

# CONVERGENCE THEOREMS FOR A COMMON FIXED POINT OF A FINITE FAMILY OF NONSELF NONEXPANSIVE MAPPINGS

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Let  $K$  be a nonempty closed convex subset of a reflexive real Banach space  $E$  which has a uniformly Gâteaux differentiable norm. Assume that  $K$  is a sunny nonexpansive retract of  $E$  with  $Q$  as the sunny nonexpansive retraction. Let  $T_i : K \rightarrow E$ ,  $i = 1, \dots, r$ , be a family of nonexpansive mappings which are weakly inward. Assume that every nonempty closed bounded convex subset of  $K$  has the fixed point property for nonexpansive mappings. A strong convergence theorem is proved for a common fixed point of a family of nonexpansive mappings provided that  $T_i$ ,  $i = 1, 2, \dots, r$ , satisfy some mild conditions.

## 1. Introduction

Let  $K$  be a nonempty closed convex subset of a real Banach space  $E$ . A mapping  $T : K \rightarrow E$  is called *nonexpansive* if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in K$ . Let  $T : K \rightarrow K$  be a nonexpansive self-mapping. For a sequence  $\{\alpha_n\}$  of real numbers in  $(0, 1)$  and an arbitrary  $u \in K$ , let the sequence  $\{x_n\}$  in  $K$  be iteratively defined by  $x_0 \in K$ ,

$$x_{n+1} := \alpha_{n+1}u + (1 - \alpha_{n+1})Tx_n, \quad n \geq 0. \quad (1.1)$$

Halpern [5] was the first to study the convergence of the algorithm (1.1) in the framework of Hilbert spaces. Lions [6] improved the result of Halpern, still in Hilbert spaces, by proving strong convergence of  $\{x_n\}$  to a fixed point of  $T$  if the real sequence  $\{\alpha_n\}$  satisfies the following conditions:

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ;
- (ii)  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ;
- (iii)  $\lim_{n \rightarrow \infty} ((\alpha_n - \alpha_{n-1})/\alpha_n^2) = 0$ .

It was observed that both Halpern's and Lions' conditions on the real sequence  $\{\alpha_n\}$  excluded the natural choice,  $\alpha_n := (n+1)^{-1}$ . This was overcome by Wittmann [12] who proved, still in Hilbert spaces, the strong convergence of  $\{x_n\}$  if  $\{\alpha_n\}$  satisfies the following conditions:

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ;
- (ii)  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ;
- (iii)\*  $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ .

Reich [9] extended this result of Wittmann to the class of Banach spaces which are uniformly smooth and have weakly sequentially continuous duality maps. Moreover, the sequence  $\{\alpha_n\}$  is required to satisfy conditions (i) and (ii) and to be decreasing (and hence also satisfying (iii)\*). Subsequently, Shioji and Takahashi [10] extended Wittmann's result to Banach spaces with uniformly Gâteaux differentiable norms and in which each nonempty closed convex subset of  $K$  has the fixed point property for nonexpansive mappings and  $\{\alpha_n\}$  satisfies conditions (i), (ii), and (iii)\*.

Xu [13] showed that the results of Halpern holds in *uniformly smooth Banach spaces* if  $\{\alpha_n\}$  satisfies the following conditions:

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ;
- (ii)  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ;
- (iii)\*\*  $\lim_{n \rightarrow \infty} ((\alpha_n - \alpha_{n-1})/\alpha_n) = 0$ .

As has been remarked in [13], conditions (iii) and (iii)\* are not comparable. Also conditions (iii)\* and (iii)\*\* are not comparable. However, condition (iii) does not permit the natural choice  $\alpha_n := (n+1)^{-1}$  for all integers  $n \geq 0$ . Hence, conditions (iii)\* and (iii)\*\* are preferred.

In [2], Chidume et al. extended the results of Xu to Banach spaces which are more general than uniformly smooth spaces.

Next consider  $r$  nonexpansive mappings  $T_1, T_2, \dots, T_r$ . For a sequence  $\{\alpha_n\} \subseteq (0, 1)$  and an arbitrary  $u_0 \in K$ , let the sequence  $\{x_n\}$  in  $K$  be iteratively defined by  $x_0 \in K$ ,

$$x_{n+1} := \alpha_{n+1}u + (1 - \alpha_{n+1})T_{n+1}x_n, \quad n \geq 0, \quad (1.2)$$

where  $T_n = T_{n(\text{mod } r)}$ .

In 1996, Bauschke [1] defined and studied the iterative process (1.2) in Hilbert spaces with conditions in (i), (ii), and (iii)\* on the parameter  $\{\alpha_n\}$ .

Recently, Takahashi et al. [11] extended Bauschke's result to uniformly convex Banach spaces. More precisely, they proved the following result.

**THEOREM 1.1** [11]. *Let  $K$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$  which has a uniformly Gâteaux differentiable norm. Let  $T_i : K \rightarrow K$ ,  $i = 1, \dots, r$ , be a family of nonexpansive mappings with  $F := \bigcap_{i=1}^r F(T_i) \neq \emptyset$  and  $\bigcap_{i=1}^r F(T_i) = F(T_r T_{r-1} \cdots T_1) = F(T_1 T_r \cdots T_2) = \cdots = F(T_{r-1} T_{r-2} \cdots T_1 T_r)$ . For given  $u, x_0 \in K$ , let  $\{x_n\}$  be generated by the algorithm*

$$x_{n+1} := \alpha_{n+1}u + (1 - \alpha_{n+1})T_{n+1}x_n, \quad n \geq 0, \quad (1.3)$$

where  $T_n := T_{n(\text{mod } r)}$  and  $\{\alpha_n\}$  is a real sequence which satisfies the following conditions: (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ; (ii)  $\sum_{n=1}^{\infty} \alpha_n = \infty$ , and (iii)\*  $\sum_{n=1}^{\infty} |\alpha_{n+r} - \alpha_n| < \infty$ . Then  $\{x_n\}$  converges strongly to a common fixed point of  $\{T_1, T_2, \dots, T_r\}$ . Further, if  $Px_0 = \lim_{n \rightarrow \infty} x_n$  for each  $x_0 \in K$ , then  $P$  is a sunny nonexpansive retraction of  $K$  onto  $F$ .

More recently, O'Hara et al. [8] proved the following complementary result to Bauschke's theorem [1] with condition (iii)\* replaced with (iii)\*\*  $\lim_{n \rightarrow \infty} ((\alpha_{n+r} - \alpha_n)/\alpha_{n+r}) = 0$  (or equivalently,  $\lim_{n \rightarrow \infty} (\alpha_n/\alpha_{n+r}) = 1$ ).

**THEOREM 1.2** [8]. Let  $K$  be a nonempty closed convex subset of a Hilbert space  $H$  and let  $T_i : K \rightarrow K$ ,  $i = 1, \dots, r$ , be a family of nonexpansive mappings with  $F := \bigcap_{i=1}^r F(T_i) \neq \emptyset$  and  $\bigcap_{i=1}^r F(T_i) = F(T_r T_{r-1} \cdots T_1) = F(T_1 T_r \cdots T_2) = \cdots = F(T_{r-1} T_{r-2} \cdots T_1 T_r)$ . For given  $u, x_0 \in K$ , let  $\{x_n\}$  be generated by the algorithm

$$x_{n+1} := \alpha_{n+1}u + (1 - \alpha_{n+1})T_{n+1}x_n, \quad n \geq 0, \quad (1.4)$$

where  $T_n := T_{n(\text{mod } r)}$  and  $\{\alpha_n\}$  is a real sequence which satisfies the following conditions: (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ; (ii)  $\sum_{n=1}^{\infty} \alpha_n = \infty$ , and (iii)\*\*  $\lim_{n \rightarrow \infty} (\alpha_n / \alpha_{n+r}) = 1$ . Then  $\{x_n\}$  converges strongly to  $Pu$ , where  $P$  is the projection of  $K$  onto  $F$ .

In the above work, the mappings  $T_1, T_2, \dots, T_r$  remain self-mappings of a nonempty closed convex subset  $K$  either of a Hilbert space or a uniformly convex space. If, however, the domain of  $T_1, T_2, \dots, T_r$ ,  $D(T_i) = K$ ,  $i = 1, 2, \dots, r$ , is a proper subset of  $E$  and  $T_i$  maps  $K$  into  $E$ , then the iteration process (1.4) may fail to be well defined (see also (1.3)).

It is our purpose in this paper to define an algorithm for nonself-mappings and to obtain a strong convergence theorem to a fixed point of a family of nonself nonexpansive mappings in Banach spaces more general than the spaces considered by Takahashi et al. [11] with  $\{\alpha_n\}$  satisfying conditions (i), (ii), and (iii)\*. We also show that our result holds if  $\{\alpha_n\}$  satisfies conditions (i), (ii), and (iii)\*\*. Our results extend and improve the corresponding results of O'Hara et al. [8], Takahashi et al. [11], and hence Bauschke [1] to more general Banach spaces and to the class of *nonself*-maps.

## 2. Preliminaries

Let  $E$  be a real Banach space with dual  $E^*$ . We denote by  $J$  the normalized duality mapping from  $E$  to  $2^{E^*}$  defined by

$$Jx := \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \quad (2.1)$$

where  $\langle \cdot, \cdot \rangle$  denotes the generalized duality pairing. It is well known that if  $E^*$  is strictly convex, then  $J$  is single valued. In the sequel, we will denote the single-valued normalized duality map by  $j$ .

The norm is said to be *uniformly Gâteaux differentiable* if for each  $y \in S_1(0) := \{x \in E : \|x\| = 1\}$ ,  $\lim_{t \rightarrow 0} ((\|x + ty\| - \|x\|)/t)$  exists uniformly for  $x \in S_1(0)$ . It is well known that  $L_p$  spaces,  $1 < p < \infty$ , have uniformly Gâteaux differentiable norm (see, e.g., [4]). Furthermore, if  $E$  has a uniformly Gâteaux differentiable norm, then the duality map is norm-to- $w^*$  uniformly continuous on bounded subsets of  $E$ .

A Banach space  $E$  is said to be *strictly convex* if  $\|(x + y)/2\| < 1$  for  $x, y \in E$  with  $\|x\| = \|y\| = 1$  and  $x \neq y$ . In a strictly convex Banach space  $E$ , we have that if  $\|x\| = \|y\| = \|\lambda x + (1 - \lambda)y\|$ , for  $x, y \in E$  and  $\lambda \in (0, 1)$ , then  $x = y$ .

Let  $K$  be a nonempty subset of a Banach space  $E$ . For  $x \in K$ , the *inward set* of  $x$ ,  $I_K(x)$ , is defined by  $I_K(x) := \{x + \lambda(u - x) : u \in K, \lambda \geq 1\}$ . A mapping  $T : K \rightarrow E$  is called *weakly inward* if  $Tx \in \text{cl}[I_K(x)]$  for all  $x \in K$ , where  $\text{cl}[I_K(x)]$  denotes the closure of the inward set. Every self-map is trivially weakly inward.

Let  $K \subseteq E$  be closed convex and  $Q$  a mapping of  $E$  onto  $K$ . Then  $Q$  is said to be *sunny* if  $Q(Qx + t(x - Qx)) = Qx$  for all  $x \in E$  and  $t \geq 0$ . A mapping  $Q$  of  $E$  into  $E$  is said to be a *retraction* if  $Q^2 = Q$ . If a mapping  $Q$  is a retraction, then  $Qz = z$  for every  $z \in R(Q)$ , range of  $Q$ . A subset  $K$  of  $E$  is said to be a *sunny nonexpansive retract* of  $E$  if there exists a sunny nonexpansive retraction of  $E$  onto  $K$  and it is said to be a *nonexpansive retract* of  $E$  if there exists a nonexpansive retraction of  $E$  onto  $K$ . If  $E = H$ , the metric projection  $P_K$  is a sunny nonexpansive retraction from  $H$  to any closed convex subset of  $H$ .

In the sequel, we will make use of the following lemma.

LEMMA 2.1. Let  $\{a_n\}$  be a sequence of nonnegative real numbers satisfying the relation

$$a_{n+1} \leq (1 - \alpha_n)a_n + \sigma_n, \quad n \geq 0, \quad (2.2)$$

where (i)  $0 < \alpha_n < 1$ ; (ii)  $\sum_{n=1}^{\infty} \alpha_n = \infty$ . Suppose, either (a)  $\sigma_n = o(\alpha_n)$ , or (b)  $\sum_{n=1}^{\infty} \sigma_n < \infty$ , or (c)  $\limsup_{n \rightarrow \infty} \sigma_n \leq 0$ . Then  $a_n \rightarrow 0$  as  $n \rightarrow \infty$  (see, e.g., [13]).

We will also need the following results.

LEMMA 2.2 (see, e.g., [7]). Let  $E$  be a real Banach space. Then the following inequality holds. For each  $x, y \in E$ ,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle \quad \forall j(x + y) \in J(x + y). \quad (2.3)$$

THEOREM 2.3 [7, Theorem 1, Proposition 2(v)]. Let  $K$  be a nonempty closed convex subset of a reflexive Banach space  $E$  which has a uniformly Gâteaux differentiable norm. Let  $T : K \rightarrow E$  be a nonexpansive mapping with  $F(T) \neq \emptyset$ . Suppose that every nonempty closed convex bounded subset of  $K$  has the fixed point property for nonexpansive mappings. Then there exists a continuous path  $t \rightarrow z_t$ ,  $0 < t < 1$ , satisfying  $z_t = tu + (1 - t)Tz_t$ , for arbitrary but fixed  $u \in K$ , which converges strongly to a fixed point of  $T$ . Further, if  $Pu = \lim_{t \rightarrow 0} z_t$  for each  $u \in K$ , then  $P$  is a sunny nonexpansive retraction of  $K$  onto  $F(T)$ .

### 3. Main results

We now prove the following theorem.

THEOREM 3.1. Let  $K$  be a nonempty closed convex subset of a reflexive real Banach space  $E$  which has a uniformly Gâteaux differentiable norm. Assume that  $K$  is a sunny nonexpansive retract of  $E$  with  $Q$  as the sunny nonexpansive retraction. Assume that every nonempty closed bounded convex subset of  $K$  has the fixed point property for nonexpansive mappings. Let  $T_i : K \rightarrow E$ ,  $i = 1, \dots, r$ , be a family of nonexpansive mappings which are weakly inward with  $F := \bigcap_{i=1}^r F(T_i) \neq \emptyset$  and  $\bigcap_{i=1}^r F(QT_i) = F(QT_r QT_{r-1} \cdots QT_1) = F(QT_1 QT_r \cdots QT_2) = \cdots = F(QT_{r-1} QT_{r-2} \cdots QT_1 QT_r)$ . For given  $u, x_0 \in K$ , let  $\{x_n\}$  be generated by the algorithm

$$x_{n+1} := \alpha_{n+1}u + (1 - \alpha_{n+1})QT_{n+1}x_n, \quad n \geq 0, \quad (3.1)$$

where  $T_n := T_{n(\text{mod } r)}$  and  $\{\alpha_n\}$  is a real sequence which satisfies the following conditions: (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ; (ii)  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ; and either (iii)\*  $\sum_{n=1}^{\infty} |\alpha_{n+r} - \alpha_n| < \infty$ , or (iii)\*\*  $\lim_{n \rightarrow \infty} ((\alpha_{n+r} - \alpha_n)/\alpha_{n+r}) = 0$ . Then  $\{x_n\}$  converges strongly to a common fixed point

of  $\{T_1, T_2, \dots, T_r\}$ . Further, if  $Pu = \lim_{n \rightarrow \infty} x_n$  for each  $u \in K$ , then  $P$  is a sunny nonexpansive retraction of  $K$  onto  $F$ .

*Proof.* For  $x^* \in F$ , one easily shows by induction that  $\|x_n - x^*\| \leq \max\{\|x_0 - x^*\|, \|u - x^*\|\}$ , for all integers  $n \geq 0$ , and hence  $\{x_n\}$  and  $\{QT_{n+1}x_n\}$  are bounded. But this implies that  $\|x_{n+1} - QT_{n+1}x_n\| = \alpha_{n+1}\|u - QT_{n+1}x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Now we show that

$$\|x_{n+r} - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.2)$$

From (3.1), we get that

$$\begin{aligned} \|x_{n+r} - x_n\| &= \|(\alpha_{n+r} - \alpha_n)(u - QT_n x_{n-1}) + (1 - \alpha_{n+r})(QT_{n+r} x_{n+r-1} - QT_n x_{n-1})\| \\ &= \|(\alpha_{n+r} - \alpha_n)(u - QT_n x_{n-1}) + (1 - \alpha_{n+r})(QT_n x_{n+r-1} - QT_n x_{n-1})\| \\ &\leq (1 - \alpha_{n+r})\|x_{n+r-1} - x_{n-1}\| + |\alpha_{n+r} - \alpha_n| M, \end{aligned} \quad (3.3)$$

for some  $M > 0$ . We consider two cases.

*Case 1.* Condition (iii)\* is satisfied. Then,

$$\|x_{n+r} - x_n\| \leq (1 - \alpha_{n+r})\|x_{n+r-1} - x_{n-1}\| + \sigma_n, \quad (3.4)$$

where  $\sigma_n := M|\alpha_{n+r} - \alpha_n|$  so that  $\sum_{n=1}^{\infty} \sigma_n < \infty$ .

*Case 2.* Condition (iii)\*\* is satisfied. Then,

$$\|x_{n+r} - x_n\| \leq (1 - \alpha_{n+r})\|x_{n+r-1} - x_{n-1}\| + \sigma_n, \quad (3.5)$$

where  $\sigma_n := \alpha_{n+r}\beta_n$  and  $\beta_n := (|\alpha_{n+r} - \alpha_n|M/\alpha_{n+r})$  so that  $\sigma_n = o(\alpha_{n+r})$ .

In either case, by Lemma 2.1, we conclude that  $\lim_{n \rightarrow \infty} \|x_{n+r} - x_n\| = 0$ . Next we prove that

$$\lim_{n \rightarrow \infty} \|x_n - QT_{n+r} \cdots QT_{n+1} x_n\| = 0. \quad (3.6)$$

In view of (3.2), it suffices to show that  $\lim_{n \rightarrow \infty} \|x_{n+r} - QT_{n+r} \cdots QT_{n+1} x_n\| = 0$ . Since  $\|x_{n+r} - QT_{n+r} x_{n+r-1}\| = \alpha_{n+r}\|u - QT_{n+r} x_{n+r-1}\|$  and  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , we have that  $x_{n+r} - QT_{n+r} x_{n+r-1} \rightarrow 0$ . From

$$\begin{aligned} \|x_{n+r} - QT_{n+r} QT_{n+r-1} x_{n+r-2}\| &\leq \|x_{n+r} - QT_{n+r} x_{n+r-1}\| \\ &\quad + \|QT_{n+r} x_{n+r-1} - QT_{n+r} QT_{n+r-1} x_{n+r-2}\| \\ &\leq \|x_{n+r} - QT_{n+r} x_{n+r-1}\| + \|x_{n+r-1} - QT_{n+r-1} x_{n+r-2}\| \\ &= \|x_{n+r} - QT_{n+r} x_{n+r-1}\| + \alpha_{n+r-1}\|u - QT_{n+r-1} x_{n+r-2}\|, \end{aligned} \quad (3.7)$$

we also have  $x_{n+r} - QT_{n+r} QT_{n+r-1} x_{n+r-2} \rightarrow 0$ . Similarly, we obtain the conclusion. Let  $z_t^n \in K$  be a continuous path satisfying

$$z_t^n = tu + (1-t)QT_{n+r} QT_{n+r-1} \cdots QT_{n+1} z_t^n \quad (3.8)$$

guaranteed by Theorem 2.3. Also by Theorem 2.3,  $z_t^n \rightarrow Pu$  as  $t \rightarrow 0^+$ , where  $P$  is the sunny nonexpansive retraction of  $K$  onto  $\bigcap_{i=1}^r F(QT_i)$  (notice  $\bigcap_{i=1}^r F(QT_i) = F(QT_{n+r}QT_{n+r-1}\dots QT_{n+1})$ ) and hence as  $T_i$ ,  $i = 1, \dots, r$ , is weakly inward by [2, Remark 2.1],  $Pu \in F = \bigcap_{i=1}^r F(T_i)$ . Let  $a = \limsup_{n \rightarrow \infty} \langle u - Pu, j(x_n - Pu) \rangle$ . Now we show that  $a \leq 0$ . We can find a subsequence  $\{x_{n_i}\}$  of  $\{x_n\}$  such that  $a = \lim_{i \rightarrow \infty} \langle u - Pu, j(x_{n_i} - Pu) \rangle$ . We assume that  $n_i \equiv k \pmod{r}$  for some  $k \in \{1, 2, \dots, r\}$ . Using Lemma 2.2, we have that

$$\begin{aligned}
\|z_t^k - x_{n_i}\|^2 &= \|t(u - x_{n_i}) + (1-t)(QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}z_t^k - x_{n_i})\|^2 \\
&\leq (1-t)^2\|QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}z_t^k - x_{n_i}\|^2 + 2t\langle u - x_{n_i}, j(z_t^k - x_{n_i}) \rangle \\
&\leq (1-t)^2(\|QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}z_t^k - QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}x_{n_i}\| \\
&\quad + \|QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}x_{n_i} - x_{n_i}\|)^2 \\
&\quad + 2t(\|z_t^k - x_{n_i}\|^2 + \langle u - z_t^k, j(z_t^k - x_{n_i}) \rangle) \\
&\leq (1+t^2)\|z_t - x_{n_i}\|^2 + \|QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}x_{n_i} - x_{n_i}\| \\
&\quad \times (2\|z_t^k - x_{n_i}\| + \|QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}x_{n_i} - x_{n_i}\|) \\
&\quad + 2t\langle u - z_t^k, j(z_t^k - x_{n_i}) \rangle,
\end{aligned} \tag{3.9}$$

and hence,

$$\begin{aligned}
\langle u - z_t^k, j(x_{n_i} - z_t^k) \rangle &\leq \frac{t}{2}\|z_t^k - x_{n_i}\|^2 + \frac{\|QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}x_{n_i} - x_{n_i}\|}{2t} \\
&\quad \times (2\|z_t^k - x_{n_i}\| + \|QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}x_{n_i} - x_{n_i}\|).
\end{aligned} \tag{3.10}$$

Since  $\{x_{n_i}\}$  is bounded, we have that  $\{QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}x_{n_i}\}$  is bounded and by (3.6),  $\|x_{n_i} - QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}x_{n_i}\| \rightarrow 0$  as  $i \rightarrow \infty$ , then it follows from the last inequality that

$$\limsup_{t \rightarrow 0^+} \limsup_{i \rightarrow \infty} \langle u - z_t^k, j(x_{n_i} - z_t^k) \rangle \leq 0. \tag{3.11}$$

Moreover,  $j$  is norm-to- $w^*$  uniformly continuous on bounded subsets of  $E$ . Thus, we obtain from (3.11) that

$$\limsup_{i \rightarrow \infty} \langle u - Pu, j(x_{n_i} - Pu) \rangle \leq 0, \tag{3.12}$$

and hence  $\limsup_{n \rightarrow \infty} \langle u - Pu, j(x_n - Pu) \rangle \leq 0$ . Furthermore, from (3.1), we have  $x_{n+1} - Pu = \alpha_{n+1}(u - Pu) + (1 - \alpha_{n+1})(QT_{n+1}x_n - Pu)$ . Thus using Lemma 2.2, we obtain that

$$\begin{aligned}
\|x_{n+1} - Pu\|^2 &\leq (1 - \alpha_{n+1})^2\|QT_{n+1}x_n - Pu\|^2 + 2\alpha_{n+1}\langle u - Pu, j(x_{n+1} - Pu) \rangle \\
&\leq (1 - \alpha_{n+1})\|x_n - Pu\|^2 + \sigma_{n+1},
\end{aligned} \tag{3.13}$$

where  $\sigma_{n+1} := \alpha_{n+1}\beta_{n+1}$  and  $\limsup_{n \rightarrow \infty} \sigma_{n+1} \leq 0$ , for  $\beta_{n+1} := \langle u - Pu, j(x_{n+1} - Pu) \rangle$ . Thus, by Lemma 2.1,  $\{x_n\}$  converges strongly to a common fixed point  $Pu$  of  $\{T_1, T_2, \dots, T_r\}$ . The proof is complete.  $\square$

If in Theorem 3.1,  $T_i$ ,  $i = 1, \dots, r$ , are self-mappings then the projection operator  $Q$  is replaced with  $I$ , the identity map on  $E$ . Moreover, each  $T_i$  for  $i \in \{1, 2, \dots, r\}$  is weakly inward. Thus, we have the following corollary.

**COROLLARY 3.2.** *Let  $K$  be a nonempty closed convex subset of a reflexive real Banach space  $E$  which has a uniformly Gâteaux differentiable norm. Assume that every nonempty closed bounded convex subset of  $K$  has the fixed point property for nonexpansive mappings. Let  $T_i : K \rightarrow K$ ,  $i = 1, \dots, r$ , be a family of nonexpansive mappings with  $\bigcap_{i=1}^r F(T_i) \neq \emptyset$  and  $\bigcap_{i=1}^r F(T_i) = F(T_r T_{r-1} \cdots T_1) = F(T_1 T_r \cdots T_2) = \cdots = F(T_{r-1} T_{r-2} \cdots T_1 T_r)$ . For given  $u, x_0 \in K$ , let  $\{x_n\}$  be generated by the algorithm*

$$x_{n+1} := \alpha_{n+1}u + (1 - \alpha_{n+1})T_{n+1}x_n, \quad n \geq 0, \quad (3.14)$$

where  $T_n := T_{n(\bmod r)}$  and  $\{\alpha_n\}$  is a real sequence which satisfies the following conditions: (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ; (ii)  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ; and either (iii)\*  $\sum_{n=1}^{\infty} |\alpha_{n+r} - \alpha_n| < \infty$ , or (iii)\*\*  $\lim_{n \rightarrow \infty} ((\alpha_{n+r} - \alpha_n)/\alpha_{n+r}) = 0$ . Then  $\{x_n\}$  converges strongly to a common fixed point of  $\{T_1, T_2, \dots, T_r\}$ . Further, if  $Pu = \lim_{n \rightarrow \infty} x_n$  for each  $u \in K$ , then  $P$  is a sunny nonexpansive retraction of  $K$  onto  $F$ .

In the sequel, we will use the following lemma.

**LEMMA 3.3.** *Let  $K$  be a nonempty closed convex subset of a strictly convex real Banach space  $E$ . Assume that  $K$  is a sunny nonexpansive retract of  $E$  with  $Q$  as the sunny nonexpansive retraction. Let  $T_i : K \rightarrow E$ ,  $i = 1, \dots, r$ , be a family of nonexpansive mappings which are weakly inward with  $\bigcap_{i=1}^r F(T_i) \neq \emptyset$ . Let  $S_i : K \rightarrow E$ ,  $i = 1, \dots, r$ , be a family of mappings defined by  $S_i := (1 - \lambda_i)I + \lambda_i T_i$ ,  $0 < \lambda_i < 1$  for each  $i = 1, 2, \dots, r$ . Then  $\bigcap_{i=1}^r F(T_i) = \bigcap_{i=1}^r F(S_i) = \bigcap_{i=1}^r F(QS_i)$  and  $\bigcap_{i=1}^r F(S_i) = F(QS_r QS_{r-1} \cdots QS_1) = F(QS_1 QS_r \cdots QS_2) = \cdots = F(QS_{r-1} QS_{r-2} \cdots QS_1 QS_r)$ .*

*Proof.* We note that, since  $T_i$  for each  $i \in \{1, 2, \dots, r\}$  is weakly inward, then by [3, Remark 3.3],  $S_i$  is weakly inward. Moreover, by [2, Remark 2.1],  $F(QS_i) = F(S_i)$ . Furthermore, one easily shows that  $F(S_i) = F(T_i)$  for each  $i = 1, 2, \dots, r$ . Now we show that  $\bigcap_{i=1}^r F(S_i) = F(QS_r QS_{r-1} \cdots QS_1) = F(QS_1 QS_r \cdots QS_2) = \cdots = F(QS_{r-1} QS_{r-2} \cdots QS_1 QS_r)$ . For simplicity, we prove for  $r = 2$ . It is clear that  $F(S_1) \cap F(S_2) \subseteq F(QS_2 QS_1)$ . Now, we show that  $F(QS_2 QS_1) \subseteq F(S_1) \cap F(S_2)$ . Let  $z \in F(QS_2 QS_1)$  and  $w \in F(S_1) \cap F(S_2) = F(T_1) \cap F(T_2)$ . Then,

$$\begin{aligned} \|z - w\| &= \|QS_2 QS_1 z - w\| \\ &\leq \|(1 - \lambda_2)Q[(1 - \lambda_1)z + \lambda_1 T_1 z] + \lambda_2 T_2(Q[(1 - \lambda_1)z + \lambda_1 T_1 z]) - w\| \\ &\leq (1 - \lambda_2)\|(1 - \lambda_1)z + \lambda_1 T_1 z - w\| + \lambda_2\|(1 - \lambda_1)z + \lambda_1 T_1 z - w\| \\ &= \|(1 - \lambda_1)(z - w) + \lambda_1(T_1 z - w)\| \\ &\leq (1 - \lambda_1)\|z - w\| + \lambda_1\|T_1 z - w\| \leq \|z - w\|. \end{aligned} \quad (3.15)$$

Thus from the preceding inequalities and strict convexity of  $E$ , we obtain that  $z - w = T_1 z - w$  and  $T_2(Q[(1 - \lambda_1)z + \lambda_1 T_1 z]) - w = z - w$ . Therefore, we obtain that  $z = T_1 z = T_2 z$ . This completes the proof.  $\square$

**THEOREM 3.4.** *Let  $K$  be a nonempty closed convex subset of a strictly convex reflexive real Banach space  $E$  which has a uniformly Gâteaux differentiable norm. Assume that  $K$  is a sunny nonexpansive retract of  $E$  with  $Q$  as the sunny nonexpansive retraction. Assume that every nonempty closed bounded convex subset of  $K$  has the fixed point property for nonexpansive mappings. Let  $T_i : K \rightarrow E$ ,  $i = 1, \dots, r$ , be a family of nonexpansive mappings which are weakly inward with  $\bigcap_{i=1}^r F(T_i) \neq \emptyset$ . Let  $S_i : K \rightarrow E$ ,  $i = 1, \dots, r$ , be a family of mappings defined by  $S_i := (1 - \lambda_i)I + \lambda_i T_i$ ,  $0 < \lambda_i < 1$  for each  $i = 1, 2, \dots, r$ . For given  $u, x_0 \in K$ , let  $\{x_n\}$  be generated by the algorithm*

$$x_{n+1} := \alpha_{n+1}u + (1 - \alpha_{n+1})QS_{n+1}x_n, \quad n \geq 0, \quad (3.16)$$

where  $S_n := S_{n(\text{mod } r)}$  and  $\{\alpha_n\}$  is a real sequence which satisfies the following conditions: (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ; (ii)  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ; and either (iii)\*  $\sum_{n=1}^{\infty} |\alpha_{n+r} - \alpha_n| < \infty$ , or (iii)\*\*  $\lim_{n \rightarrow \infty} ((\alpha_{n+r} - \alpha_n)/\alpha_{n+1}) = 0$ . Then,  $\{x_n\}$  converges strongly to a common fixed point of  $\{T_1, T_2, \dots, T_r\}$ . Further, if  $Pu = \lim_{n \rightarrow \infty} x_n$  for each  $u \in K$ , then  $P$  is a sunny nonexpansive retraction of  $K$  onto  $F$ .

*Proof.* By Lemma 3.3,  $\bigcap_{i=1}^r F(T_i) = \bigcap_{i=1}^r F(S_i) = \bigcap_{i=1}^r F(QS_i)$  and  $\bigcap_{i=1}^r F(QS_i) = F(QS_r QS_{r-1} \cdots QS_1) = F(QS_1 QS_r \cdots QS_2) = \cdots = F(QS_{r-1} QS_{r-2} \cdots QS_1 QS_r)$ . Thus, as in the proof of Theorem 3.1,  $x_n \rightarrow x^* \in \bigcap_{i=1}^r F(T_i)$ . The proof is complete.  $\square$

If in Theorem 3.4,  $T_i$ ,  $i = 1, \dots, r$ , are self-mappings, the following corollary follows.

**COROLLARY 3.5.** *Let  $K$  be a nonempty closed convex subset of a strictly convex reflexive real Banach space  $E$  which has a uniformly Gâteaux differentiable norm. Assume that every nonempty closed bounded convex subset of  $K$  has the fixed point property for nonexpansive mappings. Let  $T_i : K \rightarrow K$ ,  $i = 1, \dots, r$ , be a family of nonexpansive mappings with  $\bigcap_{i=1}^r F(T_i) \neq \emptyset$ . Let  $S_i : K \rightarrow K$ ,  $i = 1, \dots, r$ , be a family of mappings defined by  $S_i := (1 - \lambda_i)I + \lambda_i T_i$ ,  $0 < \lambda_i < 1$  for each  $i = 1, 2, \dots, r$ . For given  $u, x_0 \in K$ , let  $\{x_n\}$  be generated by the algorithm*

$$x_{n+1} := \alpha_{n+1}u + (1 - \alpha_{n+1})S_{n+1}x_n, \quad n \geq 0, \quad (3.17)$$

where  $S_n := S_{n(\text{mod } r)}$  and  $\{\alpha_n\}$  is a real sequence which satisfies the following conditions: (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ; (ii)  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ; and either (iii)\*  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ , or (iii)\*\*  $\lim_{n \rightarrow \infty} ((\alpha_{n+1} - \alpha_n)/\alpha_{n+r}) = 0$ . Then  $\{x_n\}$  converges strongly to a common fixed point of  $\{T_1, T_2, \dots, T_r\}$ . Further, if  $Pu = \lim_{n \rightarrow \infty} x_n$  for each  $u \in K$ , then  $P$  is a sunny nonexpansive retraction of  $K$  onto  $F$ .

**Remark 3.6.** Corollaries 3.2 and 3.5 are improvements of Theorems 1.1 and 1.2 to more general Banach spaces (having a uniformly Gâteaux differentiable norm) than uniformly convex spaces. Moreover, If  $E$  is a Hilbert space, Corollary 3.2 reduces to the result of Bauschke [1].

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