COMMON FIXED POINT AND INVARIANT APPROXIMATION RESULTS IN CERTAIN METRIZABLE TOPOLOGICAL VECTOR SPACES

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We obtain common fixed point results for generalized I-nonexpansive R-subweakly commuting maps on nonstarshaped domain. As applications, we establish noncommutative versions of various best approximation results for this class of maps in certain metrizable topological vector spaces.

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1. Introduction and preliminaries

Let X be a linear space. A p-norm on X is a real-valued function on X with $0 < p \leq 1$, satisfying the following conditions:

(i) $\|x\|_p \geq 0$ and $\|x\|_p = 0 \Leftrightarrow x = 0$,

(ii) $\|\alpha x\|_p = |\alpha|^p \|x\|_p$,

(iii) $\|x + y\|_p \leq \|x\|_p + \|y\|_p$

for all $x, y \in X$ and all scalars α. The pair $(X, \|,\|_p)$ is called a p-normed space. It is a metric linear space with a translation invariant metric d_p defined by $d_p(x, y) = \|x - y\|_p$ for all $x, y \in X$. If $p = 1$, we obtain the concept of the usual normed space. It is well-known that the topology of every Hausdorff locally bounded topological linear space is given by some p-norm, $0 < p \leq 1$ (see [9] and references therein). The spaces l_p and L_p, $0 < p \leq 1$ are p-normed spaces. A p-normed space is not necessarily a locally convex space. Recall that dual space X^* (the dual of X) separates points of X if for each nonzero $x \in X$, there exists $f \in X^*$ such that $f(x) \neq 0$. In this case the weak topology on X is well-defined and is Hausdorff. Notice that if X is not locally convex space, then X^* need not separate the points of X. For example, if $X = L_p[0, 1]$, $0 < p < 1$, then $X^* = \{0\}$ ([12, pages 36 and 37]). However, there are some non-locally convex spaces X (such as the p-normed spaces l_p, $0 < p < 1$) whose dual X^* separates the points of X.

Let X be a metric linear space and M a nonempty subset of X. The set $P_M(u) = \{x \in M : d(x, u) = \operatorname{dist}(u, M)\}$ is called the set of best approximants to $u \in X$ out of M, where $\operatorname{dist}(u, M) = \inf\{d(y, u) : y \in M\}$. Let $f : M \to M$ be a mapping. A mapping $T : M \to M$

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is called an f-contraction if there exists $0 \le k < 1$ such that $d(Tx,Ty) \le k\ d(fx,fy)$ for any $x,y \in M$. If $k = 1$, then T is called f-nonexpansive. A mapping $T : M \to M$ is called condensing if for any bounded subset B of M with $\alpha(B) > 0$, $\alpha(T(B)) < \alpha(B)$, where $\alpha(B) = \inf\{r > 0 : B$ can be covered by a finite number of sets of diameter $\le r\}$. A mapping $T : M \to M$ is hemicompact if any sequence $\{x_n\}$ in M has a convergent subsequence whenever $d(x_n, Tx_n) \to 0$ as $n \to \infty$. The set of fixed points of T (resp. f) is denoted by $F(T)$ (resp. $F(f)$). A point $x \in M$ is a common fixed point of f and T if $x = fx = Tx$. The pair $\{f,T\}$ is called (1) commuting if $Tfx = fTx$ for all $x \in M$; (2) R-weakly commuting [16] if for all $x \in M$ there exists $R > 0$ such that $d(fTx,Tfx) \le Rd(fx,Tx)$. If $R = 1$, then the maps are called weakly commuting. The set M is called q-starshaped with $q \in M$ if the segment $[q,x] = \{(1-k)q + kx : 0 \le k \le 1\}$ joining q to x, is contained in M for all $x \in M$. Suppose that M is q-starshaped with $q \in F(f)$ and is both T- and f-invariant. Then T and f are called R-subweakly commuting on M (see [17]) if for all $x \in M$, there exists a real number $R > 0$ such that $d(fTx,Tfx) \le R\,\mathrm{dist}(fx,[q,Tx])$. It is well-known that commuting maps are R-subweakly commuting maps and R-subweakly commuting maps are R-weakly commuting but not conversely in general (see [16, 17]).

A set M is said to have property (N) if [7, 11]

(i) $T : M \to M$,

(ii) $(1-k_n)q + k_nTx \in M$, for some $q \in M$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1 and for each $x \in M$.

A mapping f is said to have property (C) on a set M with property (N) if $f((1-k_n)q + k_nTx) = (1-k_n)fq + k_nfTx$ for each $x \in M$ and $n \in N$.

We extend the concept of R-subweakly commuting maps to nonstarshaped domain in the following way (see [7]):

Let f and T be self-maps on the set M having property (N) with $q \in F(f)$. Then f and T are called R-subweakly commuting on M, provided for all $x \in M$, there exists a real number $R > 0$ such that $d(fTx,Tfx) \le Rd(fx,T_nx)$ where $T_nx = (1-k_n)q + k_nTx$, and the sequence $\{k_n\}$ is as in definition of property (N) of M. Each T-invariant q-starshaped set has property (N) but not conversely in general. Each affine map on a q-starshaped set M satisfies condition (C).

Example 1.1 [7]. Consider $X = R^2$ and $M = \{(0,y) : y \in [-1,1]\} \cup \{(1-1/(n+1),0) : n \in N\} \cup \{(1,0)\}$ with the metric induced by the norm $\|(a,b)\| = |a| + |b|$, $(a,b) \in R^2$. Define T on M as follows:

$$T(0,y) = (0,-y), \qquad T\left(1-\frac{1}{n+1},0\right) = \left(0,1-\frac{1}{n+1}\right), \qquad T(1,0) = (0,1). \tag{1.1}$$

Clearly, M is not starshaped [11] but M has the property (N) for $q = (0,0)$ and $k_n = 1 - 1/(n+1)$. Define $I(0,y) = I(1-1/(n+1),0) = (0,0)$, $I(1,0) = (1,0)$. Then $\|TIx - ITx\| = 0$ or 1. Thus for all x in M, $\|TIx - ITx\| \le R\|k_nTx - Ix\|$ with each $R \ge 1$ and $q = (0,0) \in F(I)$. Thus I and T are R-subweakly commuting but not commuting on M.

The map $T : M \to X$ is said to be completely continuous if $\{x_n\}$ converges weakly to x implies that $\{Tx_n\}$ converges strongly to Tx.

In 1963, Meinardus [10] employed the Schauder fixed point theorem to prove a result regarding invariant approximation. In 1979, Singh [19] proved the following extension of "Meinardus" result.

Theorem 1.2. *Let T be a nonexpansive operator on a normed space X, M be a T-invariant subset of X and $u \in F(T)$. If $P_M(u)$ is nonempty compact and starshaped, then $P_M(u) \cap F(T) \neq \emptyset$.*

In 1988, Sahab et al. [13] established the following result which contains Theorem 1.2 and many others.

Theorem 1.3. *Let I and T be selfmaps of a normed space X with $u \in F(I) \cap F(T)$, $M \subset X$ with $T(\partial M) \subset M$, and $q \in F(I)$. If $P_M(u)$ is compact and q-starshaped, $I(P_M(u)) = P_M(u)$, I is continuous and linear on $P_M(u)$, I and T are commuting on $P_M(u)$ and T is I-nonexpansive on $P_M(u) \cup \{u\}$, then $P_M(u) \cap F(T) \cap F(I) \neq \emptyset$.*

Let $D = P_M(u) \cap C_M^I(u)$, where $C_M^I(u) = \{x \in M : Ix \in P_M(u)\}$.

Theorem 1.4 [1, Theorem 3.2]. *Let I and T be selfmaps of a Banach space X with $u \in F(I) \cap F(T)$, $M \subset X$ with $T(\partial M \cap M) \subset M$. Suppose that D is closed and q-starshaped with $q \in F(I)$, $I(D) = D$, I is linear and continuous on D. If I and T are commuting on D and T is I-nonexpansive on $D \cup \{u\}$ with $\mathrm{cl}(T(D))$ compact, then $P_M(u) \cap F(T) \cap F(I) \neq \emptyset$.*

Recently, by introducing the concept of non-commuting maps to this area, Shahzad [14–18], Hussain and Khan [6] and Hussain et al. [7], further extended and improved the above mentioned results to non-commuting maps.

The aim of this paper is to prove new results extending and subsuming the above mentioned invariant approximation results. To do this, we establish a general common fixed point theorem for R-subweakly commuting generalized I-nonexpansive maps on nonstarshaped domain in the setting of locally bounded topological vector spaces, locally convex topological vector spaces and metric linear spaces. We apply a new theorem to derive some results on the existence of best approximations. Our results unify and extend the results of Al-Thagafi [1], Dotson [3], Guseman and Peters [4], Habiniak [5], Hussain and Khan [6], Hussain et al. [7], Khan and Khan [9], Sahab et al. [13], Shahzad [14–18], and Singh [19].

2. Common fixed point and approximation results

The following common fixed point result is a consequence of Theorem 1 of Berinde [2], which will be needed in the sequel.

Theorem 2.1. *Let M be a closed subset of a metric space (X,d) and T and f be R-weakly commuting self-maps of M such that $T(M) \subset f(M)$. Suppose there exists $k \in (0,1)$ such that*

$$d(Tx,Ty) \leq k \max\{d(fx,fy), d(Tx,fx), d(Ty,fy), d(Tx,fy), d(Ty,fx)\} \tag{2.1}$$

for all $x, y \in M$. If $\mathrm{cl}(T(M))$ is complete and T is continuous, then there is a unique point z in M such that $Tz = fz = z$.

We can prove now the following.

THEOREM 2.2. *Let T, I be self-maps on a subset M of a p-normed space X. Assume that M has the property (N) with $q \in F(I)$, I satisfies the condition (C) and $M = I(M)$. Suppose that T and I are R-subweakly commuting and satisfy*

$$\begin{aligned}\|Tx - Ty\|_p \leq \max\{&\|Ix - Iy\|_p, \operatorname{dist}(Ix, [Tx, q]), \operatorname{dist}(Iy, [Ty, q]),\\ &\operatorname{dist}(Ix, [Ty, q]), \operatorname{dist}(Iy, [Tx, q])\}\end{aligned} \tag{2.2}$$

for all $x, y \in M$. If T is continuous, then $F(T) \cap F(I) \neq \emptyset$, provided one of the following conditions holds:

(i) *M is closed, $\operatorname{cl}(T(M))$ is compact and I is continuous,*
(ii) *M is bounded and complete, T is hemicompact and I is continuous,*
(iii) *M is bounded and complete, T is condensing and I is continuous,*
(iv) *X is complete with separating dual X^*, M is weakly compact, T is completely continuous and I is continuous.*

Proof. Define T_n by $T_n x = (1 - k_n)q + k_n Tx$ for all $x \in M$ and fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1. Then, each T_n is a well-defined self-mapping of M as M has property (N) and for each n, $T_n(M) \subset M = I(M)$. Now the property (C) of I and the R-subweak commutativity of $\{T, I\}$ imply that

$$\begin{aligned}\left\|T_n Ix - IT_n x\right\|_p = (k_n)^p \|TIx - ITx\|_p &\leq (k_n)^p R \operatorname{dist}(Ix, [Tx, q])\\ &\leq (k_n)^p R \left\|T_n x - Ix\right\|_p\end{aligned} \tag{2.3}$$

for all $x \in M$. This implies that the pair $\{T_n, I\}$ is $(k_n)^p R$-weakly commuting for each n. Also by (2.2),

$$\begin{aligned}\left\|T_n x - T_n y\right\|_p &= (k_n)^p \|Tx - Ty\|_p\\ &\leq (k_n)^p \max\{\|Ix - Iy\|_p, \operatorname{dist}(Ix, [Tx, q]), \operatorname{dist}(Iy, [Ty, q]),\\ &\qquad \operatorname{dist}(Ix, [Ty, q]), \operatorname{dist}(Iy, [Tx, q])\}\\ &\leq (k_n)^p \max\left\{\|Ix - Iy\|_p, \left\|Ix - T_n x\right\|_p, \left\|Iy - T_n y\right\|_p,\right.\\ &\qquad \left.\left\|Ix - T_n y\right\|_p, \left\|Iy - T_n x\right\|_p\right\}\end{aligned} \tag{2.4}$$

for each $x, y \in M$.

(i) Since $\operatorname{cl} T(M)$ is compact, $\operatorname{cl}(T_n(M))$ is also compact. By Theorem 2.1, for each $n \geq 1$, there exists $x_n \in M$ such that $x_n = Ix_n = T_n x_n$. The compactness of $\operatorname{cl} T(M)$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to y$ as $m \to \infty$. Then the definition of $T_m x_m$ implies $x_m \to y$, so by the continuity of T and I we have $y \in F(T) \cap F(I)$. Thus $F(T) \cap F(I) \neq \emptyset$.

(ii) As in (i) there exists $x_n \in M$ such that $x_n = Ix_n = T_n x_n$. And M is bounded, so $x_n - Tx_n = (1-(k_n)^{-1})(x_n - q) \to 0$ as $n \to \infty$ and hence $d_p(x_n, Tx_n) \to 0$ as $n \to \infty$. The hemicompactness of T implies that $\{x_n\}$ has a subsequence $\{x_j\}$ which converges to some $z \in M$. By the continuity of T and I we have $z \in F(T) \cap F(I)$. Thus $F(T) \cap F(I) \neq \emptyset$.

(iii) Every condensing map on a complete bounded subset of a metric space is hemicompact. Hence the result follows from (ii).

(iv) As in (i) there exists $x_n \in M$ such that $x_n = Ix_n = T_n x_n$. Since M is weakly compact, we can find a subsequence $\{x_m\}$ of $\{x_n\}$ in M converging weakly to $y \in M$ as $m \to \infty$. Since T is completely continuous, $Tx_m \to Ty$ as $m \to \infty$. Since $k_n \to 1$, $x_m = T_m x_m = k_m Tx_m + (1-k_m)q \to Ty$ as $m \to \infty$. Thus $Tx_m \to T^2 y$ as $m \to \infty$ and consequently $T^2 y = Ty$ implies that $Tw = w$, where $w = Ty$. Also, since $Ix_m = x_m \to Ty = w$, using the continuity of I and the uniqueness of the limit, we have $Iw = w$. Hence $F(T) \cap F(I) \neq \emptyset$. □

It is clear that each T-invariant q-starshaped set satisfies the property (N) and if I is affine, then I satisfies the condition (C) and $T_n(M) \subset I(M)$ provided $T(M) \subset I(M)$ and $q \in F(I)$.

Corollary 2.3. *Let M be a closed q-starshaped subset of a p-normed space X, and T and I continuous self-maps of M. Suppose that I is affine with $q \in F(I)$, $T(M) \subset I(M)$ and $\operatorname{cl} T(M)$ is compact. If the pair $\{T,I\}$ is R-subweakly commuting and satisfy (2.2) for all $x, y \in M$, then $F(T) \cap F(I) \neq \emptyset$.*

Corollary 2.4 [18, Theorem 2.2]. *Let M be a closed q-starshaped subset of a normed space X, and T and I continuous self-maps of M. Suppose that I is affine with $q \in F(I)$, $T(M) \subset I(M)$ and $\operatorname{cl} T(M)$ is compact. If the pair $\{T,I\}$ is R-subweakly commuting and satisfy, for all $x, y \in M$,*

$$\|Tx - Ty\| \leq \max\left\{\|Ix - Iy\|, \operatorname{dist}(Ix,[Tx,q]), \operatorname{dist}(Iy,[Ty,q]), \frac{1}{2}[\operatorname{dist}(Ix,[Ty,q]) + \operatorname{dist}(Iy,[Tx,q])]\right\}, \tag{2.5}$$

then $F(T) \cap F(I) \neq \emptyset$.

The following corollary improves and generalizes [1, Theorem 2.2].

Corollary 2.5. *Let M be a nonempty closed and q-starshaped subset of a p-normed space X and I be continuous self-map of M. Suppose that I is affine with $q \in F(I)$, $T(M) \subset I(M)$ and $\operatorname{cl} T(M)$ is compact. If the pair $\{T,I\}$ is R-subweakly commuting and T is I-nonexpansive on M, then $F(T) \cap F(I) \neq \emptyset$.*

The following corollaries improve and generalize [3, Theorem 1] and [5, Theorem 4].

Corollary 2.6. *Let M be a nonempty closed and q-starshaped subset of a p-normed space X, T and I be continuous self-maps of M. Suppose that I is affine with $q \in F(I)$, $T(M) \subset I(M)$ and $\operatorname{cl} T(M)$ is compact. If the pair $\{T,I\}$ is commuting and T and I satisfy (2.2), then $F(T) \cap F(I) \neq \emptyset$.*

Corollary 2.7 [9, Theorem 2]. *Let M be a nonempty closed and q-starshaped subset of a p-normed space X. If T is nonexpansive self-map of M and $\operatorname{cl}T(M)$ is compact, then $F(T)\neq\emptyset$.*

We now derive some approximation results.

Let $D_M^{R,I}(u)=P_M(u)\cap G_M^{R,I}(u)$, where $G_M^{R,I}(u)=\{x\in M:\|Ix-u\|_p\le(2R+1)\operatorname{dist}(u,M)\}$.

The following result extends Theorem 2.3 of Shahzad [16] from the I-nonexpansiveness of T to a more general condition.

Theorem 2.8. *Let M be subset of a p-normed space X and $I,T:X\to X$ be mappings such that $u\in F(T)\cap F(I)$ for some $u\in X$ and $T(\partial M\cap M)\subset M$. If $I(D_M^{R,I}(u))=D_M^{R,I}(u)$ and the pair $\{T,I\}$ is R-subweakly commuting and continuous on $D_M^{R,I}(u)$ and satisfy for all $x\in D_M^{R,I}(u)\cup\{u\}$,*

$$\|Tx-Ty\|_p\le\begin{cases}\|Ix-Iu\|_p & \text{if } y=u,\\ \max\{\|Ix-Iy\|_p,\operatorname{dist}(Ix,[q,Tx]),\operatorname{dist}(Iy,[q,Ty]),\\ \qquad\operatorname{dist}(Ix,[q,Ty]),\operatorname{dist}(Iy,[q,Tx])\} & \text{if } y\in D_M^{R,I}(u),\end{cases}\tag{2.6}$$

then $D_M^{R,I}(u)$ is T-invariant. Suppose that $D_M^{R,I}(u)$ is closed and $\operatorname{cl}(T(D_M^{R,I}(u)))$ is compact. If $D_M^{R,I}(u)$ has property (N) with $q\in F(I)$, and I satisfies property (C) on $D_M^{R,I}(u)$, then $P_M(u)\cap F(I)\cap F(T)\neq\emptyset$.

Proof. Let $x\in D_M^{R,I}(u)$. Then, $x\in P_M(u)$ and hence $\|x-u\|_p=\operatorname{dist}(u,M)$. Note that for any $k\in(0,1)$,

$$\|ku+(1-k)x-u\|_p=(1-k)^p\|x-u\|_p<\operatorname{dist}(u,M).\tag{2.7}$$

It follows that the line segment $\{ku+(1-k)x:0<k<1\}$ and the set M are disjoint. Thus x is not in the interior of M and so $x\in\partial M\cap M$. Since $T(\partial M\cap M)\subset M$, Tx must be in M. Also since $Ix\in P_M(u)$, $u\in F(T)\cap F(I)$ and T and I satisfy (2.6), we have

$$\|Tx-u\|_p=\|Tx-Tu\|_p\le\|Ix-Iu\|_p=\|Ix-u\|_p=\operatorname{dist}(u,M).\tag{2.8}$$

Thus $Tx\in P_M(u)$. From the R-subweak commutativity of the pair $\{T,I\}$ and (2.6), it follows that (see also proof of [16, Theorem 2.3]),

$$\begin{aligned}\|ITx-u\|_p&=\|ITx-TIx+TIx-Tu\|_p\le R\|Tx-Ix\|_p+\left\|I^2x-Iu\right\|_p\\&=R\|Tx-u+u-Ix\|_p+\left\|I^2x-u\right\|_p\\&\le R\left(\|Tx-u\|_p+\|Ix-u\|_p\right)+\left\|I^2x-u\right\|_p\\&\le(2R+1)\operatorname{dist}(u,M).\end{aligned}\tag{2.9}$$

Thus $Tx\in G_M^{R,I}(u)$. Consequently, $T(D_M^{R,I}(u))\subset D_M^{R,I}(u)=I(D_M^{R,I}(u))$. Now Theorem 2.2(i) guarantees that, $P_M(u)\cap F(I)\cap F(T)\neq\emptyset$. □

Remarks 2.9. (1) If $p = 1$ and M is q-starshaped with $q \in F(I)$, $T(M) \subset I(M)$ and I is linear on $D_M^{R,I}(u)$ in Theorem 2.8, we obtain the conclusion of a recent result [18, Theorem 2.5] for the more general inequality (2.6).

(2) Let $C_M^I(u) = \{x \in M : Ix \in P_M(u)\}$. Then $I(P_M(u)) \subset P_M(u)$ implies $P_M(u) \subset C_M^I(u) \subset G_M^{R,I}(u)$ and hence $D_M^{R,I}(u) = P_M(u)$. Consequently, Theorem 2.8 remains valid when $D_M^{R,I}(u) = P_M(u)$. Hence we obtain the following result which contains properly Theorems 1.2 and 1.3 and improves and extends Theorem 8 of [5], Theorem 4 in [9], and Theorem 6 in [14, 15].

Corollary 2.10. *Let M be subset of a p-normed space X and let $I, T : X \to X$ be mappings such that $u \in F(T) \cap F(I)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. Assume that $I(P_M(u)) = P_M(u)$ and the pair $\{T, I\}$ is R-subweakly commuting and continuous on $P_M(u)$ and satisfy for all $x \in P_M(u) \cup \{u\}$,*

$$\|Tx - Ty\|_p \leq \begin{cases} \|Ix - Iu\|_p & \text{if } y = u, \\ \max\{\|Ix - Iy\|_p, \operatorname{dist}(Ix, [q, Tx]), \operatorname{dist}(Iy, [q, Ty]), & \\ \quad \operatorname{dist}(Ix, [q, Ty]), \operatorname{dist}(Iy, [q, Tx])\} & \text{if } y \in P_M(u). \end{cases} \tag{2.10}$$

Suppose that $P_M(u)$ is closed, q-starshaped with $q \in F(I)$, I is affine and $\operatorname{cl}(T(P_M(u)))$ is compact. Then $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$.

Let $D = P_M(u) \cap C_M^I(u)$, where $C_M^I(u) = \{x \in M : Ix \in P_M(u)\}$.

The following result contains Theorem 1.4 and many others.

Theorem 2.11. *Let M be subset of a p-normed space X and $I, T : X \to X$ be mappings such that $u \in F(T) \cap F(I)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. If $I(D) = D$ and the pair $\{T, I\}$ is commuting and continuous on D and satisfy for all $x \in D \cup \{u\}$,*

$$\|Tx - Ty\|_p \leq \begin{cases} \|Ix - Iu\|_p & \text{if } y = u, \\ \max\{\|Ix - Iy\|_p, \operatorname{dist}(Ix, [q, Tx]), \operatorname{dist}(Iy, [q, Ty]), & \\ \quad \operatorname{dist}(Ix, [q, Ty]), \operatorname{dist}(Iy, [q, Tx])\} & \text{if } y \in D, \end{cases} \tag{2.11}$$

then D is T-invariant. Suppose that D is closed and $\operatorname{cl}(T(D))$ is compact. If D has property (N) with $q \in F(I)$, and I satisfies property (C) on D, then $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$.

Proof. Let $x \in D$, then proceeding as in the proof of Theorem 2.8, we obtain $Tx \in P_M(u)$. Moreover, since T commutes with I on D and T satisfies (2.11),

$$\|ITx - u\|_p = \|TIx - Tu\|_p \leq \left\|I^2x - Iu\right\|_p = \left\|I^2x - u\right\|_p = \operatorname{dist}(u, M). \tag{2.12}$$

Thus $ITx \in P_M(u)$ and so $Tx \in C_M^I(u)$. Hence $Tx \in D$. Consequently, $T(D) \subset D = I(D)$. Now Theorem 2.2(i) guarantees that $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$. □

In the following result we obtain a non-locally convex space analogue of [6, Theorem 3.3] for nonstarshaped set D.

Theorem 2.12. *Let M be subset of a p-normed space X and $I, T : X \to X$ be mappings such that $u \in F(T) \cap F(I)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. If $I(D) = D$ and the pair $\{T, I\}$ is R-subweakly commuting and continuous on D and, for all $x \in D \cup \{u\}$, satisfies the following inequality,*

$$\|Tx - Ty\|_p \leq \begin{cases} \|Ix - Iu\|_p & \text{if } y = u, \\ \max\{\|Ix - Iy\|_p, \operatorname{dist}(Ix, [q, Tx]), \operatorname{dist}(Iy, [q, Ty]), & \\ \qquad \operatorname{dist}(Ix, [q, Ty]), \operatorname{dist}(Iy, [q, Tx])\} & \text{if } y \in D, \end{cases} \tag{2.13}$$

and I is nonexpansive on $P_M(u) \cup \{u\}$, then D is T-invariant. Suppose that D is closed, has property (N) with $q \in F(I)$, $\operatorname{cl}(T(D))$ is compact and I satisfies property (C) on D. Then $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$.

Proof. Let $x \in D$, then proceeding as in the proof of Theorem 2.8, we obtain $Tx \in P_M(u)$. Moreover, since I is nonexpansive on $P_M(u) \cup \{u\}$ and T satisfies (2.13), we obtain

$$\|ITx - u\|_p \leq \|Tx - Tu\|_p \leq \|Ix - Iu\|_p = \operatorname{dist}(u, M). \tag{2.14}$$

Thus $ITx \in P_M(u)$ and so $Tx \in C_M^I(u)$. Hence $Tx \in D$. Consequently, $T(D) \subset D = I(D)$. Now Theorem 2.2(i) guarantees that $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$. □

Remark 2.13. Notice that approximation results similar to Theorems 2.8, 2.11, and 2.12 can be obtained, using Theorem 2.2(ii), (iii), and (iv).

Example 2.14. Let $X = R$ and $M = \{0, 1, 1 - 1/(n+1) : n \in N\}$ be endowed with usual metric. Define $T1 = 0$ and $T0 = T(1 - 1/(n+1)) = 1$ for all $n \in N$. Clearly, M is not starshaped but M has the property (N) for $q = 0$ and $k_n = 1 - 1/(n+1)$, $n \in N$. Let $Ix = x$ for all $x \in M$. Now I and T satisfy (2.2) together with all other conditions of Theorem 2.2(i) except the condition that T is continuous. Note that $F(I) \cap F(T) = \emptyset$.

Example 2.15. Let $X = R^2$ be endowed with the p-norm $\|,\|_p$ defined by $\|(a,b)\|_p = |a|^p + |b|^p$, $(a,b) \in R^2$.

(1) Let $M = A \cup B$, where $A = \{(a,b) \in X : 0 \leq a \leq 1, 0 \leq b \leq 4\}$ and $B = \{(a,b) \in X : 2 \leq a \leq 3, 0 \leq b \leq 4\}$. Define $T : M \to M$ by

$$T(a,b) = \begin{cases} (2,b) & \text{if } (a,b) \in A, \\ (1,b) & \text{if } (a,b) \in B \end{cases} \tag{2.15}$$

and $I(x) = x$, for all $x \in M$. All of the conditions of Theorem 2.2(i) are satisfied except that M has property (N), that is, $(1 - k_n)q + k_n T(M)$ is not contained in M for any choice of $q \in M$ and k_n. Note $F(I) \cap F(T) = \emptyset$.

(2) If $M = \{(a,b) \in X : 0 \le a < \infty, 0 \le b \le 1\}$ and $T : M \to M$ is defined by

$$T(a,b) = (a+1,b), \quad (a,b) \in M. \tag{2.16}$$

Define $I(x) = x$, for all $x \in M$. All of the conditions of Theorem 2.2(i) are satisfied except that M is compact. Note $F(I) \cap F(T) = \emptyset$. Notice that M, being convex and T-invariant, has the property (N) for any choice of q and $\{k_n\}$.

(3) If $M = \{(a,b) \in X : 0 < a < 1, 0 < b < 1\}$ and $T, I : M \to M$ are defined by $T(a,b) = (a/2, b/3)$, and $I(x) = x$ for all $x \in M$. All of the conditions of Theorem 2.2(i) are satisfied except the fact that M is closed. However $F(I) \cap F(T) = \emptyset$.

Example 2.16. Let $X = R$ and $M = [0,1]$ be endowed with the usual metric. Define $T(x) = 0$ and $I(x) = 1 - x$ for each $x \in M$. All of the conditions of Theorem 2.2(i) are satisfied except the condition that the pair $\{I, T\}$ is R-subweakly commuting. Note $F(I) \cap F(T) = \emptyset$.

3. Further results

All results of the paper (Theorem 2.2–Remark 2.13) remain valid in the setup of a metrizable locally convex topological vector space(tvs) (X,d) where d is translation invariant and $d(\alpha x, \alpha y) \le \alpha d(x,y)$, for each α with $0 < \alpha < 1$ and $x, y \in X$ (recall that d_p is translation invariant and satisfies $d_p(\alpha x, \alpha y) \le \alpha^p d_p(x,y)$ for any scalar $\alpha \ge 0$). Consequently, Theorem 2.2 (i)-(ii) and Theorem 3.3 (i)-(ii) due to Hussain and Khan [6] and Theorem 3.5 (i)-(ii) & (v), (ix)-(x) and Theorem 4.2 (i)-(ii) & (v), (ix)-(x) due to Hussain et al. [7] are extended to a class of maps satisfying a more general inequality.

From Corollary 2.3, we have the following result which extends [18, Theorem 2.2];

Corollary 3.1. *Let M be a closed q-starshaped subset of a metrizable locally convex space (X,d) where d is translation invariant and $d(\alpha x, \alpha y) \le \alpha d(x,y)$, for each α with $0 < \alpha < 1$ and $x, y \in X$. Suppose that T and I are continuous self-maps of M, I is affine with $q \in F(I)$, $T(M) \subset I(M)$ and $\operatorname{cl} T(M)$ is compact. If the pair $\{T, I\}$ is R-subweakly commuting and satisfy for all $x, y \in M$,*

$$\begin{aligned} d(Tx,Ty) \le \max\{&d(Ix,Iy), \operatorname{dist}(Ix,[Tx,q]), \operatorname{dist}(Iy,[Ty,q]), \\ &\operatorname{dist}(Ix,[Ty,q]), \operatorname{dist}(Iy,[Tx,q])\}, \end{aligned} \tag{3.1}$$

then $F(T) \cap F(I) \ne \emptyset$.

We define $C_M^I(u) = \{x \in M : Ix \in P_M(u)\}$ and denote by $\mathfrak{I}_0$ the class of closed convex subsets of X containing 0. For $M \in \mathfrak{I}_0$, we define $M_u = \{x \in M : \|x\| \le 2\|u\|\}$. It is clear that $P_M(u) \subset M_u \in \mathfrak{I}_0$.

Following result includes [1, Theorem 4.1] and [5, Theorem 8] and provides an analogue of [18, Theorem 2.8] in the setting of metrizable locally convex space and contractive condition involved is more general.

Theorem 3.2. *Let X be as in Corollary 3.1, and T be a self-mapping of X with $u \in F(T)$, $M \in \mathfrak{I}_0$ such that $T(M) \subset M$. Suppose that $\operatorname{cl} T(M)$ is compact, T is continuous on M and*

satisfies for all $x \in M \cup \{u\}$,

$$d(Tx,Ty) \le \begin{cases} d(x,u) & \text{if } y = u, \\ \max\{d(x,y), \operatorname{dist}(x,[0,Tx]), \operatorname{dist}(y,[0,Ty]), & \\ \quad \operatorname{dist}(x,[0,Ty]), \operatorname{dist}(y,[0,Tx])\} & \text{if } y \in M, \end{cases} \tag{3.2}$$

then

(i) $P_M(u)$ *is nonempty, closed, and convex,*
(ii) $T(P_M(u)) \subset P_M(u)$,
(iii) $P_M(u) \cap F(T) \neq \emptyset$.

Proof. (i) Let $r = \operatorname{dist}(u,M)$. Then there is a minimizing sequence $\{y_n\}$ in M such that $\lim_n d(u,y_n) = r$. As $\operatorname{cl} T(M)$ is compact so $\{Ty_n\}$ has a convergent subsequence $\{Ty_m\}$ with $\lim Ty_m = x_0$ (say) in M. Now by (3.2)

$$r \le d(x_0,u) = \lim d(Ty_m,u) \le \lim d(y_m,u) = \lim d(y_n,u) = r. \tag{3.3}$$

Hence $x_0 \in P_M(u)$. Thus $P_M(u)$ is nonempty closed and convex.

(ii) Let $z \in P_M(u)$. Then $d(Tz,u) = d(Tz,Tu) \le d(z,u) = \operatorname{dist}(u,M)$. This implies that $Tz \in P_M(u)$ and so $T(P_M(u)) \subset P_M(u)$.

(iii) As $\operatorname{cl} T(P_M(u)) \subset \operatorname{cl} T(M)$, so $\operatorname{cl} T(P_M(u))$ is compact. Thus by Corollary 3.1, $P_M(u) \cap F(T) \neq \emptyset$. □

Theorem 3.3. *Let* X *be as in Theorem 3.2 and* I *and* T *be self-mappings of* X *with* $u \in F(I) \cap F(T)$ *and* $M \in \mathfrak{I}_0$ *such that* $T(M_u) \subset I(M) \subset M$. *Suppose that* I *is affine and continuous on* M, $d(Ix,u) \le d(x,u)$ *for all* $x \in M$, $\operatorname{cl} I(M)$ *is compact and* I *satisfies for all* $x,y \in M$,

$$\begin{aligned} d(Ix,Iy) \le \max\{&d(x,y), \operatorname{dist}(x,[0,Ix]), \operatorname{dist}(y,[0,Iy]), \\ &\operatorname{dist}(x,[0,Iy]), \operatorname{dist}(y,[0,Ix])\}. \end{aligned} \tag{3.4}$$

If the pair $\{T,I\}$ *is R-subweakly commuting and* T *is continuous on* M_u *and satisfy for all* $x,y \in M_u \cup \{u\}$, *and* $q \in F(I)$,

$$d(Tx,Ty) \le \begin{cases} d(Ix,Iu) & \text{if } y = u, \\ \max\{d(Ix,Iy), \operatorname{dist}(Ix,[q,Tx]), \operatorname{dist}(Iy,[q,Ty]), & \\ \quad \operatorname{dist}(Ix,[q,Ty]), \operatorname{dist}(Iy,[q,Tx])\} & \text{if } y \in M_u, \end{cases} \tag{3.5}$$

then

(i) $P_M(u)$ *is nonempty, closed, and convex,*
(ii) $T(P_M(u)) \subset I(P_M(u)) \subset P_M(u)$,
(iii) $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$.

Proof. From Theorem 3.2, we obtain (i). Also we have $I(P_M(u)) \subset P_M(u)$. Let $y \in TP_M(u)$. Since $T(M_u) \subset I(M)$ and $P_M(u) \subset M_u$, there exist $z \in P_M(u)$ and $x \in M$ such

that $y = Tz = Ix$. By (3.5), we have

$$d(Ix,u) = d(Tz,Tu) \le d(Iz,Iu) \le d(z,u) = \operatorname{dist}(u,M). \tag{3.6}$$

Hence $x \in C_M^I(u) = P_M(u)$ and so (ii) holds.

(iii) Theorem 3.2 guarantees that $P_M(u) \cap F(I) \neq \emptyset$. Thus there exists $q \in P_M(u)$ such that $q \in F(I)$. Hence the conclusion follows from Corollary 3.1. □

Following corollary provides the conclusions of [1, Theorem 4.2(a)] and [17, Theorem 2.3], to the setup of metrizable locally convex space.

Corollary 3.4. *Let X be as above and I, T be self-mappings of X with $u \in F(I) \cap F(T)$ and $M \in \Im_0$ such that $T(M_u) \subset I(M) \subset M$. Suppose that I is affine and continuous on M, $d(Ix,u) \le d(x,u)$ for all $x \in M$, $\mathrm{cl}I(M)$ is compact and I is nonexpansive on M. If the pair $\{T,I\}$ is R-subweakly commuting on M_u and T is I-nonexpansive on $M_u \cup \{u\}$, then*

(i) *$P_M(u)$ is nonempty, closed and convex,*

(ii) *$T(P_M(u)) \subset I(P_M(u)) \subset P_M(u)$,*

(iii) *$P_M(u) \cap F(I) \cap F(T) \neq \emptyset$.*

Let (X,d) be a metric linear space with translation invariant metric d. We say that the metric d is strictly monotone [4], if $x \neq 0$ and $0 < t < 1$ imply $d(0,tx) < d(0,x)$. Each p-norm generates a translation invariant metric, which is strictly monotone [4].

Following the arguments of Jungck [8, Theorem 3.2] and using Theorem 2.1 instead of Theorem 3.1 of Jungck [8], we obtain,

Theorem 3.5. *Let T and f be continuous self-maps of a compact metric space (X,d) with $T(X) \subset f(X)$. If T and f are R-weakly commuting self-maps of X such that*

$$d(Tx,Ty) < \max\{d(fx,fy),d(Tx,fx),d(Ty,fy),d(Tx,fy),d(Ty,fx)\} \tag{3.7}$$

when right hand side is positive, then there is a unique point z in X such that $Tz = fz = z$.

Using Theorem 3.5, we establish common fixed point generalization of Theorem 1 of Dotson [3], and Theorem 2 of Guseman and Peters [4].

Theorem 3.6. *Let T, I be self-maps on a compact subset M of a metric linear space (X,d) with translation invariant and strictly monotone metric d. Assume that M has the property (N) with $q \in F(I)$, I satisfies the condition (C) and $M = I(M)$. Suppose that T and I are R-subweakly commuting and satisfy*

$$\begin{aligned} d(Tx,Ty) \le \max\{&d(Ix,Iy),\operatorname{dist}(Ix,[Tx,q]),\operatorname{dist}(Iy,[Ty,q]),\\ &\operatorname{dist}(Ix,[Ty,q]),\operatorname{dist}(Iy,[Tx,q])\} \end{aligned} \tag{3.8}$$

for all $x,y \in M$. If T and I are continuous, then $F(T) \cap F(I) \neq \emptyset$.

Proof. Proof is similar to Theorem 2.2(i), instead of applying Theorem 2.1, we apply Theorem 3.5. □

Similarly, all other results of Section 2 (Corollary 2.3–Theorem 2.12) hold in the setting of metric linear space (X,d) with translation invariant and strictly monotone metric d provided we replace closedness of M and compactness of $\mathrm{cl}\,T(M)$ by compactness of M and using Theorem 3.6 instead of Theorem 2.2(i). Consequently, metric linear space versions of Corollary 2.3–Corollary 2.7 improve and extend Theorem 2 and the Corollary in [4].

A metric space (X,d) is said to be S-space [20], if there exists an x_0 in X such that for every $t \in (0,1)$ there is a d-contractive self-mapping f_t of X for which the inequality $d(f_t(x),x) \leq (1-t)d(x_0,x)$ holds for every x in X. As an application of Theorem 3.5 and [20, Theorem 1], we obtain the following extension of Theorems B, K, Z and C in [2] and Theorem 3 of [20] to generalized nonexpansive mappings.

Theorem 3.7. *Let (X,d) be a compact S-space and $T : X \to X$ satisfies for all $x, y \in X$,*

$$d(Tx,Ty) \leq \max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}. \tag{3.9}$$

Then T has a fixed point.

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