/2)$ (< 1).

Thus, inductively it follows that

$$||x_{n+1} - x^*|| \leq \theta_0^n \cdot ||x_1 - x^*||. \tag{3.11}$$

Let $\eta = \min\{k/2M, k\epsilon_0^2/2\}$.

Since $\alpha_n \to 0$ as $n \to \infty$, then there is a positive integer N_0 such that $\alpha_n < \eta$ for all $n \geq N_0$.

Then from (3.8), we have

$$\begin{aligned} ||x_{n+1} - x^*|| &< [1 - k\alpha_n + \alpha_n \eta M] \cdot ||x_n - x^*|| \quad \forall n \geq N_0 \\ &\leq \left[1 - k\alpha_n + \alpha_n \frac{k}{2M} M\right] \cdot ||x_n - x^*|| \quad \forall n \geq N_0 \\ &= \left[1 - \frac{k\alpha_n}{2}\right] \cdot ||x_n - x^*|| \quad \forall n \geq N_0. \end{aligned} \tag{3.12}$$

On repeating this process, we get

$$||x_{n+1} - x^*|| < \prod_{i=N_0}^{n} \left[1 - \frac{k\alpha_i}{2}\right] \cdot ||x_{N_0} - x^*|| \quad \forall n \geq N_0. \tag{3.13}$$

On comparing the coefficients of the inequalities (3.11) and (3.13) obtained through $K(x_0, \epsilon_0, T)$ and $M(x_0, \alpha_n, T)$, respectively, we have, for $n \geq N_0$,

$$\frac{\theta_0^n}{\prod_{i=N_0}^{n} [1 - k\alpha_i/2]} \leq \frac{1}{(1 + k\epsilon_0/2)^{n-N_0}} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.14}$$

Thus by Definition 2.2, the Krasnoselskij iteration converges faster than the Mann iteration to the fixed point x^* of T. This proves (c). □

Table 3.1

x_n	$K(x_0,\epsilon_0,T)$	$K(x_0,\epsilon,T)$
x_0	1.9	1.9
x_1	1.875900277	1.887950139
x_2	1.852341980	1.876035445
x_3	1.829315930	1.864254774
x_4	1.806813062	1.852606990
x_5	1.784824424	1.841090963
x_6	1.763341175	1.829705570
x_7	1.742354579	1.818449697
x_8	1.721856008	1.807322234
x_9	1.701836931	1.796322080
x_{10}	1.682288851	1.785448141

Remark 3.2. From (3.10) of Theorem 3.1, it follows that for any $\epsilon \in (0,1)$ with $\epsilon < \epsilon_0$, the Krasnoselskij iteration $K(x_0,\epsilon_0,T)$ converges faster than $K(x_0,\epsilon,T)$ to the fixed point x^* of T for any $x_0 \in K$. This observation also is numerically shown in Table 3.1.

Theorem 3.3. *Let E, K, and T be as in Theorem 3.1. Suppose that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in $[0,1)$ such that $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ and $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$. Then*

(a) *for any $x_0 \in K$, the Ishikawa iteration $I(x_0,\alpha_n,\beta_n,T)$ converges strongly to the fixed point x^* of T, and*

(b) *the Mann iteration $M(x_0,\alpha_n,T)$ converges faster than the Ishikawa iteration $I(x_0,\alpha_n,\beta_n,T)$ to the fixed point x^* of T.*

Proof. (a) follows from [7, Theorem 1].

We now prove (b). Since $T \in \mathrm{LS}(K)$, from $I(x_0,\alpha_n,\beta_n,T)$ defined by (2.8), we have

$$\begin{aligned} x_n &= x_{n+1} + \alpha_n x_n - \alpha_n T y_n = (1+\alpha_n)x_{n+1} + \alpha_n(I - T - kI)x_{n+1} \\ &\quad - (2-k)\alpha_n x_{n+1} + \alpha_n x_n + \alpha_n(Tx_{n+1} - Ty_n) \\ &= (1+\alpha_n)x_{n+1} + \alpha_n(I - T - kI)x_{n+1} - (1-k)\alpha_n x_n \\ &\quad + (2-k)\alpha_n^2(x_n - Ty_n) + \alpha_n(Tx_{n+1} - Ty_n). \end{aligned} \tag{3.15}$$

Hence

$$\begin{aligned} x_n - x^* &= (1+\alpha_n)(x_{n+1} - x^*) + \alpha_n(I - T - kI)(x_{n+1} - x^*) \\ &\quad - (1-k)\alpha_n(x_n - x^*) + (2-k)\alpha_n^2(x_n - Ty_n) + \alpha_n(Tx_{n+1} - Ty_n). \end{aligned} \tag{3.16}$$

Thus from (2.5), we get

$$\begin{aligned} \|x_n - x^*\| &\geq (1+\alpha_n)\|x_{n+1} - x^*\| - (1-k)\alpha_n\|x_n - x^*\| \\ &\quad - (2-k)\alpha_n^2\|x_n - Ty_n\| - \alpha_n\|Tx_{n+1} - Ty_n\|. \end{aligned} \tag{3.17}$$

Then

$$\begin{aligned}&(1+\alpha_n)||x_{n+1}-x^*||\\&\quad\le[1+(1-k)\alpha_n]||x_n-x^*||+(2-k)\alpha_n^2||x_n-Ty_n||+\alpha_n||Tx_{n+1}-Ty_n||.\end{aligned}\tag{3.18}$$

We have the following estimates:

$$||y_n-x^*||\le(1-\beta_n)||x_n-x^*||+\beta_n||Tx_n-x^*||\le[1+(L-1)\beta_n]||x_n-x^*||,\tag{3.19}$$

$$\begin{aligned}||x_n-Ty_n||&\le||x_n-x^*||+||x^*-Ty_n||\le||x_n-x^*||+L||x^*-y_n||\\&\le[1+L(1+(L-1)\beta_n)]||x_n-x^*||.\end{aligned}\tag{3.20}$$

Also,

$$||Tx_{n+1}-Tx_n||\le L||x_{n+1}-y_n||\le L[(1-\alpha_n)||x_n-y_n||+\alpha_n||Ty_n-y_n||].\tag{3.21}$$

Now

$$\begin{aligned}||Ty_n-y_n||&\le||Ty_n-x^*||+||x^*-y_n||\le(1+L)||y_n-x^*||,\\||x_n-y_n||&=\beta_n||x_n-Tx_n||\le(1+L)\beta_n||x_n-x^*||.\end{aligned}\tag{3.22}$$

Now on substituting (3.22) in (3.21) and using (3.19), we have

$$\begin{aligned}||Tx_{n+1}-Ty_n||&\le L[(1+L)(1-\alpha_n)\beta_n+\alpha_n(1+L)(1+(L-1)\beta_n)]||x_n-x^*||\\&=L(1+L)[(1-\alpha_n)\beta_n+\alpha_n(1+(L-1)\beta_n)]||x_n-x^*||.\end{aligned}\tag{3.23}$$

On using (3.20) and (3.23) in (3.18), we get

$$\begin{aligned}(1+\alpha_n)||x_{n+1}-x^*||&\le[1+(1-k)\alpha_n+\alpha_n^2(2-k)[1+L(1+(L-1)\beta_n)]\\&\quad+\alpha_nL(1+L)[(1-\alpha_n)\beta_n+\alpha_n(1+(L-1)\beta_n)]]||x_n-x^*||\\&<[1+(1-k)\alpha_n+\alpha_n^2(2-k+L)(1+L)+\gamma(\alpha_n,\beta_n,L,k)]||x_n-x^*||,\end{aligned}\tag{3.24}$$

where

$$\gamma(\alpha_n,\beta_n,L,k)=\alpha_n\beta_nL[(2-k)(L-1)+(L+1)(1-\alpha_n)+(1+L)(L-1)]||x_n-x^*||.\tag{3.25}$$

Thus

$$\begin{aligned}||x_{n+1}-x^*||&\le\left[\frac{1+(1-k)\alpha_n}{1+\alpha_n}+\alpha_n^2(2-k+L)(1+L)+\gamma(\alpha_n,\beta_n,L,k)\right]||x_n-x^*||\\&=[1-k\alpha_n+\alpha_n^2+\alpha_n^2(2-k+L)(1+L)+\gamma(\alpha_n,\beta_n,L,k)]||x_n-x^*||\\&=[1-k\alpha_n+\alpha_n^2(1+(2-k+L)(1+L))+\gamma(\alpha_n,\beta_n,L,k)]||x_n-x^*||.\end{aligned}\tag{3.26}$$

Define $M_1 = (3-k+L)(1+L)$.

Since

$$\begin{aligned} 1+(2-k+L)(1+L) &\le M_1, \\ (2-k)(L-1)+(L+1)(1-\alpha_n)+(1+L)(L-1) &\le M_1, \end{aligned} \tag{3.27}$$

we have

$$\gamma(\alpha_n,\beta_n,L,k) \le \alpha_n\beta_n L M_1. \tag{3.28}$$

Now (3.26) becomes

$$\begin{aligned} ||x_{n+1}-x^*|| &\le [1-k\alpha_n+\alpha_n^2 M_1+\alpha_n\beta_n L M_1]||x_n-x^*|| \\ &= [1-k\alpha_n+\alpha_n(\alpha_n+\beta_n L)M_1]||x_n-x^*||. \end{aligned} \tag{3.29}$$

Since $\alpha_n \to 0$ as $n \to \infty$, there is a positive integer N_0 such that

$$\alpha_n < \frac{k\epsilon_0}{2M_1} \quad \forall n \ge N_0, \tag{3.30}$$

and since $\beta_n \to 0$ as $n \to \infty$, there is a positive integer N_1 such that

$$\beta_n < \frac{k\epsilon_0}{2M_1 L} \quad \forall n \ge N_1. \tag{3.31}$$

Write $N = \max\{N_0, N_1\}$. Now for any $n \ge N$, (3.29) becomes

$$\begin{aligned} ||x_{n+1}-x^*|| &< \left[1-k\alpha_n+\alpha_n\left(\frac{k\epsilon_0}{2M_1}+\frac{k\epsilon_0}{2M_1 L}L\right)M_1\right]||x_n-x^*|| \\ &= [1-k\alpha_n(1-\epsilon_0)]||x_n-x^*||. \end{aligned} \tag{3.32}$$

On repeating this process, we get

$$||x_{n+1}-x^*|| < \left\{\prod_{i=N}^{n}[1-k\alpha_i(1-\epsilon_0)]\right\}||x_N-x^*|| \quad \forall n \ge N, \tag{3.33}$$

which is an estimation for the Ishikawa iteration $I(x_0,\alpha_n,\beta_n,T)$.

On choosing $\beta_n = 0$ for all n, in (3.29), we get the following estimate for Mann iteration $M(x_0,\alpha_n,T)$:

$$\begin{aligned} ||x_{n+1}-x^*|| &\le [1-k\alpha_n+\alpha_n^2 M_1]||x_n-x^*|| \\ &< \left[1-k\alpha_n+\alpha_n M_1\frac{k\epsilon_0}{2M_1}\right]||x_n-x^*|| \quad \forall n \ge N \\ &= \left[1-k\alpha_n\left(1-\frac{\epsilon_0}{2}\right)\right]||x_n-x^*|| \quad \forall n \ge N. \end{aligned} \tag{3.34}$$

On repeating process, we get

$$||x_{n+1}-x^*|| < \left\{\prod_{i=N}^{n}\left[1-k\alpha_i\left(1-\frac{\epsilon_0}{2}\right)\right]\right\}||x_N-x^*||. \tag{3.35}$$

On comparing the coefficients of the inequalities (3.33) and (3.35), we get that for any $n \geq N$,

$$\frac{\prod_{i=N}^{n}[1-k\alpha_i(1-\epsilon_0/2)]}{\prod_{i=N}^{n}[1-k\alpha_i(1-\epsilon_0)]} \leq \prod_{i=N}^{n}\left[1-k\alpha_i\frac{\epsilon_0}{2}\right]. \tag{3.36}$$

Since $\Sigma_{n=0}^{\infty}\alpha_n = \infty$, we have $\lim_{n\to\infty}\prod_{i=N}^{n}[1-k\alpha_i(\epsilon_0/2)] = 0$. Thus the Mann iteration $M(x_0,\alpha_n,T)$ converges faster than the Ishikawa iteration $I(x_0,\alpha_n,\beta_n,T)$ to the fixed point of T. □

Remark 3.4. Under the assumptions of Theorem 3.1, it follows that for any Mann iteration $M(x_0,\alpha_n,T)$ there is a Krasnoselskij iteration $K(x_0,\epsilon_0,T)$ converges faster to the fixed point of T; and from Theorem 3.3 it follows that the Mann iteration $M(x_0,\alpha_n,T)$ converges faster than the Ishikawa iteration $I(x_0,\alpha_n,\beta_n,T)$ to the fixed point of T. Hence we conclude that the Krasnoselskij iteration converges faster than both the Mann and Ishikawa iterations to the fixed point of $T \in \mathrm{LS}(K)$.

4. Numerical examples

The following examples show the fastness of the movement of the first 10 iterates towards the fixed point.

Example 4.1 [4]. Let $X = [1/2,2]$ and $T : X \to X$ given by $Tx = 1/x$ for all $x \in X$. Then T is Lipschitz with Lipschitzian constant $L = 4$; and is strongly pseudocontractive with any positive constant $k \in (0,1)$.

We note that Picard iteration does not converge for any $x_0 \neq 1$ in X.

From Theorems 3.1 and 3.3, we have the following.

(i) The Krasnoselskij iteration $K(x_0,\epsilon_0,T)$ converges to the fixed point $x^* = 1$, where $\epsilon_0 = k/2M$, in which $k \in (0,1)$ and $M = 31-5k$. Choosing $k = 62/67$, we have $\epsilon_0 = 1/57$. For this ϵ_0, the Krasnoselskij iteration $K(x_0,\epsilon_0,T)$ is given by

$$x_{n+1} = \frac{1}{57}(56x_n + x_n^{-1}), \quad n = 0,1,2,\ldots, \tag{4.1}$$

which converges to the fixed point $x^* = 1$.

(ii) Also with $\alpha_n = 1/(n+58)$, $n = 0,1,2,\ldots$, the corresponding Mann iteration $M(x_0,\alpha_n,T)$ is given by

$$x_{n+1} = \frac{1}{n+58}[(n+57)x_n + x_n^{-1}], \quad n = 0,1,2,\ldots, \tag{4.2}$$

which converges to $x^* = 1$.

Table 4.1

x_n	$K(x_0,\epsilon_0,T)$	$M(x_0,\alpha_n,T)$	$I(x_0,\alpha_n,\alpha_n,T)$
x_0	1.9	1.9	1.9
x_1	1.875900277	1.876315789	1.876430333
x_2	1.852341980	1.853547036	1.853770048
x_3	1.829315930	1.831646354	1.831972078
x_4	1.806813062	1.810569477	1.810992457
x_5	1.784824424	1.790275008	1.790790067
x_6	1.763341175	1.770724189	1.771324237
x_7	1.742354579	1.751880697	1.752563291
x_8	1.721856008	1.733710457	1.734471166
x_9	1.701836931	1.716181474	1.717016088
x_{10}	1.682288851	1.699263676	1.700168192

(iii) The Ishikawa iteration $I(x_0,\alpha_n,\beta_n,T)$ converges to $x^*=1$ with $\alpha_n=\beta_n=1/(n+58)$, $n\geq 0$. In this case, the sequence $I(x_n,\alpha_n,\alpha_n,T)$ is given by

$$x_{n+1}=\frac{n+57}{n+58}x_n+\frac{x_n}{(n+57)x_n^2+1},\quad n=0,1,2,\ldots. \tag{4.3}$$

(iv) The comparison of the fastness of first 10 iterates of the Krasnoselskij, Mann, and Ishikawa iterations to the fixed point $x^*=1$ is given in Table 4.1 with $x_0=1.9$, and $\alpha_n=1/(n+58)$ with $\epsilon_0=1/57$.

From Table 4.1, we observe that the Krasnoselskij iteration moves faster towards the fixed point $x^*=1$.

(v) Table 3.1 shows the comparison of first 10 iterates of Krasnoselskij iterations $K(x_0,\epsilon,T)$ and $K(x_0,\epsilon_0,T)$, where $\epsilon=1/114$, $\epsilon_0=1/57$, and $x_0=1.9$. Here we observe that $K(x_0,\epsilon_0,T)$ moves faster than $K(x_0,\epsilon,T)$ to the fixed point $x^*=1$ of T (see Remark 3.2).

Example 4.2. Let $X=[0,1]$ and $T:X\to X$ given by $Tx=1-x^2$ for all $x\in X$. Then T is Lipschitz, with Lipschitzian constant $L=2$, and is strongly pseudocontractive with any positive constant $k\in(0,1)$.

(i) From Theorem 3.1, the Krasnoselskij iteration $K(x_0,\epsilon_0,T)$ converges to $x^*=(\sqrt{5}-1)/2$, where $\epsilon_0=k/2M$, $k\in(0,1)$, and $M=13-3k$.

Let $x_0=0.9$. Now for $k=26/27$, we have $\epsilon_0=1/21$; thus the Krasnoselskij iteration $K(x_0,\epsilon_0,T)$ is given by

$$x_{n+1}=\frac{1}{21}\left[1+x_n(20-x_n)\right],\quad n=0,1,2,\ldots. \tag{4.4}$$

(ii) The Mann iteration $M(x_0,\alpha_n,T)$ converges to $x^*=(\sqrt{5}-1)/2$, where $\alpha_n=1/(n+22)$, $n=0,1,2,\ldots$, and the Mann iteration $M(x_0,\alpha_n,T)$ is given by

$$x_{n+1}=\frac{1}{n+22}\left[1+x_n(n+21-x_n)\right],\quad n=0,1,2,\ldots. \tag{4.5}$$

Table 4.2

x_n	$K(x_0,\epsilon_0,T)$	$M(x_0,\alpha_n,T)$	$I(x_0,\alpha_n,\alpha_n,T)$
x_0	0.9	0.9	0.9
x_1	0.866190476	0.867727272	0.870320426
x_2	0.836834456	0.840741276	0.848058999
x_3	0.811257010	0.817925143	0.826070151
x_4	0.788904869	0.798448075	0.807059203
x_5	0.769320309	0.781680098	0.790510886
x_6	0.752121544	0.767135510	0.776017193
x_7	0.736987546	0.754434352	0.763251561
x_8	0.723647632	0.743275540	0.751950035
x_9	0.711870797	0.733417737	0.741897348
x_{10}	0.701459805	0.724665501	0.732916484

(iii) The Ishikawa iteration $I(x_0,\alpha_n,\beta_n,T)$ converges to the fixed point $x^* = (\sqrt{5}-1)/2$ by Theorem 3.3 with $\alpha_n = \beta_n = 1/(n+22)$, $n = 0,1,2,\ldots$. The Ishikawa iteration $I(x_0,\alpha_n,\alpha_n,T)$ is given by

$$x_{n+1} = \left(\frac{n+21}{n+22}\right)x_n + \frac{1}{n+22}\left[1 - \left(\frac{1}{n+22}\left(1+x_n(n+21-x_n)\right)\right)^2\right], \quad n = 0,1,2,\ldots. \tag{4.6}$$

(iv) Comparison of Krasnoselskij, Mann, and Ishikawa iterations is given for first 10 iterates in Table 4.2 for $x_0 = 0.9$, and $\alpha_n = 1/(n+22)$ with $\epsilon_0 = 1/21$.

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