

THE LEFSCHETZ-HOPF THEOREM AND AXIOMS FOR THE LEFSCHETZ NUMBER

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The reduced Lefschetz number, that is, $L(\cdot) - 1$ where $L(\cdot)$ denotes the Lefschetz number, is proved to be the unique integer-valued function λ on self-maps of compact polyhedra which is constant on homotopy classes such that (1) $\lambda(fg) = \lambda(gf)$ for $f : X \rightarrow Y$ and $g : Y \rightarrow X$; (2) if (f_1, f_2, f_3) is a map of a cofiber sequence into itself, then $\lambda(f_1) = \lambda(f_1) + \lambda(f_3)$; (3) $\lambda(f) = -(\deg(p_1 f e_1) + \cdots + \deg(p_k f e_k))$, where f is a self-map of a wedge of k circles, e_r is the inclusion of a circle into the r th summand, and p_r is the projection onto the r th summand. If $f : X \rightarrow X$ is a self-map of a polyhedron and $I(f)$ is the fixed-point index of f on all of X , then we show that $I(\cdot) - 1$ satisfies the above axioms. This gives a new proof of the normalization theorem: if $f : X \rightarrow X$ is a self-map of a polyhedron, then $I(f)$ equals the Lefschetz number $L(f)$ of f . This result is equivalent to the Lefschetz-Hopf theorem: if $f : X \rightarrow X$ is a self-map of a finite simplicial complex with a finite number of fixed points, each lying in a maximal simplex, then the Lefschetz number of f is the sum of the indices of all the fixed points of f .

1. Introduction

Let X be a finite polyhedron and denote by $\tilde{H}_*(X)$ its reduced homology with rational coefficients. Then the *reduced Euler characteristic* of X , denoted by $\tilde{\chi}(X)$, is defined by

$$\tilde{\chi}(X) = \sum_k (-1)^k \dim \tilde{H}_k(X). \quad (1.1)$$

Clearly, $\tilde{\chi}(X)$ is just the Euler characteristic minus one. In 1962, Watts [13] characterized the reduced Euler characteristic as follows. Let ϵ be a function from the set of finite polyhedra with base points to the integers such that (i) $\epsilon(S^0) = 1$, where S^0 is the 0-sphere, and (ii) $\epsilon(X) = \epsilon(A) + \epsilon(X/A)$, where A is a subpolyhedron of X . Then $\epsilon(X) = \tilde{\chi}(X)$.

Let $\mathcal{C}$ be the collection of spaces X of the homotopy type of a finite, connected CW-complex. If $X \in \mathcal{C}$, we do not assume that X has a base point except when X is a sphere or a wedge of spheres. It is not assumed that maps between spaces with base points are based. A map $f : X \rightarrow X$, where $X \in \mathcal{C}$, induces trivial homomorphisms $f_{*k} : H_k(X) \rightarrow H_k(X)$

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of rational homology vector spaces for all $j > \dim X$. The *Lefschetz number* $L(f)$ of f is defined by

$$L(f) = \sum_k (-1)^k \text{Tr } f_{*k}, \quad (1.2)$$

where Tr denotes the trace. The reduced Lefschetz number $\tilde{L}$ is given by $\tilde{L}(f) = L(f) - 1$ or, equivalently, by considering the rational, reduced homology homomorphism induced by f .

Since $\tilde{L}(\text{id}) = \tilde{\chi}(X)$, where $\text{id} : X \rightarrow X$ is the identity map, Watts's Theorem suggests an axiomatization for the reduced Lefschetz number which we state below in [Theorem 1.1](#).

For $k \geq 1$, denote by $\bigvee^k S^n$ the wedge of k copies of the n -sphere S^n , $n \geq 1$. If we write $\bigvee^k S^n$ as $S_1^n \vee S_2^n \vee \cdots \vee S_k^n$, where $S_j^n = S^n$, then we have inclusions $e_j : S_j^n \rightarrow \bigvee^k S^n$ into the j th summand and projections $p_j : \bigvee^k S^n \rightarrow S_j^n$ onto the j th summand, for $j = 1, \dots, k$. If $f : \bigvee^k S^n \rightarrow \bigvee^k S^n$ is a map, then $f_j : S_j^n \rightarrow S_j^n$ denotes the composition $p_j f e_j$. The degree of a map $f : S^n \rightarrow S^n$ is denoted by $\deg(f)$.

We characterize the reduced Lefschetz number as follows.

THEOREM 1.1. *The reduced Lefschetz number $\tilde{L}$ is the unique function λ from the set of self-maps of spaces in $\mathcal{C}$ to the integers that satisfies the following conditions.*

- (1) (*Homotopy axiom*) If $f, g : X \rightarrow X$ are homotopic maps, then $\lambda(f) = \lambda(g)$.
- (2) (*Cofibration axiom*) If A is a subpolyhedron of X , $A \rightarrow X \rightarrow X/A$ is the resulting cofiber sequence, and there exists a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & X/A \\ f' \downarrow & & f \downarrow & & \bar{f} \downarrow \\ A & \longrightarrow & X & \longrightarrow & X/A, \end{array} \quad (1.3)$$

then $\lambda(f) = \lambda(f') + \lambda(\bar{f})$.

- (3) (*Commutativity axiom*) If $f : X \rightarrow Y$ and $g : Y \rightarrow X$ are maps, then $\lambda(gf) = \lambda(fg)$.
- (4) (*Wedge of circles axiom*) If $f : \bigvee^k S^1 \rightarrow \bigvee^k S^1$ is a map, $k \geq 1$, then

$$\lambda(f) = -(\deg(f_1) + \cdots + \deg(f_k)), \quad (1.4)$$

where $f_j = p_j f e_j$.

In an unpublished dissertation [10], Hoang extended Watts's axioms to characterize the reduced Lefschetz number for basepoint-preserving self-maps of finite polyhedra. His list of axioms is different from, but similar to, those in [Theorem 1.1](#).

One of the classical results of fixed-point theory is the following theorem.

THEOREM 1.2 (Lefschetz-Hopf). *If $f : X \rightarrow X$ is a map of a finite polyhedron with a finite set of fixed points, each of which lies in a maximal simplex of X , then $L(f)$ is the sum of the indices of all the fixed points of f .*

The history of this result is described in [3], see also [8, page 458]. A proof that depends on a delicate argument due to Dold [4] can be found in [2] and, in a more condensed form, in [5]. In an appendix to his dissertation [12], McCord outlined a possibly more direct argument, but no details were published. The book of Granas and Dugundji [8, pages 441–450] presents an argument based on classical techniques of Hopf [11]. We use the characterization of the reduced Lefschetz number in [Theorem 1.1](#) to prove the Lefschetz-Hopf theorem in a quite natural manner by showing that the fixed-point index satisfies the axioms of [Theorem 1.1](#). That is, we prove the following theorem.

THEOREM 1.3 (normalization property). *If $f : X \rightarrow X$ is any map of a finite polyhedron, then $L(f) = i(X, f, X)$, the fixed-point index of f on all of X .*

The Lefschetz-Hopf theorem follows from the normalization property by the additivity property of the fixed-point index. In fact, these two statements are equivalent. The Hopf construction [2, page 117] implies that a map f from a finite polyhedron to itself is homotopic to a map that satisfies the hypotheses of the Lefschetz-Hopf theorem. Thus, the homotopy and additivity properties of the fixed-point index imply that the normalization property follows from the Lefschetz-Hopf theorem.

2. Lefschetz numbers and exact sequences

In this section, all vector spaces are over a fixed field F , which will not be mentioned, and are finite dimensional. A graded vector space $V = \{V_n\}$ will always have the following properties: (1) each V_n is finite dimensional and (2) $V_n = 0$, for $n < 0$ and for $n > N$, for some nonnegative integer N . A map $f : V \rightarrow W$ of graded vector spaces $V = \{V_n\}$ and $W = \{W_n\}$ is a sequence of linear transformations $f_n : V_n \rightarrow W_n$. For a map $f : V \rightarrow V$, the *Lefschetz number* is defined by

$$L(f) = \sum_n (-1)^n \text{Tr } f_n. \quad (2.1)$$

The proof of the following lemma is straightforward, and hence omitted.

LEMMA 2.1. *Given a map of short exact sequences of vector spaces*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0, \end{array} \quad (2.2)$$

then $\text{Tr } g = \text{Tr } f + \text{Tr } h$.

THEOREM 2.2. *Let A , B , and C be graded vector spaces with maps $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$ and self-maps $f : A \rightarrow A$, $g : B \rightarrow B$, and $h : C \rightarrow C$. If, for every n , there is a linear transformation*

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$\partial_n : C_n \rightarrow A_{n-1}$ such that the following diagram is commutative and has exact rows:

$$\begin{array}{ccccccccccc}
 0 & \longrightarrow & A_N & \xrightarrow{\alpha_N} & B_N & \xrightarrow{\beta_N} & C_N & \xrightarrow{\partial_N} & A_{N-1} & \xrightarrow{\alpha_{N-1}} & \cdots \\
 & & \downarrow f_N & & \downarrow g_N & & \downarrow h_N & & \downarrow f_{N-1} & & \\
 0 & \longrightarrow & A_N & \xrightarrow{\alpha_N} & B_N & \xrightarrow{\beta_N} & C_N & \xrightarrow{\partial_N} & A_{N-1} & \xrightarrow{\alpha_{N-1}} & \cdots \\
 & & & & & & & & & & \\
 \cdots & \xrightarrow{\partial_1} & A_0 & \xrightarrow{\alpha_0} & B_0 & \xrightarrow{\beta_0} & C_0 & \longrightarrow & 0 & & \\
 & & \downarrow f_0 & & \downarrow g_0 & & \downarrow h_0 & & & & \\
 \cdots & \xrightarrow{\partial_1} & A_0 & \xrightarrow{\alpha_0} & B_0 & \xrightarrow{\beta_0} & C_0 & \longrightarrow & 0, & &
 \end{array} \tag{2.3}$$

then

$$L(g) = L(f) + L(h). \tag{2.4}$$

Proof. Let Im denote the image of a linear transformation and consider the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Im} & \longrightarrow & C_n & \longrightarrow & \text{Im } \partial_n \longrightarrow 0 \\
 & & \downarrow h_n | \text{Im } \beta_n & & \downarrow h_n & & \downarrow f_{n-1} | \text{Im } \partial_n \\
 0 & \longrightarrow & \text{Im } \beta_n & \longrightarrow & C_n & \longrightarrow & \text{Im } \partial_n \longrightarrow 0.
 \end{array} \tag{2.5}$$

By [Lemma 2.1](#), $\text{Tr}(h_n) = \text{Tr}(h_n | \text{Im } \beta_n) + \text{Tr}(f_{n-1} | \text{Im } \partial_n)$. Similarly, the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Im } \partial_n & \longrightarrow & A_{n-1} & \longrightarrow & \text{Im } \alpha_{n-1} \longrightarrow 0 \\
 & & \downarrow f_{n-1} | \text{Im } \partial_n & & \downarrow f_{n-1} & & \downarrow g_{n-1} | \text{Im } \alpha_{n-1} \\
 0 & \longrightarrow & \text{Im } \partial_n & \longrightarrow & A_{n-1} & \longrightarrow & \text{Im } \alpha_{n-1} \longrightarrow 0
 \end{array} \tag{2.6}$$

yields $\text{Tr}(f_{n-1} | \text{Im } \partial_n) = \text{Tr}(f_{n-1}) - \text{Tr}(g_{n-1} | \text{Im } \alpha_{n-1})$. Therefore,

$$\text{Tr}(h_n) = \text{Tr}(h_n | \text{Im } \beta_n) + \text{Tr}(f_{n-1}) - \text{Tr}(g_{n-1} | \text{Im } \alpha_{n-1}). \tag{2.7}$$

Now consider

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Im } \alpha_{n-1} & \longrightarrow & B_{n-1} & \longrightarrow & \text{Im } \beta_{n-1} \longrightarrow 0 \\
 & & \downarrow g_{n-1} | \text{Im } \alpha_{n-1} & & \downarrow g_{n-1} & & \downarrow h_{n-1} | \text{Im } \beta_{n-1} \\
 0 & \longrightarrow & \text{Im } \alpha_{n-1} & \longrightarrow & B_{n-1} & \longrightarrow & \text{Im } \beta_{n-1} \longrightarrow 0.
 \end{array} \tag{2.8}$$

So $\text{Tr}(g_{n-1} | \text{Im } \alpha_{n-1}) = \text{Tr}(g_{n-1}) - \text{Tr}(h_{n-1} | \text{Im } \beta_{n-1})$. Putting this all together, we obtain

$$\text{Tr}(h_n) = \text{Tr}(h_n | \text{Im } \beta_n) + \text{Tr}(f_{n-1}) - \text{Tr}(g_{n-1}) + \text{Tr}(h_{n-1} | \text{Im } \beta_{n-1}). \quad (2.9)$$

We next look at the left end of diagram (2.3) and get

$$0 = \text{Tr}(h_{N+1}) = \text{Tr}(f_N) - \text{Tr}(g_N) + \text{Tr}(h_N | \text{Im } \beta_N), \quad (2.10)$$

and at the right end which gives

$$\text{Tr}(h_1) = \text{Tr}(h_1 | \text{Im } \beta_1) + \text{Tr}(f_0) - \text{Tr}(g_0) + \text{Tr}(h_0). \quad (2.11)$$

A simple calculation now yields (where a homomorphism with a negative subscript is the zero homomorphism)

$$\begin{aligned} & \sum_{n=0}^N (-1)^n \text{Tr}(h_n) \\ &= \sum_{n=0}^{N+1} (-1)^n (\text{Tr}(h_n | \text{Im } \beta_n) + \text{Tr}(f_{n-1}) - \text{Tr}(g_{n-1}) + \text{Tr}(h_{n-1} | \text{Im } \beta_{n-1})) \quad (2.12) \\ &= - \sum_{n=0}^N (-1)^n \text{Tr}(f_n) + \sum_{n=0}^N (-1)^n \text{Tr}(g_n). \end{aligned}$$

Therefore, $L(h) = -L(f) + L(g)$. □

A more condensed version of this argument has recently been published, see [8, page 420].

We next give some simple consequences of [Theorem 2.2](#).

If $f : (X, A) \rightarrow (X, A)$ is a self-map of a pair, where $X, A \in \mathcal{C}$, then f determines $f_X : X \rightarrow X$ and $f_A : A \rightarrow A$. The map f induces homomorphisms $f_{*k} : H_k(X, A) \rightarrow H_k(X, A)$ of relative homology with coefficients in F . The *relative Lefschetz number* $L(f; X, A)$ is defined by

$$L(f; X, A) = \sum_k (-1)^k \text{Tr } f_{*k}. \quad (2.13)$$

Applying [Theorem 2.2](#) to the homology exact sequence of the pair (X, A) , we obtain the following corollary.

COROLLARY 2.3. *If $f : (X, A) \rightarrow (X, A)$ is a map of pairs, where $X, A \in \mathcal{C}$, then*

$$L(f; X, A) = L(f_X) - L(f_A). \quad (2.14)$$

This result was obtained by Bouszyc [1].

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COROLLARY 2.4. *Suppose $X = P \cup Q$, where $X, P, Q \in \mathcal{C}$ and $(X; P, Q)$ is a proper triad [6, page 34]. If $f : X \rightarrow X$ is a map such that $f(P) \subseteq P$ and $f(Q) \subseteq Q$, then, for f_P , f_Q , and $f_{P \cap Q}$ being the restrictions of f to P , Q , and $P \cap Q$, respectively, there exists*

$$L(f) = L(f_P) + L(f_Q) - L(f_{P \cap Q}). \quad (2.15)$$

Proof. The map f and its restrictions induce a map of the Mayer-Vietoris homology sequence [6, page 39] to itself, so the result follows from [Theorem 2.2](#). $\square$

A similar result was obtained by Ferrario [7, Theorem 3.2.1].

Our final consequence of [Theorem 2.2](#) will be used in the characterization of the reduced Lefschetz number.

COROLLARY 2.5. *If A is a subpolyhedron of X , $A \rightarrow X \rightarrow X/A$ is the resulting cofiber sequence of spaces in $\mathcal{C}$ and there exists a commutative diagram*

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & X/A \\ f' \downarrow & & f \downarrow & & \tilde{f} \downarrow \\ A & \longrightarrow & X & \longrightarrow & X/A, \end{array} \quad (2.16)$$

then

$$L(f) = L(f') + L(\tilde{f}) - 1. \quad (2.17)$$

Proof. We apply [Theorem 2.2](#) to the homology cofiber sequence. The “minus one” on the right-hand side arises because such sequence ends with

$$\longrightarrow H_0(A) \longrightarrow H_0(X) \longrightarrow \tilde{H}_0(X/A) \longrightarrow 0. \quad (2.18)$$

$\square$

3. Characterization of the Lefschetz number

Throughout this section, all spaces are assumed to lie in $\mathcal{C}$.

We let λ be a function from the set of self-maps of spaces in $\mathcal{C}$ to the integers that satisfies the homotopy axiom, cofibration axiom, commutativity axiom, and wedge of circles axiom of [Theorem 1.1](#) as stated in the introduction.

We draw a few simple consequences of these axioms. From the commutativity and homotopy axioms, we obtain the following lemma.

LEMMA 3.1. *If $f : X \rightarrow X$ is a map and $h : X \rightarrow Y$ is a homotopy equivalence with homotopy inverse $k : Y \rightarrow X$, then $\lambda(f) = \lambda(hfk)$.*

LEMMA 3.2. *If $f : X \rightarrow X$ is homotopic to a constant map, then $\lambda(f) = 0$.*

Proof. Let $*$ be a one-point space and $*$: $*$ $\rightarrow$ $*$ the unique map. From the map of cofiber sequences

$$\begin{array}{ccccc} * & \longrightarrow & * & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & * & \longrightarrow & * \end{array} \quad (3.1)$$

and the cofibration axiom, we have $\lambda(*) = \lambda(*) + \lambda(*)$, and therefore $\lambda(*) = 0$. Write any constant map $c : X \rightarrow X$ as $c(x) = *$, for some $*$ $\in$ X , let $e : * \rightarrow X$ be inclusion and $p : X \rightarrow *$ projection. Then $c = ep$ and $pe = *$, and so $\lambda(c) = 0$ by the commutativity axiom. The lemma follows from the homotopy axiom. $\square$

If X is a based space with base point $*$, that is, a sphere or wedge of spheres, then the cone and suspension of X are defined by $CX = X \times I / (X \times 1 \cup * \times I)$ and $\Sigma X = CX / (X \times 0)$, respectively.

LEMMA 3.3. *If X is a based space, $f : X \rightarrow X$ is a based map, and $\Sigma f : \Sigma X \rightarrow \Sigma X$ is the suspension of f , then $\lambda(\Sigma f) = -\lambda(f)$.*

Proof. Consider the maps of cofiber sequences

$$\begin{array}{ccccc} X & \longrightarrow & CX & \longrightarrow & \Sigma X \\ f \downarrow & & Cf \downarrow & & \Sigma f \downarrow \\ X & \longrightarrow & CX & \longrightarrow & \Sigma X. \end{array} \quad (3.2)$$

Since CX is contractible, Cf is homotopic to a constant map. Therefore, by Lemma 3.2 and the cofibration axiom,

$$0 = \lambda(Cf) = \lambda(\Sigma f) + \lambda(f). \quad (3.3)$$

$\square$

LEMMA 3.4. *For any $k \geq 1$ and $n \geq 1$, if $f : \bigvee^k S^n \rightarrow \bigvee^k S^n$ is a map, then*

$$\lambda(f) = (-1)^n (\deg(f_1) + \cdots + \deg(f_k)), \quad (3.4)$$

where $e_j : S^n \rightarrow \bigvee^k S^n$ and $p_j : \bigvee^k S^n \rightarrow S^n$, for $j = 1, \dots, k$, are the inclusions and projections, respectively, and $f_j = p_j f e_j$.

Proof. The proof is by induction on the dimension n of the spheres. The case $n = 1$ is the wedge of circles axiom. If $n \geq 2$, then the map $f : \bigvee^k S^n \rightarrow \bigvee^k S^n$ is homotopic to a based map $f' : \bigvee^k S^n \rightarrow \bigvee^k S^n$. Then f' is homotopic to Σg , for some map $g : \bigvee^k S^{n-1} \rightarrow \bigvee^k S^{n-1}$. Note that if $g_j : S_j^{n-1} \rightarrow S_j^{n-1}$, then Σg_j is homotopic to $f_j : S_j^n \rightarrow S_j^n$. Therefore, by

Lemma 3.3 and the induction hypothesis,

$$\begin{aligned}\lambda(f) &= \lambda(f') = -\lambda(g) = -(-1)^{n-1}(\deg(g_1) + \cdots + \deg(g_k)) \\ &= (-1)^n(\deg(f_1) + \cdots + \deg(f_k)).\end{aligned}\tag{3.5}$$

□

Proof of Theorem 1.1. Since $\tilde{L}(f) = L(f) - 1$, **Corollary 2.5** implies that $\tilde{L}$ satisfies the cofibration axiom. We next show that $\tilde{L}$ satisfies the wedge of circles axiom. There is an isomorphism $\theta: \bigoplus^k H_1(S^1) \rightarrow H_1(\bigvee^k S^1)$ defined by $\theta(x_1, \dots, x_k) = e_{1*}(x_1) + \cdots + e_{k*}(x_k)$, where $x_i \in H_1(S^1)$. The inverse $\theta^{-1}: H_1(\bigvee^k S^1) \rightarrow \bigoplus^k H_1(S^1)$ is given by $\theta^{-1}(y) = (p_{1*}(y), \dots, p_{k*}(y))$. If $u \in H_1(S^1)$ is a generator, then a basis for $H_1(\bigvee^k S^1)$ is $e_{1*}(u), \dots, e_{k*}(u)$. By calculating the trace of $f_*: H_1(\bigvee^k S^1) \rightarrow H_1(\bigvee^k S^1)$ with respect to this basis, we obtain $\tilde{L}(f) = -(\deg(f_1) + \cdots + \deg(f_k))$. The remaining axioms are obviously satisfied by $\tilde{L}$. Thus $\tilde{L}$ satisfies the axioms of **Theorem 1.1**.

Now suppose λ is a function from the self-maps of spaces in $\mathcal{C}$ to the integers that satisfies the axioms. We regard X as a connected, finite CW-complex and proceed by induction on the dimension of X . If X is 1-dimensional, then it is the homotopy type of a wedge of circles. By **Lemma 3.1**, we can regard f as a self-map of $\bigvee^k S^1$, and so the wedge of circles axiom gives

$$\lambda(f) = -(\deg(f_1) + \cdots + \deg(f_k)) = \tilde{L}(f).\tag{3.6}$$

Now suppose that X is n -dimensional and let X^{n-1} denote the $(n-1)$ -skeleton of X . Then f is homotopic to a cellular map $g: X \rightarrow X$ by the cellular approximation theorem [9, Theorem 4.8, page 349]. Thus $g(X^{n-1}) \subseteq X^{n-1}$, and so we have a commutative diagram

$$\begin{array}{ccccc} X^{n-1} & \longrightarrow & X & \longrightarrow & X/X^{n-1} = \bigvee^k S^n \\ g' \downarrow & & g \downarrow & & \tilde{g} \downarrow \\ X^{n-1} & \longrightarrow & X & \longrightarrow & X/X^{n-1} = \bigvee^k S^n. \end{array}\tag{3.7}$$

Then, by the cofibration axiom, $\lambda(g) = \lambda(g') + \lambda(\tilde{g})$. **Lemma 3.4** implies that $\lambda(\tilde{g}) = \tilde{L}(\tilde{g})$. So, applying the induction hypothesis to g' , we have $\lambda(g) = \tilde{L}(g') + \tilde{L}(\tilde{g})$. Since we have seen that the reduced Lefschetz number satisfies the cofibration axiom, we conclude that $\lambda(g) = \tilde{L}(g)$. By the homotopy axiom, $\lambda(f) = \tilde{L}(f)$. □

4. The normalization property

Let X be a finite polyhedron and $f: X \rightarrow X$ a map. Denote by $I(f)$ the fixed-point index of f on all of X , that is, $I(f) = i(X, f, X)$ in the notation of [2] and let $\tilde{I}(f) = I(f) - 1$.

In this section, we prove **Theorem 1.3** by showing that, with rational coefficients, $I(f) = L(f)$.

Proof of Theorem 1.3. We will prove that $\tilde{I}$ satisfies the axioms, and therefore, by **Theorem 1.1**, $\tilde{I}(f) = \tilde{L}(f)$. The homotopy and commutativity axioms are well-known properties of the fixed-point index (see [2, pages 59–62]).

To show that $\tilde{I}$ satisfies the cofibration axiom, it suffices to consider A a subpolyhedron of X and $f(A) \subseteq A$. Let $f' : A \rightarrow A$ denote the restriction of f and $\tilde{f} : X/A \rightarrow X/A$ the map induced on quotient spaces. Let $r : U \rightarrow A$ be a deformation retraction of a neighborhood of A in X onto A and let L be a subpolyhedron of a barycentric subdivision of X such that $A \subseteq \text{int } L \subseteq L \subseteq U$. By the homotopy extension theorem, there is a homotopy $H : X \times I \rightarrow X$ such that $H(x, 0) = f(x)$ for all $x \in X$, $H(a, t) = f(a)$ for all $a \in A$, and $H(x, 1) = fr(x)$ for all $x \in L$. If we set $g(x) = H(x, 1)$, then, since there are no fixed points of g on $L - A$, the additivity property implies that

$$I(g) = i(X, g, \text{int } L) + i(X, g, X - L). \quad (4.1)$$

We discuss each summand of (4.1) separately. We begin with $i(X, g, \text{int } L)$. Since $g(L) \subseteq A \subseteq L$, it follows from the definition of the index (see [2, page 56]) that $i(X, g, \text{int } L) = i(L, g, \text{int } L)$. Moreover, $i(L, g, \text{int } L) = i(L, g, L)$ since there are no fixed points on $L - \text{int } L$ (the excision property of the index). Let $e : A \rightarrow L$ be inclusion, then, by the commutativity property [2, page 62], we have

$$i(L, g, L) = i(L, eg, L) = i(A, ge, A) = I(f') \quad (4.2)$$

because $f(a) = g(a)$ for all $a \in A$.

Next we consider the summand $i(X, g, X - L)$ of (4.1). Let $\pi : X \rightarrow X/A$ be the quotient map, set $\pi(A) = *$, and note that $\pi^{-1}(*) = A$. If $\bar{g} : X/A \rightarrow X/A$ is induced by g , the restriction of $\bar{g}$ to the neighborhood $\pi(\text{int } L)$ of $*$ in X/A is constant, so $i(X/A, \bar{g}, \pi(\text{int } L)) = 1$. If we denote the set of fixed points of $\bar{g}$ with $*$ deleted by $\text{Fix}_* \bar{g}$, then $\text{Fix}_* \bar{g}$ is in the open subset $X/A - \pi(L)$ of X/A . Let W be an open subset of X/A such that $\text{Fix}_* \bar{g} \subseteq W \subseteq X/A - \pi(L)$ with the property $\bar{g}(W) \cap \pi(L) = \emptyset$. By the additivity property, we have

$$I(\bar{g}) = i(X/A, \bar{g}, \pi(\text{int } L)) + i(X/A, \bar{g}, W) = 1 + i(X/A, \bar{g}, W). \quad (4.3)$$

Now, identifying $X - L$ with the corresponding subset $\pi(X - L)$ of X/A and identifying the restrictions of $\bar{g}$ and g to those subsets, we have $i(X/A, \bar{g}, W) = i(X, g, \pi^{-1}(W))$. The excision property of the index implies that $i(X, g, \pi^{-1}(W)) = i(X, g, X - L)$. Thus we have determined the second summand of (4.1): $i(X, g, X - L) = I(\bar{g}) - 1$.

Therefore, from (4.1) we obtain $I(g) = I(f') + I(\bar{g}) - 1$. The homotopy property then tells us that

$$I(f) = I(f') + I(\tilde{f}) - 1 \quad (4.4)$$

since f is homotopic to g and $\tilde{f}$ is homotopic to $\bar{g}$. We conclude that $\tilde{I}$ satisfies the cofibration axiom.

It remains to verify the wedge of circles axiom. Let $X = \bigvee^k S^1 = S^1_1 \vee \cdots \vee S^1_k$ be a wedge of circles with basepoint $*$ and $f : X \rightarrow X$ a map. We first verify the axiom in the case $k = 1$. We have $f : S^1 \rightarrow S^1$ and we denote its degree by $\deg(f) = d$. We regard $S^1 \subseteq \mathbb{C}$, the complex numbers. Then f is homotopic to g_d , where $g_d(z) = z^d$ has $|d - 1|$ fixed points for $d \neq 1$. The fixed-point index of g_d in a neighborhood of a fixed point that contains no other fixed point of g_d is -1 if $d \geq 2$ and is 1 if $d \leq 0$. Since g_1 is homotopic to

a map without fixed points, we see that $I(g_d) = -d + 1$ for all integers d . We have shown that $I(f) = -\deg(f) + 1$.

Now suppose $k \geq 2$. If $f(*) = *$, then, by the homotopy extension theorem, f is homotopic to a map which does not fix $*$. Thus we may assume, without loss of generality, that $f(*) \in S_1^1 - \{*\}$. Let V be a neighborhood of $f(*)$ in $S_1^1 - \{*\}$ such that there exists a neighborhood U of $*$ in X , disjoint from V , with $f(\bar{U}) \subseteq V$. Since $\bar{U}$ contains no fixed point of f and the open subsets $S_j^1 - \bar{U}$ of X are disjoint, the additivity property implies

$$I(f) = i(X, f, S_1^1 - \bar{U}) + \sum_{j=2}^k i(X, f, S_j^1 - \bar{U}). \quad (4.5)$$

The additivity property also implies that

$$I(f_j) = i(S_j^1, f_j, S_j^1 - \bar{U}) + i(S_j^1, f_j, S_j^1 \cap U). \quad (4.6)$$

There is a neighborhood W_j of $(\text{Fix } f) \cap S_j^1$ in S_j^1 such that $f(\bar{W}_j) \subseteq S_j^1$. Thus $f_j(x) = f(x)$ for $x \in W_j$, and therefore, by the excision property,

$$i(S_j^1, f_j, S_j^1 - \bar{U}) = i(S_j^1, f_j, W_j) = i(X, f, W_j) = i(X, f, S_j^1 - \bar{U}). \quad (4.7)$$

Since $f(\bar{U}) \subseteq S_1^1$, then $f_1(x) = f(x)$ for all $x \in \bar{U} \cap S_1^1$. There are no fixed points of f in $\bar{U}$, so $i(S_1^1, f_1, S_1^1 \cap U) = 0$, and thus, $I(f_1) = i(X, f, S_1^1 - \bar{U})$ by (4.6) and (4.7).

For $j \geq 2$, the fact that $f_j(U) = *$ gives us $i(S_j^1, f_j, S_j^1 \cap U) = 1$, so $I(f_j) = i(X, f, S_j^1 - \bar{U}) + 1$ by (4.6) and (4.7). Since $f_j : S_j^1 \rightarrow S_j^1$, the $k = 1$ case of the argument tells us that $I(f_j) = -\deg(f_j) + 1$ for $j = 1, 2, \dots, k$. In particular, $i(X, f, S_1^1 - \bar{U}) = -\deg(f_1) + 1$, whereas, for $j \geq 2$, we have $i(X, f, S_j^1 - \bar{U}) = -\deg(f_j)$. Therefore, by (4.5),

$$I(f) = i(X, f, S_1^1 - \bar{U}) + \sum_{j=2}^k i(X, f, S_j^1 - \bar{U}) = -\sum_{j=1}^k \deg(f_j) + 1. \quad (4.8)$$

This completes the proof of [Theorem 1.3](#). □

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