

FIXED-POINT-LIKE THEOREMS ON SUBSPACES

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Received 8 June 2004

We prove a fixed-point-like theorem for multivalued mappings defined on the finite Cartesian product of Grassmannian manifolds and convex sets. Our result generalizes two different kinds of theorems: the fixed-point-like theorem by Hirsch et al. (1990) or Husseini et al. (1990) and the fixed-point theorem by Gale and Mas-Colell (1975) (which generalizes Kakutani's theorem (1941)).

1. Introduction

In this paper, we prove a fixed-point-like theorem for multivalued mappings defined on the finite Cartesian product of Grassmannian manifolds and convex sets. Let k be an integer and let V be a Euclidean space such that $0 \leq k \leq \dim V$, then the k -Grassmannian manifold of V , denoted $G^k(V)$, is the set of all the k -dimensional subspaces of V . The set $G^k(V)$ is a smooth compact manifold but, in general, it does not satisfy properties such as convexity or acyclicity and its Euler characteristic may be null. This prevents the use of classical fixed-point theorems as Brouwer's [2], Kakutani's [14], or Eilenberg-Montgomery's theorem [7].

Our main result generalizes two different kinds of theorems: the fixed-point-like theorem by Hirsch et al. [11] or Husseini et al. [13] and the fixed-point theorem by Gale and Mas-Colell [8] (which generalizes Kakutani's theorem [14]). As in [11, 13], we will mainly use techniques from degree theory. As a consequence of our main result, we first deduce the standard fixed-point theorems when the variable is in a convex domain (such as Brouwer and Kakutani's theorem) and second Borsuk-Ulam's theorem.

The main result of this paper is directly motivated by the existence problem of equilibria in economic models with incomplete markets; in [1], it is used to extend the classical existence result by Duffie and Shafer [6] to the nontransitive case.

The paper is organized as follows. The main result is stated in Section 2 together with some direct consequences of it, namely, the results by Hirsch et al. [11], Gale and Mas-Colell [8] and Borsuk-Ulam's theorem. The proof of the main result is given in Section 3

and the appendix recalls the main properties of the Grassmannian manifold, used in this paper.

2. Statement of the results

2.1. Preliminaries. A correspondence Φ from a set X to a set Y is a map from X to the set of all the subsets of Y , and the graph of Φ , denoted $G(\Phi)$, is defined by $G(\Phi) = \{(x, y) \in X \times Y \mid y \in \Phi(x)\}$. A mapping $\varphi: X \rightarrow Y$ is said to be a selection of Φ if $\varphi(x) \in \Phi(x)$ for all $x \in X$. If A is a subset of X , we let $\Phi(A) = \bigcup_{x \in A} \Phi(x)$, and the restriction of Φ to A , denoted $\Phi|_A$, is the correspondence from A to Y defined by $\Phi|_A(x) = \Phi(x)$ if $x \in A$. If X and Y are topological spaces, the correspondence Φ is said to be lower semicontinuous (l.s.c.) (resp., upper semicontinuous (u.s.c.)) if for every open set $U \subset Y$, the set $\{x \in X \mid \Phi(x) \cap U \neq \emptyset\}$ is open in X (resp., the set $\{x \in X \mid \Phi(x) \subset U\}$ is open in X and, for every $x \in X$, $\Phi(x)$ is compact).

If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ belong to $\mathbb{R}^n$, we denote by $x \cdot y = \sum_{i=1}^n x_i y_i$ the scalar product of $\mathbb{R}^n$, $\|x\| = \sqrt{x \cdot x}$ the Euclidian norm. If $x \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, we let $B(x, r) = \{y \in \mathbb{R}^n \mid \|x - y\| < r\}$ and $\bar{B}(x, r) = \{y \in \mathbb{R}^n \mid \|x - y\| \leq r\}$. If E is a vector subspace of $\mathbb{R}^n$, we denote by $E^\perp = \{u \in \mathbb{R}^n \mid \forall x \in E, x \cdot u = 0\}$ the orthogonal space to E . If $u_1, \dots, u_k$ belong to E , a vector space, we denote by $\text{span}\{u_1, \dots, u_k\}$ the vector subspace of E spanned by $u_1, \dots, u_k$.

Let V be a Euclidean space and let k be an integer such that $0 \leq k \leq \dim V$; we denote by $G^k(V)$ the set consisting of all the linear subspaces of V of dimension k , called the (k -)Grassmannian manifold of V . Then it is known that $G^k(V)$ is a smooth manifold of dimension $k(\dim V - k)$ and we refer to the appendix for the properties we will use hereafter, together with the precise definition of the manifold structure on $G^k(V)$.

2.2. The main result and some consequences. The aim of this paper is to prove the following result.

THEOREM 2.1. *Let I, J be two finite disjoint sets. For every $i \in I$, let k_i be an integer and let V_i be a Euclidean space such that $0 \leq k_i \leq \dim V_i$. For every $j \in J$, let C_j be a nonempty, convex, compact subset of a Euclidean space V_j , and let $M = \prod_{i \in I} G^{k_i}(V_i) \times \prod_{j \in J} C_j$.*

For $i \in I$ and $k = 1, \dots, k_i$, let F_i^k be a correspondence from M to V_i with convex values, for $j \in J$, let F_j be a correspondence from M to C_j with convex values, and suppose that, for every $i \in I$ and $k = 1, \dots, k_i$ (resp., $j \in J$), the correspondence F_i^k (resp., F_j) is either l.s.c or u.s.c.

Then, there exists $\bar{x} = ((\bar{x}_i)_{i \in I}, (\bar{x}_j)_{j \in J}) \in M$ such that

- (i) *either $F_i^k(\bar{x}) \cap \bar{x}_i \neq \emptyset$ or $F_i^k(\bar{x}) = \emptyset$ for every $i \in I$ and $k = 1, \dots, k_i$;*
- (ii) *either $F_j(\bar{x}) \cap \{\bar{x}_j\} \neq \emptyset$ or $F_j(\bar{x}) = \emptyset$ for every $j \in J$.*

The proof of [Theorem 2.1](#) is given in [Section 3](#). A first consequence of [Theorem 2.1](#) is the following theorem by Hirsch et al. [[11](#)].

COROLLARY 2.2. *Let V_1 be a Euclidean space, let k_1 be an integer such that $0 \leq k_1 \leq \dim V_1$, and for every $k = 1, \dots, k_1$, let $f^k: G^{k_1}(V_1) \rightarrow V_1$ be a continuous mapping. Then, there exists $\bar{x} \in G^{k_1}(V_1)$ such that for every $k = 1, \dots, k_1$, $f^k(\bar{x}) \in \bar{x}$.*

Proof. Take $I = \{1\}$, $J = \emptyset$, and $F_1^k(x) = \{f^k(x)\}$ for every $x \in G^{k_1}(V_1)$ and for every $k = 1, \dots, k_1$. From [Theorem 2.1](#), there exists $\bar{x} \in M = G^{k_1}(V_1)$ such that for every $k = 1, \dots, k_1$, $F_1^k(\bar{x}) \cap \bar{x} \neq \emptyset$, that is, $f^k(\bar{x}) \in \bar{x}$. $\square$

A second consequence of [Theorem 2.1](#) is the following generalization of Gale and Mas-Colell's theorem [8], which is also a generalization of Kakutani's theorem. Hereafter, we use the formulation by Gourdel [9] allowing each correspondence to be either u.s.c. or l.s.c.

COROLLARY 2.3. *Let J be a finite set, for $j \in J$, let C_j be a nonempty, convex, compact subset of a Euclidean space, and let F_j be a correspondence from $M := \prod_{j \in J} C_j$ to C_j with convex values, such that the correspondence F_j is either l.s.c or u.s.c. Then, there exists $\bar{x} = (\bar{x}_j)_{j \in J} \in M$ such that for every $j \in J$, either $\bar{x}_j \in F_j(\bar{x})$ or $F_j(\bar{x}) = \emptyset$.*

Proof. Take $I = \emptyset$ and apply [Theorem 2.1](#). $\square$

Remark 2.4. According to our definition, an u.s.c. correspondence has compact values and without this requirement, [Theorem 2.1](#) may not be true, as we can see in the following counterexample. Let $M := G^1(\mathbb{R}^2)$. Each element D of $G^1(\mathbb{R}^2)$ can be written as $D_t = \{\lambda(\cos t, \sin t) \mid \lambda \in \mathbb{R}\}$, for some $t \in [0, \pi[$. We define the correspondence F from M to $\mathbb{R}^2$ by $F(D_0) = \mathbb{R} \times \{1\}$ and $F(D_t) = D_t \cap (\mathbb{R} \times \{1\}) + \{(1, 0)\}$ if $t \in]0, \pi[$. We let the reader check that for every open set $U \subset \mathbb{R}^2$, the set $\{x \in M \mid F(x) \subset U\}$ is open in M and that F has nonempty, convex (and closed) values. Yet, it is straightforward that $F(x) \cap x = \emptyset$ for every $x \in G^1(\mathbb{R}^2)$.

Another consequence of our main result is the following multivalued version of Borsuk and Ulam's theorem. We denote by S^n the unit sphere of $\mathbb{R}^{n+1}$.

COROLLARY 2.5. *For $k = 1, \dots, n$, let F^k be a correspondence from S^n to $\mathbb{R}$ with nonempty and convex values such that for every $k = 1, \dots, n$, F^k is either l.s.c or u.s.c. Then, there exists $\bar{x} \in S^n$ such that*

$$\forall k \in \{1, \dots, n\}, \quad F^k(\bar{x}) \cap F^k(-\bar{x}) \neq \emptyset. \quad (2.1)$$

Proof. For every $k = 1, \dots, n$, let $\hat{F}^k$ be the correspondence from S^n to $\mathbb{R}$ defined by

$$\hat{F}^k(x) = \{u - v \mid u \in F^k(x), v \in F^k(-x)\}. \quad (2.2)$$

We let the reader check that for every $k = 1, \dots, n$, the correspondence $\hat{F}^k$ has nonempty, convex values and that it is u.s.c. (resp., l.s.c.) if F^k is u.s.c. (resp., l.s.c.). So, to prove [Corollary 2.5](#), it suffices to show the existence of $\bar{x} \in S^n$ such that $0 \in \hat{F}^k(\bar{x})$ for every $k = 1, \dots, n$.

We define, for every $k = 1, \dots, n$, the correspondence H^k from $G^n(\mathbb{R}^{n+1})$ to $\mathbb{R}^{n+1}$ as follows: for every $E \in G^n(\mathbb{R}^{n+1})$, we let $H^k(E) = \hat{F}^k(x)x$, where x is an arbitrary element of $E^\perp \cap S^n$. The correspondence H^k is well defined since $E^\perp \cap S^n = \{x, -x\}$ for some element $x \in S^n$ and since $\hat{F}^k(x)x = \hat{F}^k(-x)(-x)$.

Take $I = \{1\}$, $V_1 = \mathbb{R}^{n+1}$, $k_1 = n$, $J = \emptyset$, and apply [Theorem 2.1](#) to the correspondences H^k , which clearly satisfy the assumptions of [Theorem 2.1](#). So there exists $\bar{E} \in G^n(\mathbb{R}^{n+1})$ such that $\bar{E} \cap H^k(\bar{E}) \neq \emptyset$ for every $k = 1, \dots, n$.

Now, if $\bar{x}$ is an arbitrary point of $\bar{E}^\perp \cap S^n$, then we have $\bar{E} \cap \hat{F}^k(\bar{x})\bar{x} \neq \emptyset$; from $\bar{x} \in \bar{E}^\perp$ and $\bar{x} \neq 0$, we finally obtain $0 \in \hat{F}^k(\bar{x})$ for every $k = 1, \dots, n$, which ends the proof of [Corollary 2.5](#). $\square$

3. Proof of [Theorem 2.1](#)

The proof is given in three steps, corresponding to the following three subsections. The first step gives the proof under the additional assumptions that $J = \emptyset$ and the correspondences F_i^k are single-valued. The second step only assumes in addition that $J = \emptyset$. Finally, the third step gives the proof under the assumptions of [Theorem 2.1](#).

3.1. Proof when $J = \emptyset$ and F_i^k are single-valued. We first prove [Theorem 2.1](#) under the additional assumptions that $J = \emptyset$ and the F_i^k are single-valued. This is exactly the statement below.

THEOREM 3.1. *Let I be a finite set and for $i \in I$, let k_i be an integer and let V_i be a Euclidean space such that $0 \leq k_i \leq \dim V_i$. Let $M := \prod_{i \in I} G^{k_i}(V_i)$ and for $i \in I$, let $f_i: M \rightarrow (V_i)^{k_i}$ be a continuous mapping. Then, there exists $\bar{x} = (\bar{x}_i)_{i \in I} \in M$ such that*

$$\forall i \in I, \quad f_i(\bar{x}) \in (\bar{x}_i)^{k_i}. \quad (3.1)$$

The proof of [Theorem 3.1](#) is given in two steps. In the first step, we additionally assume that the mappings are smooth, and the second step gives the proof in the general case.

3.1.1. Proof of [Theorem 3.1](#) when the f_i are smooth. Let $M := \prod_{i \in I} G^{k_i}(V_i)$ and define $f: M \rightarrow \prod_{i \in I} V_i^{k_i}$ by

$$f(x) = (\text{proj}_{(x_i^{k_i})^\perp} f_i(x))_{i \in I} \quad \text{for } x = (x_i)_{i \in I} \in M, \quad (3.2)$$

and the subsets Z , Z_1 , and Z_2 of $M \times \prod_{i \in I} V_i^{k_i}$ by

$$\begin{aligned} Z &= \left\{ (x, y) \in M \times \prod_{i \in I} V_i^{k_i} \mid \forall i \in I, y_i \in (x_i^{k_i})^\perp \right\}, \\ Z_1 &= \left\{ (x, y) \in M \times \prod_{i \in I} V_i^{k_i} \mid y = f(x) \right\}, \\ Z_2 &= \left\{ (x, y) \in M \times \prod_{i \in I} V_i^{k_i} \mid y = 0 \right\}. \end{aligned} \quad (3.3)$$

Proving [Theorem 3.1](#) amounts to showing the existence of $\bar{x} \in M$ such that $f(\bar{x}) = 0$ or, equivalently, such that $Z_1 \cap Z_2 \neq \emptyset$. For this, we will use the following [Intersection Theorem 3.2](#), which is a direct consequence of mod 2 intersection theory (see, e.g., [10, page 79] and [5, page 127]).

INTERSECTION THEOREM 3.2. *Let Z be a smooth boundaryless manifold of dimension $2m$ and let Z_1, Z_2 be two compact boundaryless submanifolds of Z of dimension m . If $\bar{Z}_1$ is a compact boundaryless m -submanifold of Z homotopic to Z_1 and if the manifolds $\bar{Z}_1$ and Z_2 intersect transversally in a unique point $\bar{z}$ (which means that $T_{\bar{z}}\bar{Z}_1 + T_{\bar{z}}Z_2 = T_{\bar{z}}Z$), then $Z_1 \cap Z_2 \neq \emptyset$.*

The proof of [Theorem 3.1](#) consists of checking that the above-defined sets Z, Z_1 , and Z_2 (together with the set $\bar{Z}_1$ defined below) satisfy the assumptions of [Intersection Theorem 3.2](#).

The sets Z, Z_1 , and Z_2 satisfy the assumptions of [Intersection Theorem 3.2](#). We recall that for every $i \in I$, $G^{k_i}(V_i)$ is a smooth, boundaryless, compact manifold of dimension $k_i(\dim V_i - k_i)$ (see [Lemma A.1](#) in the appendix). Thus $M := \prod_{i \in I} G^{k_i}(V_i)$ is a boundaryless, smooth, compact manifold of dimension $m = \sum_{i \in I} k_i(\dim V_i - k_i)$. Clearly Z is a fiber bundle whose base space is M and whose fiber at $x = (x_i)_{i \in I} \in M$ is the vector space $\prod_{i \in I} (x_i^{k_i})^\perp$ which has the dimension of M . Hence, Z is a smooth manifold of dimension $2m$.

The mapping $f : M \rightarrow \prod_{i \in I} V_i^{k_i}$ is a smooth mapping from Parts (c), (d), and (e) of [Lemma A.1](#) in the appendix. Consequently, Z_1 is a smooth compact boundaryless submanifold of Z of dimension m . Finally, Z_2 is clearly a smooth boundaryless compact submanifold of Z of dimension m .

The manifold Z_1 is homotopic to the manifold $\bar{Z}_1$ that we now define. For every $i \in I$, let $\bar{x}_i \in G^{k_i}(V_i)$ and let $\{\bar{e}_i^1, \dots, \bar{e}_i^{k_i}\}$ be an orthonormal basis of $\bar{x}_i$. For every $i \in I$, let $g_i : G^{k_i}(V_i) \rightarrow V_i^{k_i}$ and $g : M \rightarrow \prod_{i \in I} V_i^{k_i}$ be the mappings defined as follows:

$$\begin{aligned} \forall x_i \in G^{k_i}(V_i), \quad g_i(x_i) &= (\text{proj}_{x_i^\perp}(\bar{e}_i^1), \dots, \text{proj}_{x_i^\perp}(\bar{e}_i^{k_i})) \in (x_i^\perp)^{k_i} = (x_i^{k_i})^\perp, \\ \forall x &= (x_i)_{i \in I} \in M, \quad g(x) = (g_i(x_i))_{i \in I}. \end{aligned} \quad (3.4)$$

We let

$$\bar{Z}_1 := \left\{ (x, y) \in M \times \prod_{i \in I} V_i^{k_i} \mid y = g(x) \right\}. \quad (3.5)$$

To show that the manifold Z_1 is homotopic to $\bar{Z}_1$, we let $H : [0, 1] \times Z_1 \rightarrow Z$ be the continuous mapping defined by $H(t, (x, f(x))) = (x, (1-t)f(x) + tg(x))$. Then $H(0, \cdot)$ is the canonical inclusion from Z_1 to Z , and $H(1, \cdot)(Z_1) = \bar{Z}_1$.

The manifolds $\bar{Z}_1$ and Z_2 intersect transversally in a unique point. First, notice that $\bar{Z}_1 \cap Z_2 = \{(x, 0) \in M \times \prod_{i \in I} V_i^{k_i} \mid g(x) = 0\}$ is the singleton $(\bar{x}, 0) = ((\bar{x}_i)_{i \in I}, 0)$. But that $\bar{Z}_1$ and Z_2 intersect each other transversally in Z means that $T_{(\bar{x}, 0)}\bar{Z}_1 + T_{(\bar{x}, 0)}Z_2 = T_{(\bar{x}, 0)}Z$. Recalling that $\dim T_{(\bar{x}, 0)}\bar{Z}_1 + \dim T_{(\bar{x}, 0)}Z_2 = \dim T_{(\bar{x}, 0)}Z = 2m$, we only have to show that $T_{(\bar{x}, 0)}\bar{Z}_1 \cap T_{(\bar{x}, 0)}Z_2 = \{0\}$. Finally, noticing that $T_{(\bar{x}, 0)}\bar{Z}_1 = \{(u, Dg(\bar{x})(u)) \mid u \in T_{\bar{x}}M\}$ and $T_{(\bar{x}, 0)}Z_2 = \{(u, 0) \mid u \in T_{\bar{x}}M\}$, we only have to prove that $Dg(\bar{x})$ is injective, which is proved in the following lemma.

LEMMA 3.3. *$Dg(\bar{x})$ is injective.*

Proof. Recalling that for every $x = (x_i)_{i \in I} \in M$, $g(x) = (g_i(x_i))_{i \in I}$, the mapping $Dg(\bar{x})$ is injective if and only if for every $i \in I$, $Dg_i(\bar{x}_i)$ is injective. $\square$

So, let $i \in I$, let (φ, U) be a local chart of $G^{k_i}(V_i)$ at $\bar{x}_i$, and let $\psi : (\bar{x}_i^\perp)^{k_i} \rightarrow G^{k_i}(V_i)$ be the inverse mapping of $\varphi : U \rightarrow (\bar{x}_i^\perp)^{k_i}$. From the appendix, if $\{\bar{e}_1^1, \dots, \bar{e}_i^{k_i}\}$ is a given orthonormal basis of $\bar{x}_i$, ψ can be defined by

$$\psi(u^1, \dots, u^{k_i}) = \text{span} \{ \bar{e}_i^1 + u^1, \dots, \bar{e}_i^{k_i} + u^{k_i} \} \quad \text{for every } (u^1, \dots, u^{k_i}) \in (\bar{x}_i^\perp)^{k_i}. \quad (3.6)$$

Since the mapping $g_i \circ \psi$ is the local representation g_i in the chart (φ, U) , proving that $Dg_i(\bar{x}_i)$ is injective amounts to proving that $D(g_i \circ \psi)(0)$ is injective. This is a consequence of the following claim.

CLAIM 3.4. *For all $(h_1, \dots, h_{k_i}) \in (\bar{x}_i^\perp)^{k_i}$, $D(g_i \circ \psi)(0)(h_1, \dots, h_{k_i}) = -(h_1, \dots, h_{k_i})$.*

Proof of Claim 3.4. Let $p : V_i \times (\bar{x}_i^\perp)^{k_i} \rightarrow V_i$ be defined by

$$p(y, u) := \text{proj}_{\psi(u)} y. \quad (3.7)$$

If we prove that for every $y \in V_i$, the derivative of the mapping $p_y : u \rightarrow p(y, u)$ is the linear mapping $Dp_y(0) : (\bar{x}_i^\perp)^{k_i} \rightarrow V_i$ defined by

$$Dp_y(0)(h) = \sum_{k=1}^{k_i} (y \cdot \bar{e}_i^k) h_k, \quad \forall h = (h_1, \dots, h_{k_i}) \in (\bar{x}_i^\perp)^{k_i}, \quad (3.8)$$

then Claim 3.4 will be proved. Indeed, taking $y = \bar{e}_i^k$ for every $k = 1, \dots, k_i$, we would obtain $D_{\bar{e}_i^k} p(0)(h_1, \dots, h_{k_i}) = h_k$. Thus, since $g_i \circ \psi(u) = (\bar{e}_i^1, \dots, \bar{e}_i^{k_i}) - (p_{\bar{e}_i^1}(u), \dots, p_{\bar{e}_i^{k_i}}(u))$, it would entail Claim 3.4.

Now, for every $u = (u_1, \dots, u_{k_i}) \in (\bar{x}_i^\perp)^{k_i}$, there exists $\lambda(y, u) = (\lambda_k(y, u))_{k=1, \dots, k_i} \in \mathbb{R}^{k_i}$ such that

$$p(y, u) = \text{proj}_{\psi(u)} y = \sum_{k=1}^{k_i} \lambda_k(y, u) (\bar{e}_i^k + u^k), \quad (3.9)$$

with $(\lambda_k(y, u))$ satisfying

$$\left(-y + \sum_{k=1}^{k_i} \lambda_k(y, u) (\bar{e}_i^k + u^k) \right) \cdot (\bar{e}_i^j + u^j) = 0 \quad \text{for every } j = 1, \dots, k_i. \quad (3.10)$$

This can be equivalently rewritten as follows:

$$(I_{k_i} + G(u))\lambda(y, u) = (y \cdot (\bar{e}_i^1 + u^1), \dots, y \cdot (\bar{e}_i^{k_i} + u^{k_i})), \quad (3.11)$$

where I_{k_i} is the $k_i \times k_i$ identity matrix and $G(u)$ is the $k_i \times k_i$ matrix $G(u) = (u^j \cdot u^k)_{j,k=1, \dots, k_i}$. Besides, for u in a neighborhood $\mathcal{N}$ of 0 small enough, the matrix $(I_{k_i} + G(u))$ is invertible. Consequently, the mapping $\lambda(\cdot, \cdot)$ is smooth on $V \times \mathcal{N}$, which implies

that the mapping $p(\cdot, \cdot)$ is smooth on $V \times \mathcal{N}$. Differentiating, with respect to u , the above equality at $u = 0$, we obtain, for every $h = (h_1, \dots, h_{k_i}) \in (\tilde{x}_i^\perp)^{k_i}$,

$$DG(0)(h)\lambda(y, 0) + D_u\lambda(y, 0)(h) = 0. \quad (3.12)$$

But it is clear that $DG(0) = 0$. Consequently, $D_u\lambda(y, 0) = 0$.

Finally, differentiating the equality $p(y, u) = \sum_{k=1}^{k_i} \lambda_k(y, u)(\tilde{e}_i^k + u^k)$ at $(y, 0)$, one obtains, for every $h = (h_k)_{k=1}^{k_i} \in (\tilde{x}_i^\perp)^{k_i}$,

$$D_u p(y, 0)(h) = \sum_{k=1}^{k_i} \lambda_k(y, 0) h_k = \sum_{k=1}^{k_i} (y \cdot \tilde{e}_i^k) h_k, \quad (3.13)$$

which ends the proof of [Claim 3.4](#). $\square$

3.1.2. Proof of [Theorem 3.1](#) in the general case. Since M is a compact manifold and $V_i^{k_i}$ is a Euclidean space, for every $i \in I$, each continuous mapping $f_i : M \rightarrow V_i^{k_i}$ can be approximated by a sequence of smooth mappings $f_i^n : M \rightarrow V_i^{k_i}$ converging to f_i , in the sense that $\lim_{n \rightarrow \infty} \|f_i^n - f_i\|_\infty = 0$ (see, e.g., Hirsch [12]). Applying the first step to the smooth mappings f_i^n , we deduce the existence of $(x_i^n)_{i \in I} \in M$ such that

$$\forall i \in I, \quad f_i^n(x_i^n) \in (x_i^n)^{k_i} \quad (3.14)$$

or, equivalently,

$$\text{proj}_{(x_i^n)^{k_i}} f_i(x_i^n) = 0. \quad (3.15)$$

From the compactness of M , without any loss of generality, one can suppose that the sequence $(x_i^n)_{i \in I}$ converges to some element $(\tilde{x}_i)_{i \in I} \in M$. We have

$$\begin{aligned} & \left\| \text{proj}_{(\tilde{x}_i^\perp)^{k_i}} f_i(\tilde{x}_i) - \text{proj}_{(\tilde{x}_i^\perp)^{k_i}} f_i^n(x_i^n) \right\| \\ & \leq \left\| \text{proj}_{(\tilde{x}_i^\perp)^{k_i}} f_i(\tilde{x}_i) - \text{proj}_{(\tilde{x}_i^\perp)^{k_i}} f_i(x_i^n) \right\| + \|f_i^n - f_i\|_\infty. \end{aligned} \quad (3.16)$$

Consequently, from the convergence of f_i^n to f_i and the continuity of the mapping $(u, v) \rightarrow \text{proj}_{(u^\perp)^{k_i}} v$ (see [Lemma A.1](#) in the appendix), we obtain

$$\text{proj}_{(\tilde{x}_i^\perp)^{k_i}} f_i(\tilde{x}_i) = 0 \quad (3.17)$$

or, equivalently,

$$\forall i \in I, \quad f_i(\tilde{x}_i) \in \left((\tilde{x}_i^\perp)^{k_i} \right)^\perp = (\tilde{x}_i)^{k_i}, \quad (3.18)$$

which ends the proof of [Theorem 3.1](#).

3.2. Proof of Theorem 2.1 when $J = \emptyset$. We now prove Theorem 2.1 when $J = \emptyset$. The proof rests on the following claim.

CLAIM 3.5. *For every $i \in I$ and every $k \in \{1, \dots, k_i\}$, there exists an u.s.c. correspondence $\hat{F}_i^k$ from M to V_i , with nonempty convex values, such that*

$$\forall x \in M, [F_i^k(x) \neq \emptyset] \implies [\forall y \in \hat{F}_i^k(x), \exists \lambda \in \mathbb{R}, \lambda y \in F_i^k(x)]. \quad (3.19)$$

Proof of Claim 3.5. Let $i \in I$ and $k \in \{1, \dots, k_i\}$. We distinguish two cases.

Assume first that F_i^k is l.s.c. Let $U_i^k = \{x \in M \mid F_i^k(x) \neq \emptyset\}$. Then U_i^k is an open subset of M and $F_i^k|_{U_i^k}$ is a l.s.c. correspondence with nonempty convex values. By Michael [15], there exists a continuous selection f_i^k of $F_i^k|_{U_i^k}$, that is, $f_i^k : U_i^k \rightarrow V_i$ is a continuous mapping such that $f_i^k(x) \in F_i^k(x)$ for every $x \in U_i^k$. Let B_i be the closed unit ball of V_i , and we define the correspondence $\hat{F}_i^k$ from M to B_i by $\hat{F}_i^k(x) = \{f_i^k(x)/\|f_i^k(x)\|\}$ if $x \in U_i^k$ and $f_i^k(x) \neq 0$ and $\hat{F}_i^k(x) = B_i$ otherwise. We let the reader check that the correspondence $\hat{F}_i^k$ satisfies the conclusion of Claim 3.5.

We now consider the case where F_i^k is u.s.c. Let $U_i^k = \{x \in M \mid F_i^k(x) \neq \emptyset\}$. Then U_i^k is a closed subset of M . By Cellina [4], one can extend $F_i^k|_{U_i^k}$ as follows: there exists a correspondence $\hat{F}_i^k$ from M to V_i which is u.s.c., with nonempty, convex, and compact values, such that for every $x \in U_i^k$, $\hat{F}_i^k(x) = F_i^k(x)$. $\square$

We now come back to the proof of Theorem 2.1 when $J = \emptyset$. For every $i \in I$ and $k = 1, \dots, k_i$, let $\hat{F}_i^k$ be the u.s.c. correspondence from M to V_i with nonempty convex (compact) values defined in Claim 3.5. By Cellina [3], for every integer n , there exists a continuous mapping $f_i^{k,n} : M \rightarrow V_i$ such that

$$G(f_i^{k,n}) \subset G(\hat{F}_i^k) + B\left(0, \frac{1}{n}\right). \quad (3.20)$$

Now, for $i \in I$, let $f_i^n : M \rightarrow (V_i)^{k_i}$ be defined as follows:

$$\forall x \in M, f_i^n(x) = (f_i^{1,n}(x), \dots, f_i^{k_i,n}(x)). \quad (3.21)$$

Applying Theorem 3.1 to the mappings f_i^n , we deduce the existence of $(\bar{x}_i^n)_{i \in I} \in M$ such that for every $i \in I$, $f_i^n(\bar{x}^n) \in (\bar{x}_i^n)^{k_i}$, hence

$$y_i^{k,n} := f_i^{k,n}(\bar{x}^n) \in \bar{x}_i^n. \quad (3.22)$$

Since the correspondence $\hat{F}_i^k$ is bounded (M is compact and $\hat{F}_i^k$ is u.s.c.), the sequence $(y_i^{k,n})$ is bounded. Thus, without any loss of generality, one can suppose that the sequence $(y_i^{k,n})$ converges to some $y_i^k \in V_i$ when n tends to $+\infty$.

Besides, from the compactness of M , without any loss of generality, one can suppose that $(\bar{x}_i^n)_{i \in I}$ converges to $\bar{x} = (\bar{x}_i)_{i \in I} \in M$ when n tends to $+\infty$.

Moreover, from Lemma A.1(d) in the appendix and from $y_i^{k,n} \in \bar{x}_i^{k,n}$, at the limit we have that

$$\forall i \in I, \forall k = 1, \dots, k_i, y_i^k \in \bar{x}_i. \quad (3.23)$$

Since the graph of $\hat{F}_i^k$ is closed (it is u.s.c. with compact values) and from $G(f_i^{k,n}) \subset G(\hat{F}_i^k) + B(0, 1/n)$, one obtains

$$y_i^k \in \hat{F}_i^k(\bar{x}). \quad (3.24)$$

To end the proof, we assume that $F_i^k(\bar{x}) \neq \emptyset$. Since $y_i^k \in \hat{F}_i^k(\bar{x})$, by [Claim 3.5](#), there exists $\lambda \in \mathbb{R}$ such that $\lambda y_i^k \in F_i^k(\bar{x})$. Hence $\lambda y_i^k \in F_i^k(\bar{x}) \cap \bar{x}_i \neq \emptyset$ (since $y_i^k \in \bar{x}_i$). This ends the proof of [Theorem 2.1](#).

3.3. Proof of [Theorem 2.1](#) in the general case. We first prove the following lemma.

LEMMA 3.6. *Let C be a nonempty, convex, compact subset of a Euclidean space V . Then there exists a continuous mapping $\rho : G^1(V \times \mathbb{R}) \rightarrow C$ such that*

$$\forall x \in G^1(V \times \mathbb{R}), \quad x \cap [C \times \{1\}] \neq \emptyset \implies x \cap [C \times \{1\}] = \{(\rho(x), 1)\}. \quad (3.25)$$

Proof. Since C is compact, it is included in a closed ball $\bar{B}(0, k)$ of V . We let $r : V \rightarrow \bar{B}(0, k+1)$ be defined by $r(u) = \alpha(\|u\|)u$, where $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined by

$$\begin{aligned} \alpha(t) &= 1 && \text{if } t \in [0, k], \\ \alpha(t) &= k+1-t && \text{if } t \in [k, k+1], \\ \alpha(t) &= 0 && \text{if } t \geq k+1. \end{aligned} \quad (3.26)$$

Let $\pi_1 : V \times \mathbb{R} \rightarrow V$ and $\rho : G^1(V \times \mathbb{R}) \rightarrow C$ be defined by $\pi_1(x, t) = x$ and

$$\rho(x) = \begin{cases} \text{proj}_C \circ r \circ \pi_1(x \cap [V \times \{1\}]) & \text{if } x \cap [V \times \{1\}] \neq \emptyset, \\ \text{proj}_C(0) & \text{if } x \cap [V \times \{1\}] = \emptyset, \end{cases} \quad (3.27)$$

where $\text{proj}_C : V \rightarrow C$ denotes the projection from V to C . Then, one easily sees that ρ satisfies the conclusion of [Lemma 3.6](#). $\square$

Proof of [Theorem 2.1](#). Using [Lemma 3.6](#), we first modify the correspondences F_j for every $j \in J$ and replace each nonempty compact convex set $C_j \subset V_j$ by the Grassmannian manifold $G^1(V_j \times \mathbb{R})$. For every $j \in J$, let $\rho_j : G^1(V_j \times \mathbb{R}) \rightarrow C_j$ be the mapping associated to $C_j \subset V_j$ by [Lemma 3.6](#). Let

$$\rho : \tilde{M} := \prod_{i \in I} G^{k_i}(V_i) \times \prod_{j \in J} G^1(V_j \times \mathbb{R}) \longrightarrow M := \prod_{i \in I} G^{k_i}(V_i) \times \prod_{j \in J} C_j \quad (3.28)$$

be defined by

$$\rho(x) = \left((x_i)_{i \in I}, \rho_j(x_j)_{j \in J} \right), \quad \text{for } x = \left((x_i)_{i \in I}, (x_j)_{j \in J} \right). \quad (3.29)$$

For $i \in I$ and $k = 1, \dots, k_i$, let $\tilde{F}_i^k$ be the correspondence from $\tilde{M}$ to V_i defined by

$$\tilde{F}_i^k(x) = F_i^k(\rho(x)). \quad (3.30)$$

For $j \in J$, let $\tilde{F}_j$ be the correspondence from $\tilde{M}$ to $V_j \times \mathbb{R}$ defined by

$$\tilde{F}_j(x) = F_j(\rho(x)) \times \{1\}. \quad (3.31)$$

Now, applying the result proved in [Section 3.2](#) (i.e., [Theorem 2.1](#) when $J = \emptyset$) to the correspondences $\tilde{F}_i^k$ and $\tilde{F}_j$, there exists $x = ((x_i)_{i \in I}, (x_j)_{j \in J}) \in \tilde{M}$ such that

- (i) either $\tilde{F}_i^k(x) \cap x_i \neq \emptyset$ or $\tilde{F}_i^k(x) = \emptyset$ for every $i \in I$ and $i = 1, \dots, k_i$,
- (ii) either $\tilde{F}_j(x) \cap x_j \neq \emptyset$ or $\tilde{F}_j(x) = \emptyset$ for every $j \in J$.

Let $\bar{x} = \rho(x) \in M$; we end the proof by showing that it satisfies the conclusion of [Theorem 2.1](#). From the above, it is clearly the case for $i \in I$ and $k = 1, \dots, k_i$, that is, we have that

- (i) either $F_i^k(x) \cap x_i \neq \emptyset$ or $F_i^k(x) = \emptyset$ for every $i \in I$ and $i = 1, \dots, k_i$.

Now, let $j \in J$. We first notice that $\tilde{F}_j(x) = \emptyset$ if and only if $F_j(x) = \emptyset$. Assume now that $\tilde{F}_j(x) \cap x_j \neq \emptyset$ and recall that $x_j \cap \tilde{F}_j(x) = x_j \cap (F_j(\bar{x}) \times \{1\})$ and $F_j(\bar{x}) \subset C_j$. Consequently, $x_j \cap (C_j \times \{1\}) \neq \emptyset$ and from [Lemma 3.6](#) we get

$$\emptyset \neq x_j \cap (F_j(\bar{x}) \times \{1\}) \subset x_j \cap (C_j \times \{1\}) = \{(\rho_j(x_j), 1)\}. \quad (3.32)$$

Hence, the equality holds and $\bar{x}_j = \rho_j(x_j) \in F_j(\bar{x})$. This ends the proof of [Theorem 2.1](#). $\square$

Appendix

The Grassmannian manifold $G^k(V)$

Let V be a Euclidean space and let k be an integer such that $0 \leq k \leq \dim V$. In this section, we recall the properties of $G^k(V)$ which are used in this paper.

First, we recall that $G^k(V)$ is a smooth boundaryless manifold of dimension $k(\dim V - k)$ (see, e.g., Hirsch [12] and [Lemma A.1](#)). The local charts can be defined as follows. Let $\bar{E} \in G^k(V)$ and let $\{\bar{e}_1, \dots, \bar{e}_k\}$ be some given orthonormal basis of $\bar{E}$; we define the mapping $\psi_{\bar{E}} : (\bar{E}^\perp)^k \rightarrow G^k(V)$ by

$$\psi_{\bar{E}}(u) = \text{span} \{\bar{e}_1 + u_1, \dots, \bar{e}_k + u_k\}, \quad \text{for } u = (u_1, \dots, u_k) \in (\bar{E}^\perp)^k. \quad (\text{A.1})$$

Then it is easy to check that the mapping $\psi_{\bar{E}}$ is injective (see [Claim A.2](#)); so $\psi_{\bar{E}}$ is a bijection from $(\bar{E}^\perp)^k$ onto $U_{\bar{E}} = \psi_{\bar{E}}((\bar{E}^\perp)^k)$. We can now consider the inverse mapping $\varphi_{\bar{E}} : U_{\bar{E}} \rightarrow (\bar{E}^\perp)^k$ defined by $\varphi_{\bar{E}}(E) = \psi_{\bar{E}}^{-1}(E)$, which is clearly a bijection.

LEMMA A.1. (a) $G^k(V)$ is a smooth boundaryless (i.e., C^∞) manifold of dimension $k(\dim V - k)$ without boundary and $(U_{\bar{E}}, \varphi_{\bar{E}})_{\bar{E} \in G^k(V)}$ defines a C^∞ atlas of $G^k(V)$.

(b) The set $G^k(V)$ is compact.

(c) The mapping $E \rightarrow E^\perp$ from $G^k(V)$ to $G^\ell(V)$ ($\ell = \dim V - k$) is a smooth diffeomorphism.

(d) The mapping $p : V \times G^k(V) \rightarrow V$ defined by $p(x, E) = \text{proj}_E(x)$ is smooth. Hence, the set $\{(x, E) \in V \times G^k(V) \mid x \in E\}$ is a closed subset of $V \times G^k(V)$.

(e) The mapping $x \rightarrow x^p$ from $G^k(V)$ to $(G^k(V))^p$ is smooth.

We prepare the proof of the lemma with a claim.

CLAIM A.2. Let $\bar{E} \in G^k(V)$ and let $\{\bar{e}_1, \dots, \bar{e}_k\}$ be an orthonormal basis of $\bar{E}$.

(a) The mapping $\psi_{\bar{E}}$ is injective.

(b) For every $u \in (\bar{E}^\perp)^k$, $\psi(0) \cap \psi(u)^\perp = \psi(0)^\perp \cap \psi(u) = \{0\}$.

Proof of Claim A.2. Part (a). Let $u = (u_1, \dots, u_k)$ and $v = (v_1, \dots, v_k)$ in $(\bar{E}^\perp)^k$ such that

$$\psi_{\bar{E}}(u) = \text{span}\{\bar{e}_1 + u_1, \dots, \bar{e}_k + u_k\} = \psi_{\bar{E}}(v) = \text{span}\{\bar{e}_1 + v_1, \dots, \bar{e}_k + v_k\}. \quad (\text{A.2})$$

Then, there exist some real numbers λ_{ij} ($i = 1, \dots, k$, $j = 1, \dots, k$) such that

$$\bar{e}_i + u_i = \sum_{j=1}^k \lambda_{ij}(\bar{e}_j + v_j), \quad \text{for every } i = 1, \dots, k. \quad (\text{A.3})$$

Taking, for each inequality, the scalar product with $\bar{e}_l$, where $l = 1, \dots, k$, we obtain $\lambda_{il} = 0$ if $i \neq l$ and $\lambda_{il} = 1$. Hence $u_l = v_l$ for every $l = 1, \dots, k$, and finally $u = v$.

Part (b). Let $x \in \bar{E} \cap \psi(u)^\perp$. Then there exists some real numbers λ_i ($i = 1, \dots, k$) such that $x = \sum_{i=1}^k \lambda_i \bar{e}_i$. Taking the scalar product with $\bar{e}_j + u_j$ ($j = 1, \dots, k$), we obtain $\lambda_j = 0$ for every $j = 1, \dots, k$, which proves $\psi(0) \cap \psi(u)^\perp = \{0\}$. Similarly, let $x \in \bar{E}^\perp \cap \psi(u)$. Then there exists some real numbers λ_i ($i = 1, \dots, k$) such that $x = \sum_{i=1}^k \lambda_i(\bar{e}_i + u_i)$. Taking the scalar product with $\bar{e}_j$ ($j = 1, \dots, k$), we obtain $\lambda_j = 0$ for every $j = 1, \dots, k$, which proves $\psi(0) \cap \psi(u)^\perp = \{0\}$ and ends the proof of the claim. $\square$

Proof of Lemma A.1. Part (a). We prove that $(U_E, \varphi_E)_{E \in G^k(V)}$ is a smooth (i.e., C^∞) atlas of $G^k(V)$ and it is then clear that $\dim M = \dim(E^\perp)^k = k(\dim V - k)$. Let (U_E, φ_E) and (U_F, φ_F) be two local charts at E and F , respectively, such that $U_E \cap U_F \neq \emptyset$. We will prove that $\varphi_F \circ \varphi_E^{-1}$ is smooth (i.e., C^∞). We let $\{e_1, \dots, e_k\}$ and $\{f_1, \dots, f_k\}$ be two orthonormal bases of E and F , respectively. Let $(v_1, \dots, v_k) = \varphi_F \circ \varphi_E^{-1}(u_1, \dots, u_k)$ and $u = (u_1, \dots, u_k)$; then there exist real numbers $\lambda_{ij}(u)$ ($i, j = 1, \dots, k$) such that

$$f_i + v_i = \sum_{j=1}^k \lambda_{ij}(u)(e_j + u_j) \quad (i = 1, \dots, k). \quad (\text{A.4})$$

The proof will be complete by showing that the real-valued functions $\lambda_{ij}(u)$ are differentiable with respect to u . Taking the scalar product with f_l ($l = 1, \dots, k$), we obtain

$$f_l \cdot f_l = \sum_{j=1}^k \lambda_{lj}(u)(e_j + u_j) \cdot f_l \quad (l = 1, \dots, k). \quad (\text{A.5})$$

Thus, for every $i = 1, \dots, k$, the vector $\lambda_i(u) = (\lambda_{ij}(u))_{j=1}^k$ is the solution of a linear system whose matrix $G(u) = ((e_j + u_j) \cdot f_l)_{j,l=1,\dots,k}$ is now shown to be invertible (which clearly implies the differentiability of $\lambda_i(u)$). Indeed, if $G(u)\lambda = 0$ for some $\lambda \in \mathbb{R}^k$, then $\sum_{j=1}^k \lambda_j(e_j + u_j) \cdot f_l = 0$ (for $l = 1, \dots, k$), thus $\sum_{j=1}^k \lambda_j(e_j + u_j) \in F^\perp$. Besides, since $\sum_{j=1}^k \lambda_j(e_j + u_j) \in \psi_E(u_1, \dots, u_j) = \psi_F(v_1, \dots, v_j)$, we finally obtain $\sum_{j=1}^k \lambda_j(e_j + u_j) \in F^\perp \cap \psi_F(v_1, \dots, v_k) = \{0\}$ (from Claim A.2). Now, since $(e_j + u_j)_{j=1,\dots,k}$ is a basis, we obtain $\lambda_j = 0$ for every $j = 1, \dots, k$.

Part (b). Let (E^ν) be a sequence in $G^k(V)$ and for every ν , let $\{e_1^\nu, \dots, e_k^\nu\}$ be an orthonormal basis of E^ν . Without any loss of generality, we can assume that for every $i = 1, \dots, k$, the sequence (e_i^ν) converges to some element e_i in V . Clearly, $\{e_1, \dots, e_k\}$ is an orthonormal family in V , and we let $E = \text{span}\{e_1, \dots, e_k\}$. We will now prove that the sequence (E^ν) converges to E . Indeed, for ν large enough, there exists $u^\nu = (u_1^\nu, \dots, u_k^\nu) \in (E^\perp)^\perp$ such that $E^\nu = \psi_E(u_1^\nu, \dots, u_k^\nu)$. It can be written as $e_i + u_i^\nu = \sum_{j=1}^k \lambda_{ij}^\nu e_j^\nu$ ($i = 1, \dots, k$). Multiplying these equalities by e_l ($l = 1, \dots, k$), we obtain $\sum_{j=1}^k \lambda_{ij}^\nu e_j^\nu \cdot e_l = 0$ if $i \neq l$ and $\sum_{j=1}^k \lambda_{ij}^\nu e_j^\nu \cdot e_i = 1$. This can be written (for every $i = 1, \dots, k$) as $(e_j^\nu \cdot e_l)_{j,l=1,\dots,k} (\lambda_{ij}^\nu)_{j=1}^k = (e_i \cdot e_l)_{l=1,\dots,k}$. If ν is large enough, then $(e_j^\nu \cdot e_l)_{j,l=1,\dots,k}$ is invertible and converges to Id , which proves that the sequence $(\lambda_{ij}^\nu)_{j=1}^k$ converges to $(e_i \cdot e_l)_{l=1,\dots,k}$ for every $i = 1, \dots, k$, that is, (λ_{ii}^ν) converges to 1 and (λ_{ij}^ν) converges to 0 for $i \neq j$. We finally obtain that (u_i^ν) converges to 0, which proves that E^ν converges to E .

Part (c). Let $\bar{E} \in G^k(V)$ and let $(\bar{e}_1, \dots, \bar{e}_k)$ and $(\bar{f}_1, \dots, \bar{f}_\ell)$ be orthonormal bases of $\bar{E}$ and $\bar{E}^\perp$, respectively. Let $(u_1, \dots, u_k) \in (\bar{E}^\perp)^\perp$ and let $E = \psi(u)$. Then it is easy to see that $E^\perp = \text{span}\{f_1 + v_1, \dots, f_\ell + v_\ell\}$, where $v_i = -\sum_{j=1}^k (f_i \cdot u_j) \bar{e}_j$. So each v_i is a smooth mapping with respect to the u_i and, conversely, from $(E^\perp)^\perp = E$, each u_i is a smooth mapping with respect to the v_i . This ends the proof of part (c).

Part (d). The differentiability of the mapping p is left to the reader. Then notice that $\{(x, E) \in V \times G^k(V) \mid x \in E\} = \{(x, E) \in V \times G^k(V) \mid x = \text{proj}_E(x)\}$, which is clearly closed since the mapping $p : (x, E) \rightarrow \text{proj}_E(x)$ is continuous. $\square$

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