

QUASICONTRACTION NONSELF-MAPPINGS ON CONVEX METRIC SPACES AND COMMON FIXED POINT THEOREMS

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We consider quasicontraction nonself-mappings on Takahashi convex metric spaces and common fixed point theorems for a pair of maps. Results generalizing and unifying fixed point theorems of Ivanov, Jungck, Das and Naik, and Ćirić are established.

1. Introduction and preliminaries

Let X be a complete metric space. A map $T : X \rightarrow X$ such that for some constant $\lambda \in (0, 1)$ and for every $x, y \in X$

$$d(Tx, Ty) \leq \lambda \cdot \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \quad (1.1)$$

is called *quasicontraction*. Let us remark that Ćirić [1] introduced and studied quasicontraction as one of the most general contractive type map. The well known Ćirić's result (see, e.g., [1, 6, 11]) is that quasicontraction T possesses a unique fixed point.

For the convenience of the reader we recall the following recent Ćirić's result.

THEOREM 1.1 [2, Theorem 2.1]. *Let X be a Banach space, C a nonempty closed subset of X , and ∂C the boundary of C . Let $T : C \rightarrow X$ be a nonself mapping such that for some constant $\lambda \in (0, 1)$ and for every $x, y \in C$*

$$d(Tx, Ty) \leq \lambda \cdot \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \quad (1.2)$$

Suppose that

$$T(\partial C) \subset C. \quad (1.3)$$

Then T has a unique fixed point in C .

Following Ćirić [3], let us remark that *problem to extend the known fixed point theorem for self mappings $T : C \rightarrow C$, defined by (1.1), to corresponding nonself mappings $T : C \rightarrow X$, $C \neq X$, was open more than 20 years.*

In 1970, Takahashi [15] introduced the definition of convexity in metric space and generalized same important fixed point theorems previously proved for Banach spaces. In

this paper we consider quasicontraction nonself-mappings on Takahashi convex metric spaces and common fixed point theorems for a pair of maps. Results generalizing and unifying fixed point theorems of Ivanov [7], Jungck [8], Das and Naik [3], Ćirić [2], Gajić [5] and Rakočević [12] are established.

Let us recall that (see Jungck [9]) the self maps f and g on a metric space (X, d) are said to be a *compatible pair* if

$$\lim_{n \rightarrow \infty} d(gfx_n, fgx_n) = 0 \quad (1.4)$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} gx_n = \lim_{n \rightarrow \infty} fx_n = x \quad (1.5)$$

for some x in X .

Following Sessa [14] we will say that $f, g : X \rightarrow X$ are *weakly commuting* if

$$d(fgx, gfx) \leq d(fx, gx) \quad \text{for every } x \in X. \quad (1.6)$$

Clearly weak commutativity of f and g is a generalization of the conventional commutativity of f and g , and the concept of compatibility of two mappings includes weakly commuting mappings as a proper subclass.

We recall the following definition of a convex metric space (see [15]).

Definition 1.2. Let X be a metric space and $I = [0, 1]$ the closed unit interval. A Takahashi convex structure on X is a function $W : X \times X \times I \rightarrow X$ which has the property that for every $x, y \in X$ and $\lambda \in I$

$$d(z, W(x, y, \lambda)) \leq \lambda d(z, x) + (1 - \lambda)d(z, y) \quad (1.7)$$

for every $z \in X$. If (X, d) is equipped with a Takahashi convex structure, then X is called a Takahashi convex metric space.

If (X, d) is a Takahashi convex metric space, then for $x, y \in X$ we set

$$\text{seg}[x, y] = \{W(x, y, \lambda) : \lambda \in [0, 1]\}. \quad (1.8)$$

Let us remark that any convex subset of normed space is a convex metric space with $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$.

2. Main results

The next theorem is our main result.

THEOREM 2.1. *Let (X, d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable, C a nonempty closed subset of X and ∂C the boundary of C . Let $g : C \rightarrow X$, $f : X \rightarrow X$ and $f : C \rightarrow C$. Suppose that $\partial C \neq \emptyset$, f is continuous, and let us assume that f and g satisfy the following conditions.*

(i) *For every $x, y \in C$*

$$d(gx, gy) \leq M_\omega(x, y), \quad (2.1)$$

where

$$M_\omega(x, y) = \max \{ \omega[d(fx, fy)], \omega[d(fx, gx)], \omega[d(fy, gy)], \omega[d(fx, gx)], \omega[d(fy, gx)] \}, \quad (2.2)$$

$\omega : [0, +\infty) \rightarrow [0, +\infty)$ is a nondecreasing semicontinuous function from the right, such that $\omega(r) < r$, for $r > 0$, and $\lim_{r \rightarrow \infty} [r - \omega(r)] = +\infty$.

(ii) *f and g are a compatible pair on C , that is,*

$$\lim_{n \rightarrow \infty} d(gfx_n, fgx_n) = 0 \quad (2.3)$$

whenever $\{x_n\}$ is a sequence in C such that

$$\lim_{n \rightarrow \infty} gx_n = \lim_{n \rightarrow \infty} fx_n = x \quad (2.4)$$

for some x in X .

(iii)

$$g(C) \cap C \subset f(C). \quad (2.5)$$

(iv)

$$g(\partial C) \subset C. \quad (2.6)$$

(v)

$$f(\partial C) \supset \partial C. \quad (2.7)$$

Then f and g have a unique common fixed point z in C .

Proof. Starting with an arbitrary $x_0 \in \partial C$, we construct a sequence $\{x_n\}$ of points in C as follows. By (2.6) $g(x_0) \in C$. Hence, (2.5) implies that there is $x_1 \in C$ such that $f(x_1) = g(x_0)$. Let us consider $g(x_1)$. If $g(x_1) \in C$, again by (2.5) there is $x_2 \in C$ such that $f(x_2) = g(x_1)$. Suppose that $g(x_1) \notin C$. Now, because W is continuous in the third

variable, there exists $\lambda_{11} \in [0, 1]$ such that

$$W(f(x_1), g(x_1), \lambda_{11}) \in \partial C \cap \text{seg}[f(x_1), g(x_1)]. \quad (2.8)$$

By (2.7) there is $x_2 \in \partial C$ such that $f(x_2) = W(f(x_1), g(x_1), \lambda_{11})$.

Hence, by induction we construct a sequence $\{x_n\}$ of points in C as follows. If $g(x_n) \in C$, then by (2.5) $f(x_{n+1}) = g(x_n)$ for some $x_{n+1} \in C$; if $g(x_n) \notin C$, then there exists $\lambda_{nn} \in [0, 1]$ such that

$$W(f(x_n), g(x_n), \lambda_{nn}) \in \partial C \cap \text{seg}[f(x_n), g(x_n)]. \quad (2.9)$$

Now, by (2.7) pick $x_{n+1} \in \partial C$ such that

$$f(x_{n+1}) = W(f(x_n), g(x_n), \lambda_{nn}). \quad (2.10)$$

Let us remark (see [6]) that for every $x, y \in X$ and every $\lambda \in [0, 1]$

$$d(x, y) = d(x, W(x, y, \lambda)) + d(W(x, y, \lambda), y). \quad (2.11)$$

Furthermore, if $u \in X$ and $z = W(x, y, \lambda) \in \text{seg}[x, y]$ then

$$d(u, z) = d(u, W(x, y, \lambda)) \leq \max\{d(u, x), d(u, y)\}. \quad (2.12)$$

First let us prove that

$$f(x_{n+1}) \neq g(x_n) \implies f(x_n) = g(x_{n-1}). \quad (2.13)$$

Suppose the contrary that $f(x_n) \neq g(x_{n-1})$. Then $x_n \in \partial C$. Now, by (2.5) $g(x_n) \in C$, hence $f(x_{n+1}) = g(x_n)$, a contradiction. Thus we prove (2.13).

We will prove that $g(x_n)$ and $f(x_n)$ are Cauchy sequences. First we will prove that these sequences are bounded, that is that the set

$$A = \left(\bigcup_{i=0}^{\infty} \{f(x_i)\} \right) \cup \left(\bigcup_{i=0}^{\infty} \{g(x_i)\} \right) \quad (2.14)$$

is bounded.

For each $n \geq 1$ set

$$A_n = \left(\bigcup_{i=0}^{n-1} \{f(x_i)\} \right) \cup \left(\bigcup_{i=0}^{n-1} \{g(x_i)\} \right), \quad (2.15)$$

$$a_n = \text{diam}(A_n).$$

We will prove that

$$a_n = \max\{d(f(x_0), g(x_i)) : 0 \leq i \leq n-1\}. \quad (2.16)$$

If $a_n = 0$, then $f(x_0) = g(x_0)$. We will prove that $g(x_0)$ is a common fixed point for f and g . By (2.3) it follows that

$$fg(x_0) = gf(x_0) = gg(x_0). \quad (2.17)$$

Now we obtain

$$d(gg(x_0), g(x_0)) \leq M_\omega(gx_0, x_0) = \omega(d(gg(x_0), g(x_0))), \quad (2.18)$$

and hence $gg(x_0) = g(x_0)$. From (2.17), we conclude that $g(x_0) = z$ is also a fixed point of f . To prove the uniqueness of the common fixed point, let us suppose that $fu = gu = u$ for some $u \in C$. Now, by (2.1) we have

$$d(z, u) = d(gz, gu) \leq M_\omega(z, u) = \omega(d(z, u)), \quad (2.19)$$

and so, $z = u$.

Suppose that $a_n > 0$. To prove (2.16) we have to consider three cases.

Case 1. Suppose that $a_n = d(fx_i, gx_j)$ for some $0 \leq i, j \leq n-1$.

(1i) Now, if $i \geq 1$ and $fx_i = gx_{i-1}$, we have

$$a_n = d(fx_i, gx_j) = d(gx_{i-1}, gx_j) \leq M_\omega(x_{i-1}, x_j) \leq \omega(a_n) < a_n. \quad (2.20)$$

and we get a contradiction. Hence $i = 0$.

(1ii) If $i \geq 1$ and $fx_i \neq gx_{i-1}$, we have $i \geq 2$, and $fx_{i-1} = gx_{i-2}$. Hence

$$fx_i \in \text{seg}[g(x_{i-2}), g(x_{i-1})], \quad (2.21)$$

we have

$$\begin{aligned} a_n = d(fx_i, gx_j) &\leq \max \{d(gx_{i-2}, gx_j), d(gx_{i-1}, gx_j)\} \\ &\leq \max \{M_\omega(x_{i-2}, x_j), M_\omega(x_{i-1}, x_j)\} \leq \omega(a_n) < a_n \end{aligned} \quad (2.22)$$

and we get a contradiction.

Case 2. Suppose that $a_n = d(fx_i, fx_j)$ for some $0 \leq i, j \leq n-1$.

(2i) If $fx_j = gx_{j-1}$, then Case (2i) reduces to Case (1i).

(2ii) If $fx_j \neq gx_{j-1}$, then as in the Case (1ii) we have $j \geq 2$, $fx_{j-1} = gx_{j-2}$, and

$$fx_j \in \partial C \cap \text{seg}[gx_{j-2}, gx_{j-1}]. \quad (2.23)$$

Hence

$$a_n = d(fx_i, fx_j) \leq \max \{d(fx_i, gx_{j-2}), d(fx_i, gx_{j-1})\} \quad (2.24)$$

and Case (2ii) reduces to Case (1i).

Case 3. The remaining case $a_n = d(gx_i, gx_j)$ for some $0 \leq i, j \leq n-1$, is not possible (see Case (1i)). Hence we proved (2.16).

Now

$$a_n = d(fx_0, gx_i) \leq d(fx_0, gx_0) + d(gx_0, gx_i) \leq d(fx_0, gx_0) + \omega(a_n), \quad (2.25)$$

$$a_n - \omega(a_n) \leq d(fx_0, gx_0). \quad (2.26)$$

By (i) there is $r_0 \in [0, +\infty)$ such that

$$r - \omega(r) > d(fx_0, gx_0), \quad \text{for } r > r_0. \quad (2.27)$$

Thus, by (2.26)

$$a_n \leq r_0, \quad n = 1, 2, \dots, \quad (2.28)$$

and clearly

$$a = \lim_{n \rightarrow \infty} a_n = \text{diam}(A) \leq r_0. \quad (2.29)$$

Hence we proved that gx_n and fx_n are bounded sequences.

To prove that gx_n and fx_n are Cauchy sequences, let us consider the set

$$B_n = \left(\bigcup_{i=n}^{\infty} \{fx_i\} \right) \cup \left(\bigcup_{i=n}^{\infty} \{gx_i\} \right), \quad n = 2, 3, \dots \quad (2.30)$$

By (2.16) we have

$$b_n \equiv \text{diam}(B_n) = \sup_{j \geq n} d(fx_n, gx_j), \quad n = 1, 2, \dots \quad (2.31)$$

If $fx_n = gx_{n-1}$, then as in Case (1i) for each $j \geq n$

$$b_n = d(fx_n, gx_j) = d(gx_{n-1}, gx_j) \leq \omega(b_{n-1}), \quad n = 1, 2, \dots \quad (2.32)$$

If $fx_n \neq gx_{n-1}$, then as in Case (1ii) for each $n \geq 1$ and $j \geq n$

$$b_n = d(fx_n, gx_j) \leq \max \{d(gx_{n-2}, gx_j), d(gx_{n-1}, gx_j)\} \leq \omega(b_{n-2}). \quad (2.33)$$

By (2.32) and (2.33) we get

$$b_n \leq \omega(b_{n-2}), \quad n = 2, 3, \dots \quad (2.34)$$

Clearly, $b_n \geq b_{n+1}$ for each n , and set $\lim_n b_n = b$. We will prove that $b = 0$. If $b > 0$, then (2.34) and (i) imply $b \leq \omega(b) < b$, and we get a contradiction. It follows that both fx_n and gx_n are Cauchy sequences. Since $fx_n \in C$ and C is a closed subset of a complete metric space X we conclude that $\lim_n fx_n = y \in C$. Furthermore,

$$d(f(x_n), g(x_n)) \rightarrow 0, \quad n \rightarrow \infty, \quad (2.35)$$

implies $\lim g(x_n) = y$. Hence,

$$\lim g(x_n) = \lim f(x_n) = y \in C. \quad (2.36)$$

By continuity of f

$$\lim f(g(x_n)) = \lim f(f(x_n)) = f(y) \in C. \quad (2.37)$$

Now, by (2.3), we have

$$d(gf(x_n), f(y)) \leq d(gf(x_n), fg(x_n)) + d(fg(x_n), f(y)) \rightarrow 0, \quad n \rightarrow \infty, \quad (2.38)$$

that is

$$\lim(gf)(x_n) = f(y). \quad (2.39)$$

Now,

$$\begin{aligned} M_\omega(fx_n, y) &\rightarrow \omega(d(fy, gy)) \quad n \rightarrow \infty, \\ d(gfx_n, gy) &\leq M_\omega(fx_n, y) \quad n \rightarrow \infty, \end{aligned} \quad (2.40)$$

implies

$$d(fy, gy) \leq \omega(d(fy, gy)). \quad (2.41)$$

Hence, $f(y) = g(y)$, and gy is a common fixed point of f and g (see (2.17)). $\square$

In the special case, when $\omega(r) = \lambda \cdot r$ where $0 < \lambda < 1$, we obtain the following result.

THEOREM 2.2. *Let (X, d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable, C a nonempty closed subset of X and ∂C the boundary of C . Let $g : C \rightarrow X$, $f : X \rightarrow X$ and $f : C \rightarrow C$. Suppose that $\partial C \neq \emptyset$, f is continuous, and let us assume that f and g satisfy the following conditions.*

(i) *There exists a constant $\lambda \in (0, 1)$ such that for every $x, y \in C$*

$$d(gx, gy) \leq \lambda \cdot M(x, y), \quad (2.42)$$

where

$$M(x, y) = \max \{d(fx, fy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}. \quad (2.43)$$

Suppose that the conditions (ii)–(v) in Theorem 2.1 are satisfied. Then f and g have a unique common fixed point z in C and g is continuous at z . Moreover, if $z_n \in C$, $n = 1, 2, \dots$, then

$$\lim d(fz_n, gz_n) = 0 \quad \text{iff} \quad \lim_n z_n = z. \quad (2.44)$$

Proof. By Theorem 2.1 we know that f and g have a unique common fixed point z in C . Now, we show that g is continuous at z . Let $\{y_n\}$ be a sequence in C such that $y_n \rightarrow z$.

Now we have

$$\begin{aligned}
 d(gy_n, gz) &\leq \lambda \cdot M(y_n, z) \\
 &= \lambda \cdot \max \{d(fy_n, fz), d(fy_n, gy_n), d(fz, gy_n)\} \\
 &= \lambda \cdot \max \{d(fy_n, fz), d(fy_n, gy_n)\} \\
 &\leq \lambda \cdot (d(fy_n, fz) + d(fz, gy_n)),
 \end{aligned} \tag{2.45}$$

that is

$$d(gy_n, gz) \leq (1 - \lambda)^{-1} \lambda \cdot d(fy_n, fz). \tag{2.46}$$

Therefore, we have $gy_n \rightarrow gz$ and so g is continuous at z . To prove (2.44), let us suppose that $w \in C$. Now, since $fz = gz = z$, we have

$$\begin{aligned}
 d(fw, gw) &\leq d(fw, fz) + d(gw, gz) \leq d(fw, fz) + \lambda \cdot M(w, z) \\
 &\leq d(fw, fz) + \lambda \cdot \max \{d(fw, fz), d(fw, gw), d(fz, gw)\} \\
 &\leq d(fw, fz) + \lambda \cdot (d(fw, fz) + d(fw, gw)),
 \end{aligned} \tag{2.47}$$

that is

$$(1 - \lambda)d(fw, gw) \leq (1 + \lambda)d(fw, fz). \tag{2.48}$$

Let us remark that

$$\begin{aligned}
 d(fw, fz) &\leq d(fw, gw) + d(gw, gz) \leq d(fw, gw) + \lambda \cdot M(w, z) \\
 &\leq d(fw, gw) + \lambda \cdot \max \{d(fw, fz), d(fw, gw), d(fz, gw)\} \\
 &\leq d(fw, gw) + \lambda \cdot (d(fw, fz) + d(fw, gw)),
 \end{aligned} \tag{2.49}$$

that is

$$(1 - \lambda)d(fw, fz) \leq (1 + \lambda)d(fw, gw). \tag{2.50}$$

By (2.48) and (2.50) we obtain

$$\begin{aligned}
 (1 - \lambda)d(fw, gw) &\leq (1 + \lambda)d(fw, fz) \\
 &\leq (1 - \lambda)^{-1} (1 + \lambda)^2 d(fw, gw).
 \end{aligned} \tag{2.51}$$

Clearly (2.51) implies (2.44). □

Remark 2.3. Let (K, ρ) be a bounded metric space. It is said that the fixed point problem for a mapping $A : K \rightarrow K$ is *well posed* if there exists a unique $x_A \in K$ such that $Ax_A = x_A$ and the following property holds: If $\{x_n\} \subset K$ and $\rho(x_n, Ax_n) \rightarrow 0$ as $n \rightarrow \infty$, then $\rho(x_n, x_A) \rightarrow 0$ as $n \rightarrow \infty$. Let us remark that condition (2.44) is related to the notion

of well posed fixed point problem, and the notion of well-posedness is of central importance in many areas of Mathematics and its applications ([4, 10, 13]).

Remark 2.4. If in Theorem 2.1 we let f be the identity map on X and $\omega(r) = \lambda \cdot r$ where $0 < \lambda < 1$, we get Ćirić's Theorem 1.1 (Gajić's theorem [5]) stated for a Banach (convex complete metric) space X .

Remark 2.5. If in Theorem 2.1 we let f be the identity map on X and $C = X$, we get Ivanov's result [6, 7] stated for a Banach space X .

Remark 2.6. Let us recall that the first part of Theorem 2.2, that is the existence of the unique common fixed point of f and g was proved by Rakočević [12].

By the proof of Theorem 2.1 we can recover some results of Das and Naik [3] and Jungck [8].

COROLLARY 2.7 [3, Theorem 2.1]. *Let X be a complete metric space. Let f be a continuous self-map on X and g be any self-map on X that commutes with f . Further let f and g satisfy*

$$g(X) \subset f(X) \quad (2.52)$$

and there exists a constant $\lambda \in (0, 1)$ such that for every $x, y \in X$

$$d(gx, gy) \leq \lambda \cdot M(x, y), \quad (2.53)$$

where

$$M(x, y) = \max \{d(fx, fy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}. \quad (2.54)$$

Then f and g have a unique fixed point.

Proof. We follow the proof of Theorem 2.1. Let us remark that the condition (2.52) implies that starting with an arbitrary $x_0 \in X$, we construct a sequence $\{x_n\}$ of points in X such that $f(x_{n+1}) = g(x_n)$, $n = 0, 1, 2, \dots$. The rest of the proof follows by the proof of Theorem 2.1. $\square$

COROLLARY 2.8 [3, Theorem 3.1]. *Let X be a complete metric space. Let f^2 be a continuous self-map on X and g be any self-map on X that commutes with f . Further let f and g satisfy*

$$gf(X) \subset f^2(X) \quad (2.55)$$

and $f(g(x)) = g(f(x))$ whenever both sides are defined. Further, let there exist a constant $\lambda \in (0, 1)$ such that for every $x, y \in f(X)$

$$d(gx, gy) \leq \lambda \cdot M(x, y), \quad (2.56)$$

where

$$M(x, y) = \max \{d(fx, fy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}. \quad (2.57)$$

Then f and g have a unique common fixed point.

Proof. Again, we follow the proof of Theorem 2.1. By (2.55) starting with an arbitrary $x_0 \in f(X)$, we construct a sequence $\{x_n\}$ of points in $f(X)$ such that $f(x_{n+1}) = g(x_n) = y_n$, $n = 0, 1, 2, \dots$. Now $f(y_n) = f(g(x_n)) = g(f(x_n)) = g(y_{n-1}) = z_n$, $n = 1, 2, \dots$, and from the proof of Theorem 2.1 we conclude that $\{z_n\}$ is a Cauchy sequence in X and hence convergent to some $z \in X$. Now, for each $n \geq 1$

$$\begin{aligned} & d(f^2g(x_n), gf(z)) \\ &= d(gf^2(x_n), gf(z)) \leq \lambda \cdot M(f^2(x_n), f(z)) \\ &= \lambda \cdot \max \left\{ d(f^2f(x_n), f^2(z)), d(f^2f(x_n), f^2g(x_n)), \right. \\ &\quad \left. d(f^2(z), gf(z)), d(f^2f(x_n), gf(z)), d(f^2(z), f^2g(x_n)) \right\}. \end{aligned} \quad (2.58)$$

Now, by continuity of f^2

$$d(f^2(z), gf(z)) \leq \lambda \cdot d(f^2(z), gf(z)). \quad (2.59)$$

Whence, $f^2(z) = gf(z)$, and gfz is a unique common fixed of f and g . $\square$

Let us remark that from Theorem 2.1 and the proof of Corollary 2.7, we get the following.

COROLLARY 2.9. *Let X be a complete metric space. Let f be a continuous self-map on X and g be any self-map on X that weakly commutes with f . Further let f and g satisfy (2.52) and (2.53). Then f and g have a unique common fixed point.*

Now as a corollary we get the following result of Jungck [8].

COROLLARY 2.10. *Let X be a complete metric space. Let f be a continuous self-map on X and g be any self-map on X that commutes with f . Further let f and g satisfy (2.52) and there exists a constant $\lambda \in (0, 1)$ such that for every $x, y \in X$*

$$d(gx, gy) \leq \lambda \cdot d(fx, fy). \quad (2.60)$$

Then f and g have a unique common fixed point.

COROLLARY 2.11. *Let X be a convex complete metric space, C a nonempty compact subset of X , and ∂C the boundary of C . Let $g : C \rightarrow X$, $f : X \rightarrow X$ and $f : C \rightarrow C$. Suppose that g and f are continuous, f and g satisfy the conditions (ii)–(v) in Theorem 2.1, and for all $x, y \in C$, $x \neq y$*

$$d(gx, gy) < M(x, y), \quad (2.61)$$

where

$$M(x, y) = \max \{d(fx, fy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}. \quad (2.62)$$

Then f and g have a unique common fixed point in C .

Proof. By Theorem 2.2 and the proof of [12, Theorem 4]. $\square$

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