

FIXED POINT THEORY ON EXTENSION-TYPE SPACES AND ESSENTIAL MAPS ON TOPOLOGICAL SPACES

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Received 19 November 2003

We present several new fixed point results for admissible self-maps in extension-type spaces. We also discuss a continuation-type theorem for maps between topological spaces.

1. Introduction

In [Section 2](#), we begin by presenting most of the up-to-date results in the literature [[3](#), [5](#), [6](#), [7](#), [8](#), [12](#)] concerning fixed point theory in extension-type spaces. These results are then used to obtain a number of new fixed point theorems, one concerning approximate neighborhood extension spaces and another concerning inward-type maps in extension-type spaces. Our first result was motivated by ideas in [[12](#)] whereas the second result is based on an argument of Ben-El-Mechaiekh and Kryszewski [[9](#)]. Also in [Section 2](#) we present a new continuation theorem for maps defined between Hausdorff topological spaces, and our theorem improves results in [[3](#)].

For the remainder of this section we present some definitions and known results which will be needed throughout this paper. Suppose X and Y are topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X, Y)$ denotes the set of maps $F : X \rightarrow 2^Y$ (nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. We let

$$\mathcal{F}(\mathcal{X}) = \{Z : \text{Fix } F \neq \emptyset \ \forall F \in \mathcal{X}(Z, Z)\}, \quad (1.1)$$

where $\text{Fix } F$ denotes the set of fixed points of F .

The class $\mathcal{A}$ of maps is defined by the following properties:

- (i) $\mathcal{A}$ contains the class $\mathcal{C}$ of single-valued continuous functions;
- (ii) each $F \in \mathcal{A}_c$ is upper semicontinuous and closed valued;
- (iii) $B^n \in \mathcal{F}(\mathcal{A}_c)$ for all $n \in \{1, 2, \dots\}$; here $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$.

Remark 1.1. The class $\mathcal{A}$ is essentially due to Ben-El-Mechaiekh and Deguire [[7](#)]. It includes the class of maps $\mathcal{U}$ of Park ($\mathcal{U}$ is the class of maps defined by (i), (iii), and (iv) each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued). Thus if each $F \in \mathcal{A}_c$ is compact

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valued, the classes $\mathcal{A}$ and $\mathcal{U}$ coincide and this is what occurs in [Section 2](#) since our maps will be compact.

The following result can be found in [7, Proposition 2.2] (see also [11, page 286] for a special case).

THEOREM 1.2. *The Hilbert cube I^∞ (subset of l^2 consisting of points $(x_1, x_2, \dots)$ with $|x_i| \leq 1/2^i$ for all i) and the Tychonoff cube T (Cartesian product of copies of the unit interval) are in $\mathcal{F}(\mathcal{A}_c)$.*

We next consider the class $\mathcal{U}_c^k(X, Y)$ (resp., $\mathcal{A}_c^k(X, Y)$) of maps $F : X \rightarrow 2^Y$ such that for each F and each nonempty compact subset K of X , there exists a map $G \in \mathcal{U}_c(K, Y)$ (resp., $G \in \mathcal{A}_c(K, Y)$) such that $G(x) \subseteq F(x)$ for all $x \in K$.

THEOREM 1.3. *The Hilbert cube I^∞ and the Tychonoff cube T are in $\mathcal{F}(\mathcal{A}_c^k)$ (resp., $\mathcal{F}(\mathcal{U}_c^k)$).*

Proof. Let $F \in \mathcal{A}_c^k(I^\infty, I^\infty)$. We must show that $\text{Fix } F \neq \emptyset$. Now, by definition, there exists $G \in \mathcal{A}_c(I^\infty, I^\infty)$ with $G(x) \subseteq F(x)$ for all $x \in I^\infty$, so [Theorem 1.2](#) guarantees that there exists $x \in I^\infty$ with $x \in Gx$. In particular, $x \in Fx$ so $\text{Fix } F \neq \emptyset$. Thus $I^\infty \in \mathcal{F}(\mathcal{A}_c^k)$. $\square$

Notice that $\mathcal{U}_c^k$ is closed under compositions. To see this, let X, Y , and Z be topological spaces, $F_1 \in \mathcal{U}_c^k(X, Y)$, $F_2 \in \mathcal{U}_c^k(Y, Z)$, and K a nonempty compact subset of X . Now there exists $G_1 \in \mathcal{U}_c(K, Y)$ with $G_1(x) \subseteq F_1(x)$ for all $x \in K$. Also [4, page 464] guarantees that $G_1(K)$ is compact so there exists $G_2 \in \mathcal{U}_c^k(G_1(K), Z)$ with $G_2(y) \subseteq F_2(y)$ for all $y \in G_1(K)$. As a result,

$$G_2 G_1(x) \subseteq F_2 G_1(x) \subseteq F_2 F_1(x) \quad \forall x \in K \quad (1.2)$$

and $G_2 G_1 \in \mathcal{U}_c(X, Z)$.

For a subset K of a topological space X , we denote by $\text{Cov}_X(K)$ the set of all coverings of K by open sets of X (usually we write $\text{Cov}(K) = \text{Cov}_X(K)$). Given a map $F : X \rightarrow 2^X$ and $\alpha \in \text{Cov}(X)$, a point $x \in X$ is said to be an α -fixed point of F if there exists a member $U \in \alpha$ such that $x \in U$ and $F(x) \cap U \neq \emptyset$. Given two maps $F, G : X \rightarrow 2^Y$ and $\alpha \in \text{Cov}(Y)$, F and G are said to be α -close if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$, and $w \in G(x) \cap U_x$.

The following results can be found in [5, Lemmas 1.2 and 4.7].

THEOREM 1.4. *Let X be a regular topological space and $F : X \rightarrow 2^X$ an upper semicontinuous map with closed values. Suppose there exists a cofinal family of coverings $\theta \subseteq \text{Cov}_X(F(X))$ such that F has an α -fixed point for every $\alpha \in \theta$. Then F has a fixed point.*

THEOREM 1.5. *Let T be a Tychonoff cube contained in a Hausdorff topological vector space. Then T is a retract of $\text{span}(T)$.*

Remark 1.6. From [Theorem 1.4](#) in proving the existence of fixed points in uniform spaces for upper semicontinuous compact maps with closed values, it suffices [6, page 298] to prove the existence of approximate fixed points (since open covers of a compact set A

admit refinements of the form $\{U[x] : x \in A\}$ where U is a member of the uniformity [14, page 199], so such refinements form a cofinal family of open covers). Note also that uniform spaces are regular (in fact completely regular) [10, page 431] (see also [10, page 434]). Note in [Theorem 1.4](#) if F is compact valued, then the assumption that X is regular can be removed. For convenience in this paper we will apply [Theorem 1.4](#) only when the space is uniform.

2. Extension-type spaces

We begin this section by recalling some results we established in [3]. By a space we mean a Hausdorff topological space. Let Q be a class of topological spaces. A space Y is an *extension space* for Q (written $Y \in \text{ES}(Q)$) if for all $X \in Q$ and all $K \subseteq X$ closed in X , any continuous function $f_0 : K \rightarrow Y$ extends to a continuous function $f : X \rightarrow Y$.

Using (i) the fact that every compact space is homeomorphic to a closed subset of the Tychonoff cube and (ii) [Theorem 1.3](#), we established the following result in [3].

THEOREM 2.1. *Let $X \in \text{ES}(\text{compact})$ and $F \in \mathcal{U}_c^k(X, X)$ a compact map. Then F has a fixed point.*

Remark 2.2. If $X \in \text{AR}$ (an absolute retract as defined in [11]), then of course $X \in \text{ES}(\text{compact})$.

A space Y is an *approximate extension space* for Q (written $Y \in \text{AES}(Q)$) if for all $\alpha \in \text{Cov}(Y)$, all $X \in Q$, all $K \subseteq X$ closed in X , and any continuous function $f_0 : K \rightarrow Y$, there exists a continuous function $f : X \rightarrow Y$ such that $f|_K$ is α -close to f_0 .

THEOREM 2.3. *Let $X \in \text{AES}(\text{compact})$ be a uniform space and $F \in \mathcal{U}_c^k(X, X)$ a compact upper semicontinuous map with closed values. Then F has a fixed point.*

Remark 2.4. This result was established in [3]. However, we excluded some assumptions (X uniform and F upper semicontinuous with closed values) so the proof in [3] has to be adjusted slightly.

Proof. Let $\alpha \in \text{Cov}_X(K)$ where $K = \overline{F(X)}$. From [Theorem 1.4](#) (see [Remark 1.6](#)), it suffices to show that F has an α -fixed point. We know (see [13]) that K can be embedded as a closed subset K^* of T ; let $s : K \rightarrow K^*$ be a homeomorphism. Also let $i : K \hookrightarrow X$ and $j : K^* \hookrightarrow T$ be inclusions. Next let $\alpha' = \alpha \cup \{X \setminus K\}$ and note that α' is an open covering of X . Let the continuous map $h : T \rightarrow X$ be such that $h|_{K^*}$ and s^{-1} are α' -close (guaranteed since $X \in \text{AES}(\text{compact})$). Then it follows immediately from the definition (note that $\alpha' = \alpha \cup \{X \setminus K\}$) that $hs : K \rightarrow X$ and $i : K \rightarrow X$ are α -close. Let $G = jsFh$ and notice that $G \in \mathcal{U}_c^k(T, T)$. Now [Theorem 1.3](#) guarantees that there exists $x \in T$ with $x \in Gx$. Let $y = h(x)$, and so, from the above, we have $y \in hjsF(y)$, that is, $y = hjs(q)$ for some $q \in F(y)$. Now since hs and i are α -close, there exists $U \in \alpha$ with $hs(q) \in U$ and $i(q) \in U$, that is, $q \in U$ and $y = hjs(q) = hs(q) \in U$ since $s(q) \in K^*$. Thus $q \in U$ and $y \in U$, so $y \in U$ and $F(y) \cap U \neq \emptyset$ since $q \in F(y)$. As a result, F has an α -fixed point. $\square$

Definition 2.5. Let V be a uniform space. Then V is *Schauder admissible* if for every compact subset K of V and every covering $\alpha \in \text{Cov}_V(K)$, there exists a continuous function (called the Schauder projection) $\pi_\alpha : K \rightarrow V$ such that

- (i) π_α and $i : K \hookrightarrow V$ are α -close;
- (ii) $\pi_\alpha(K)$ is contained in a subset $C \subseteq V$ with $C \in \text{AES}(\text{compact})$.

THEOREM 2.6. *Let V be a uniform space and Schauder admissible and $F \in \mathcal{U}_c^\kappa(V, V)$ a compact upper semicontinuous map with closed values. Then F has a fixed point.*

Proof. Let $K = \overline{F(X)}$ and let $\alpha \in \text{Cov}_V(K)$. From [Theorem 1.4](#) (see [Remark 1.6](#)), it suffices to show that F has an α -fixed point. There exists $\pi_\alpha : K \rightarrow V$ (as described in [Definition 2.5](#)) and a subset $C \subseteq V$ with $C \in \text{AES}(\text{compact})$ such that (here $F_\alpha = \pi_\alpha F$)

$$F_\alpha(V) = \pi_\alpha F(V) \subseteq C. \quad (2.1)$$

Notice that $F_\alpha \in \mathcal{U}_c^\kappa(C, C)$ is a compact upper semicontinuous map with closed (in fact compact) values. So [Theorem 2.3](#) guarantees that there exists $x \in C$ with $x \in \pi_\alpha F(x)$, that is, $x = \pi_\alpha q$ for some $q \in F(x)$. Now [Definition 2.5\(i\)](#) guarantees that there exists $U \in \alpha$ with $\pi_\alpha(q) \in U$ and $i(q) \in U$, that is, $x \in U$ and $q \in U$. Thus $x \in U$ and $F(x) \cap U \neq \emptyset$ since $q \in F(x)$, so F has an α -fixed point. $\square$

A space Y is a *neighborhood extension space* for Q (written $Y \in \text{NES}(Q)$) if for all $X \in Q$, all $K \subseteq X$ closed in X , and any continuous function $f_0 : K \rightarrow Y$, there exists a continuous extension $f : U \rightarrow Y$ of f_0 over a neighborhood U of K in X .

Let $X \in \text{NES}(Q)$ and $F \in \mathcal{U}_c^\kappa(X, X)$ a compact map. Now let K, K^*, s , and i be as in the proof of [Theorem 2.3](#). Let U be an open neighborhood of K^* in T and let $h_U : U \rightarrow X$ be a continuous extension of $is^{-1} : K^* \rightarrow X$ on U (guaranteed since $X \in \text{NES}(\text{compact})$). Let $j_U : K^* \hookrightarrow U$ be the natural embedding, so $h_U j_U = is^{-1}$. Now consider $\text{span}(T)$ in a Hausdorff locally convex topological vector space containing T . Now [Theorem 1.5](#) guarantees that there exists a retraction $r : \text{span}(T) \rightarrow T$. Let $i^* : U \hookrightarrow r^{-1}(U)$ be an inclusion and consider $G = i^* j_U s F h_U r$. Notice that $G \in \mathcal{U}_c^\kappa(r^{-1}(U), r^{-1}(U))$. We now assume that

$$G \in \mathcal{U}_c^\kappa(r^{-1}(U), r^{-1}(U)) \text{ has a fixed point.} \quad (2.2)$$

Now there exists $x \in r^{-1}(U)$ with $x \in Gx$. Let $y = h_U r(x)$, so $y \in h_U r i^* j_U s F(y)$, that is, $y = h_U r i^* j_U s(q)$ for some $q \in F(y)$. Since $h_U(z) = is^{-1}(z)$ for $z \in K^*$, we have

$$h_U r i^* j_U s(q) = (h_U r i^* j_U) s(q) = i(q), \quad (2.3)$$

so $y \in F(y)$.

THEOREM 2.7. *Let $X \in \text{NES}(\text{compact})$ and $F \in \mathcal{U}_c^\kappa(X, X)$ a compact map. Also assume that (2.2) holds with $K, K^*, s, i, i^*, j_U, h_U$, and r as described above. Then F has a fixed point.*

Remark 2.8. [Theorem 2.7](#) was also established in [\[3\]](#). Note that if F is admissible in the sense of Gorniewicz and the Lefschetz set $\Lambda(F) \neq \{0\}$, then we know [\[11\]](#) that (2.2) holds. Note that if $X \in \text{ANR}$ (see [\[11\]](#)), then of course $X \in \text{NES}(\text{compact})$.

A space Y is an *approximate neighborhood extension space* for Q (written $Y \in \text{ANES}(Q)$) if for all $\alpha \in \text{Cov}(Y)$, all $X \in Q$, all $K \subseteq X$ closed in X , and any continuous function $f_0 : K \rightarrow Y$, there exists a neighborhood U_α of K in X and a continuous function $f_\alpha : U_\alpha \rightarrow Y$ such that $f_\alpha|_K$ and f_0 are α .

Let $X \in \text{ANES}(\text{compact})$ be a uniform space and $F \in \mathcal{U}_c^\kappa(X, X)$ a compact upper semicontinuous map with closed values. Also let $\alpha \in \text{Cov}_X(K)$ where $K = \overline{F(X)}$. To show that F has a fixed point, it suffices (Theorem 1.4 and Remark 1.6) to show that F has an α -fixed point. Let $\alpha' = \alpha \cup \{X \setminus K\}$ and let K^* , s , and i be as in the proof of Theorem 2.3. Since $X \in \text{ANES}(\text{compact})$, there exists an open neighborhood U_α of K^* in T and $f_\alpha : U_\alpha \rightarrow X$ a continuous function such that $f_\alpha|_{K^*}$ and s^{-1} are α' -close and as a result $f_\alpha s : K \rightarrow X$ and $i : K \rightarrow X$ are α -close. Let $j_{U_\alpha} : K^* \hookrightarrow U_\alpha$ be the natural imbedding. We know (see [5, page 426]) that $U_\alpha \in \text{NES}(\text{compact})$. Also notice that $G_\alpha = j_{U_\alpha} s F f_\alpha \in \mathcal{U}_c^\kappa(U_\alpha, U_\alpha)$ is a compact upper semicontinuous map with closed values. We now assume that

$$G_\alpha = j_{U_\alpha} s F f_\alpha \in \mathcal{U}_c^\kappa(U_\alpha, U_\alpha) \text{ has a fixed point for each } \alpha \in \text{Cov}_X(\overline{F(X)}). \quad (2.4)$$

We still have $\alpha \in \text{Cov}_X(K)$ fixed and we let x be a fixed point of G_α . Now let $y_\alpha = f_\alpha(x)$, so $y = f_\alpha j_{U_\alpha} s F(y)$, that is, $y = f_\alpha j_{U_\alpha} s(q)$ for some $q \in F(y)$. Now since $f_\alpha s$ and i are α -close, there exists $U \in \alpha$ with $f_\alpha s(q) \in U$ and $i(q) \in U$, that is, $q \in U$ and $y = f_\alpha j_{U_\alpha} s(q) = f_\alpha s(q) \in U$ since $s(q) \in K^*$. Thus $q \in U$ and $y \in U$, so

$$y \in U, \quad F(y) \cap U \neq \emptyset \quad \text{since } q \in F(y). \quad (2.5)$$

THEOREM 2.9. *Let $X \in \text{ANES}(\text{compact})$ be a uniform space and $F \in \mathcal{U}_c^\kappa(X, X)$ a compact upper semicontinuous map with closed values. Also assume that (2.4) holds with K , s , U_α , j_{U_α} , and f_α as described above. Then F has a fixed point.*

Next we present continuation results for multimaps. Let Y be a completely regular topological space and U an open subset of Y . We consider a subclass $\mathcal{D}$ of $\mathcal{U}_c^\kappa$. This subclass must have the following property: for subsets X_1, X_2 , and X_3 of Hausdorff topological spaces, if $F \in \mathcal{D}(X_2, X_3)$ is compact and $f \in \mathcal{C}(X_1, X_2)$, then $F \circ f \in \mathcal{D}(X_1, X_3)$.

Definition 2.10. The map $F \in \mathcal{D}_{\partial U}(\overline{U}, Y)$ if $F \in \mathcal{D}(\overline{U}, Y)$ with F compact and $x \notin Fx$ for $x \in \partial U$; here $\overline{U}$ (resp., ∂U) denotes the closure (resp., the boundary) of U in Y .

Definition 2.11. A map $F \in \mathcal{D}_{\partial U}(\overline{U}, Y)$ is essential in $\mathcal{D}_{\partial U}(\overline{U}, Y)$ if for every $G \in \mathcal{D}_{\partial U}(\overline{U}, Y)$ with $G|_{\partial U} = F|_{\partial U}$, there exists $x \in U$ with $x \in Gx$.

THEOREM 2.12 (homotopy invariance). *Let Y and U be as above. Suppose $F \in \mathcal{D}_{\partial U}(\overline{U}, Y)$ is essential in $\mathcal{D}_{\partial U}(\overline{U}, Y)$ and $H \in \mathcal{D}(\overline{U} \times [0, 1], Y)$ is a closed compact map with $H(x, 0) = F(x)$ for $x \in \overline{U}$. Also assume that*

$$x \notin H_t(x) \quad \text{for any } x \in \partial U, t \in (0, 1] \quad (H_t(\cdot) = H(\cdot, t)). \quad (2.6)$$

Then H_1 has a fixed point in U .

Proof. Let

$$B = \{x \in \overline{U} : x \in H_t(x) \text{ for some } t \in [0, 1]\}. \quad (2.7)$$

When $t = 0$, $H_t = F$, and since $F \in \mathcal{D}_{\partial U}(\overline{U}, Y)$ is essential in $\mathcal{D}_{\partial U}(\overline{U}, Y)$, there exists $x \in U$ with $x \in Fx$. Thus $B \neq \emptyset$ and note that B is closed, in fact compact (recall that H is a closed, compact map). Notice also that (2.6) implies $B \cap \partial U = \emptyset$. Thus, since Y is

completely regular, there exists a continuous function $\mu : \overline{U} \rightarrow [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(B) = 1$. Define a map R by $R(x) = H(x, \mu(x))$ for $x \in \overline{U}$. Let $j : \overline{U} \rightarrow \overline{U} \times [0, 1]$ be given by $j(x) = (x, \mu(x))$. Note that j is continuous, so $R = H \circ j \in \mathcal{D}(\overline{U}, Y)$ (see the description of the class $\mathcal{D}$ before [Definition 2.10](#)). In addition, R is compact, and for $x \in \partial U$, we have $R(x) = H_0(x) = F(x)$. As a result, $R \in \mathcal{D}_{\partial U}(\overline{U}, Y)$ with $R|_{\partial U} = F|_{\partial U}$. Now since F is essential in $\mathcal{D}_{\partial U}(\overline{U}, Y)$, there exists $x \in U$ with $x \in R(x)$, that is, $x \in H_{\mu(x)}(x)$. Thus $x \in B$ and so $\mu(x) = 1$. Consequently, $x \in H_1(x)$. $\square$

Next we give an example of an essential map.

THEOREM 2.13 (normalization). *Let Y and U be as above with $0 \in U$. Suppose the following conditions are satisfied:*

for any map $\theta \in \mathcal{D}_{\partial U}(\overline{U}, Y)$ with $\theta|_{\partial U} = \{0\}$, the map J is in $\mathcal{U}_c^k(Y, Y)$;

$$J(x) = \begin{cases} \theta(x), & x \in \overline{U}, \\ \{0\}, & x \in Y \setminus \overline{U}, \end{cases} \quad (2.8)$$

and

$$J \in \mathcal{U}_c^k(Y, Y) \text{ has a fixed point.} \quad (2.9)$$

Then the zero map is essential in $\mathcal{D}_{\partial U}(\overline{U}, Y)$.

Remark 2.14. Note that examples of spaces Y for (2.9) to be true can be found in Theorems 2.1, 2.3, 2.6, 2.7, and 2.9 (notice that J is compact).

Proof of Theorem 2.13. Let $\theta \in \mathcal{D}_{\partial U}(\overline{U}, Y)$ with $\theta|_{\partial U} = \{0\}$. We must show that there exists $x \in U$ with $x \in \theta(x)$. Define a map J as in (2.8). From (2.8) and (2.9), we know that there exists $x \in Y$ with $x \in J(x)$. Now if $x \notin U$, we have $x \in J(x) = \{0\}$, which is a contradiction since $0 \in U$. Thus $x \in U$ so $x \in J(x) = \theta(x)$. $\square$

Remark 2.15. Other homotopy and essential map results in a topological vector space setting can be found in [1, 2].

To conclude this paper, we discuss inward-type maps for a general class of admissible maps. The proof presented involves minor modifications of an argument due to Ben-El-Mechaiekh and Kryszewski [9]. Let Y be a normed space and $X \subseteq Y$, and consider a subclass $\mathcal{R}(X, Y)$ of $\mathcal{U}_c^k(X, Y)$. This subclass must have the following properties: (i) if $X \subseteq Z \subseteq Y$ and if $I : X \hookrightarrow Z$ is an inclusion, $t > 0$, and $F \in \mathcal{R}(X, Y)$ with $(I + tF)(X) \subseteq Z$, then $I + tF \in \mathcal{U}_c^k(X, Z)$, and (ii) each $F \in \mathcal{R}(X, Y)$ is upper semicontinuous and compact valued.

In our next result we assume that Ω is a compact $\mathcal{L}$ -retract [9], that is,

- (A) Ω is a compact neighborhood retract of a normed space $E = (E, \|\cdot\|)$ and there exist $\beta > 0$, $r : B(\Omega, \beta) \rightarrow \Omega$ a retraction, and $L > 0$ such that $\|r(x) - x\| \leq Ld(x; \Omega)$ for $x \in B(\Omega, \beta)$.

As a result,

$$\exists \eta > 0, \quad \eta < \frac{\beta}{2} \quad \text{with } \|r(x) - x\| < \eta \quad \forall x \in B(\Omega, \eta). \quad (2.10)$$

THEOREM 2.16. *Let $E = (E, \|\cdot\|)$ be a normed space and Ω as in assumption (A), and assume either (i) Ω is Schauder admissible or (ii) (2.2) holds with $X = \Omega$. In addition, suppose $F \in \mathcal{R}(\Omega, E)$ with*

$$F(x) \subseteq C_\Omega(x) \quad \forall x \in \Omega. \quad (2.11)$$

Then there exists $x \in \Omega$ with $0 \in Fx$.

Remark 2.17. Here C_Ω is the Clarke tangent cone, that is,

$$C_\Omega(x) = \{v \in E : c(x, v) = 0\}, \quad (2.12)$$

where

$$c(x, y) = \limsup_{\substack{y \rightarrow x, y \in \Omega \\ t \downarrow 0}} \frac{d(x + tv; \Omega)}{t}. \quad (2.13)$$

Remark 2.18. If Ω is a compact neighborhood retract, then of course $\Omega \in \text{NES}(\text{compact})$.

Remark 2.19. The proof is basically due to Ben-El-Mechaiekh and Kryszewski [9] and is based on [9, Lemma 5.1] (this lemma is a modification of a standard argument in the literature using partitions of unity).

Proof. Now [9, Lemma 5.1] (choose $\Psi(x) = \{x \in E : c(x, v) < \delta\}$ ($\delta > 0$ appropriately chosen), $\Phi(x) = co(F(x))$ and apply the argument in [9, page 4176]) implies that there exists $M > 0$ such that for each $x \in K$ and each $y \in Fx$, we have $\|y\| \leq M$. Choose $\tau > 0$ with $M\tau < \eta$ (here η is as in (2.10)) and a sequence $(t_n)_{n \in \mathbb{N}}$ in $(0, \tau]$ with $t_n \downarrow 0$; here $N = \{1, 2, \dots\}$. Define a sequence of maps ψ_n , $n \in N$, by

$$\psi_n(x) = r(x + t_n F(x)) \quad \text{for } x \in \Omega; \quad (2.14)$$

note that $d(x + t_n y; \Omega) < \eta$ for $x \in \Omega$ and $y \in F(x)$ since $M\tau < \eta$. Fix $n \in N$ and notice that $\psi_n \in \mathcal{U}_c^\kappa(\Omega, \Omega)$ is a compact map (note that Ω is compact and ψ_n is upper semicontinuous with compact values). Now Theorem 2.6 or Theorem 2.7 guarantees that there exists $x_n \in \Omega$ and $y_n \in Fx_n$ with

$$x_n = r(x_n + t_n y_n). \quad (2.15)$$

Also notice from (2.15) and assumption (A) (note that $M\tau < \eta < \beta/2 < \beta$) that

$$t_n \|y_n\| = \|x_n + t_n y_n - r(x_n + t_n y_n)\| \leq Ld(x_n + t_n y_n; \Omega). \quad (2.16)$$

Now Ω is compact so $F(\Omega)$ is compact, and as a result, there exists a subsequence S of N with $(x_n, y_n) \in \text{Graph } F$ and $(x_n, y_n) \rightarrow (\bar{x}, \bar{y})$ as $n \rightarrow \infty$ in S . Of course, since F is upper

semicontinuous, we have $\bar{y} \in F(\bar{x})$. Also from (2.11), we have $F(\bar{x}) \subseteq C_\Omega(\bar{x})$ and as a result, $\bar{y} \in F(\bar{x}) \subseteq C_\Omega(\bar{x})$, so $c(\bar{x}, \bar{y}) = 0$. Note also that

$$d(x_n + t_n y_n; \Omega) \leq d(x_n + t_n \bar{y}; \Omega) + t_n \|y_n - \bar{y}\| \quad (2.17)$$

and this together with (2.16) yields

$$\|\bar{y}\| = \limsup_{n \rightarrow \infty} \|y_n\| \leq \limsup \left(\frac{Ld(x_n + t_n \bar{y}; \Omega)}{t_n} + \|y_n - \bar{y}\| \right) = c(\bar{x}, \bar{y}) = 0, \quad (2.18)$$

so $0 \in F(\bar{x})$. □

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