

Research Article

Positive Solution to Nonzero Boundary Values Problem for a Coupled System of Nonlinear Fractional Differential Equations

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We consider the existence and uniqueness of positive solution to nonzero boundary values problem for a coupled system of fractional differential equations. The differential operator is taken in the standard Riemann-Liouville sense. By using Banach fixed point theorem and nonlinear differentiation of Leray-Schauder type, the existence and uniqueness of positive solution are obtained. Two examples are given to demonstrate the feasibility of the obtained results.

1. Introduction

Fractional differential equation can describe many phenomena in various fields of science and engineering such as control, porous media, electrochemistry, viscoelasticity, and electromagnetic. There are many papers dealing with the existence and uniqueness of solution for nonlinear fractional differential equation; see, for example, [1–5]. In [1], the authors investigated a singular coupled system with initial value problems of fractional order. In [2], Su discussed a boundary value problem of coupled system with zero boundary values. By means of Schauder fixed point theorem, the existence of the solution is obtained. The nonzero boundary values problem of nonlinear fractional differential equations is more difficult and complicated. No contributions exist, as far as we know, concerning the existence of positive solution for coupled system of nonlinear fractional differential equations with nonzero boundary values.

In this paper, we consider the existence and uniqueness of positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations:

$$\begin{aligned}
D^\alpha u(t) + f(t, v(t)) &= 0, & 0 < t < 1, \\
D^\beta v(t) + g(t, u(t)) &= 0, & 0 < t < 1, \\
u(0) = 0, & u(1) = au(\xi), \\
v(0) = 0, & v(1) = bv(\xi),
\end{aligned} \tag{1.1}$$

where $1 < \alpha < 2, 1 < \beta < 2, 0 \leq a, b \leq 1, 0 < \xi < 1, f, g : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ are given functions, and D is the standard Riemann-Liouville differentiation. By using Banach fixed point theorem and nonlinear differentiation of Leray-Schauder type, some sufficient conditions for the existence and uniqueness of positive solution to the above coupled boundary values problem are obtained.

The rest of the paper is organized as follows. In Section 2, we introduce some basic definitions and preliminaries used in later. In Section 3, the existence and uniqueness of positive solution for the coupled boundary values problem (1.1) will be discussed, and examples are given to demonstrate the feasibility of the obtained results.

2. Basic Definitions and Preliminaries

In this section, we introduce some basic definitions and lemmas which are used throughout this paper.

Definition 2.1 (see [6, 7]). The fractional integral of order α ($\alpha > 0$) of a function $y : (0, \infty) \rightarrow R$ is given by

$$I^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds, \tag{2.1}$$

provided that the right side is pointwise defined on $(0, \infty)$.

Definition 2.2 (see [6, 7]). The fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \rightarrow R$ is given by

$$D^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} y(s) ds, \tag{2.2}$$

where $n = [\alpha] + 1$ provided that the right side is pointwise defined on $(0, \infty)$.

Remark 2.3 (see [3]). The following properties are useful for our discussion:

- (1) $I^\alpha D^\alpha u(t) = u(t) - \sum_{k=1}^N C_k t^{\alpha-k}, D^\alpha u(t) \in C(0, 1) \cap L(0, 1), C_k \in R, N = [\alpha] + 1,$
- (2) $D^\alpha I^\alpha u(t) = u(t),$
- (3) $D^\alpha t^\gamma = \Gamma(\gamma + 1) / \Gamma(\gamma + 1 - \alpha) t^{\gamma-\alpha}, \alpha > 0, \gamma > -1, \gamma > \alpha - 1, t > 0.$

Lemma 2.4 (the nonlinear alternative of Leray and Schauder type [8]). *Let E be a Banach space with $C \subseteq E$ closed and convex. Let U be a relatively open subset of C with $0 \in U$ and let $T : \overline{U} \rightarrow C$ be a continuous and compact mapping. Then either*

- (a) *the mapping T has a fixed point in \overline{U} , or*
- (b) *there exist $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda Tu$.*

Consider

$$\begin{aligned} D^\alpha u(t) + y(t) &= 0, \quad 0 < t < 1, \\ u(0) &= 0, \quad u(1) = au(\xi), \end{aligned} \tag{2.3}$$

then one has the following lemma.

Lemma 2.5. *Let $y \in C[0, 1]$ and $1 < \alpha < 2$, then $u(t)$ is a solution of BVP (2.3) if and only if $u(t)$ is a solution of the integral equation:*

$$u(t) = \int_0^1 G_1(t, s)y(s)ds, \tag{2.4}$$

where

$$G_1(t, s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1} - at^{\alpha-1}(\xi-s)^{\alpha-1} - (t-s)^{\alpha-1}(1-a\xi^{\alpha-1})}{(1-a\xi^{\alpha-1})\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, s \leq \xi, \\ \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}(1-a\xi^{\alpha-1})}{(1-a\xi^{\alpha-1})\Gamma(\alpha)}, & 0 < \xi \leq s \leq t \leq 1, \\ \frac{[t(1-s)]^{\alpha-1} - at^{\alpha-1}(\xi-s)^{\alpha-1}}{(1-a\xi^{\alpha-1})\Gamma(\alpha)}, & 0 \leq t \leq s \leq \xi \leq 1, \\ \frac{[t(1-s)]^{\alpha-1}}{(1-a\xi^{\alpha-1})\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \xi \leq s. \end{cases} \tag{2.5}$$

Proof. Assume that $u(t)$ is a solution of BVP (2.3), then by Remark 2.3, we have

$$\begin{aligned} u(t) &= -I^\alpha y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} \\ &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}. \end{aligned} \tag{2.6}$$

By (2.3), we have

$$C_2 = 0, \quad C_1 = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-a\xi^{\alpha-1})} y(s)ds - a \int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)(1-a\xi^{\alpha-1})} y(s)ds. \tag{2.7}$$

Therefore, we obtain

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-a\xi^{\alpha-1})} y(s) ds - a \int_0^\xi \frac{t^{\alpha-1}(\xi-s)^{\alpha-1}}{\Gamma(\alpha)(1-a\xi^{\alpha-1})} y(s) ds \\ &= \int_0^1 G_1(t,s) y(s) ds. \end{aligned} \quad (2.8)$$

Conversely, if $u(t)$ is a solution of integral equation (2.4), using the relation $D^\alpha t^{\alpha-m} = 0$, $m = 1, 2, \dots, N$, where N is the smallest integer greater than or equal to α [3, Remark 2.1], we have

$$\begin{aligned} D^\alpha u(t) &= -D^\alpha \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \right) \\ &\quad + D^\alpha t^{\alpha-1} \left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-a\xi^{\alpha-1})} y(s) ds - a \int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)(1-a\xi^{\alpha-1})} y(s) ds \right] \\ &= -D^\alpha I^\alpha y(t) = -y(t). \end{aligned} \quad (2.9)$$

A simple computation showed $u(0) = 0$, $u(1) = au(\xi)$. The proof is complete. \square

Let

$$G_2(t,s) = \begin{cases} \frac{[t(1-s)]^{\beta-1} - bt^{\beta-1}(\xi-s)^{\beta-1} - (t-s)^{\beta-1}(1-b\xi^{\beta-1})}{(1-b\xi^{\beta-1})\Gamma(\beta)}, & 0 \leq s \leq t \leq 1, s \leq \xi, \\ \frac{[t(1-s)]^{\beta-1} - (t-s)^{\beta-1}(1-b\xi^{\beta-1})}{(1-b\xi^{\beta-1})\Gamma(\beta)}, & 0 < \xi \leq s \leq t \leq 1, \\ \frac{[t(1-s)]^{\beta-1} - bt^{\beta-1}(\xi-s)^{\beta-1}}{(1-b\xi^{\beta-1})\Gamma(\beta)}, & 0 \leq t \leq s \leq \xi \leq 1, \\ \frac{[t(1-s)]^{\beta-1}}{(1-b\xi^{\beta-1})\Gamma(\beta)}, & 0 \leq t \leq s \leq 1, \xi \leq s, \end{cases} \quad (2.10)$$

we call $G(t,s) = (G_1(t,s), G_2(t,s))$ Green's function of the boundary value problem (1.1).

Lemma 2.6. *Let $0 \leq a, b \leq 1$, then the function $G(t,s)$ is continuous and satisfies*

- (1) $G(t,s) > 0$, for $t, s \in (0,1)$,
- (2) $G(t,s) \leq G(s,s)$, for $t, s \in (0,1)$.

Proof. It is easy to prove that $G(t, s)$ is continuous on $[0, 1] \times [0, 1]$, here we omit it. Now we prove $G_1(t, s) > 0$. Let

$$\begin{aligned} g_1(t, s) &= \frac{[t(1-s)]^{\alpha-1} - at^{\alpha-1}(\xi-s)^{\alpha-1} - (t-s)^{\alpha-1}(1-a\xi^{\alpha-1})}{(1-a\xi^{\alpha-1})\Gamma(\alpha)}, \quad 0 < s \leq t \leq 1, s \leq \xi, \\ g_2(t, s) &= \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}(1-a\xi^{\alpha-1})}{(1-a\xi^{\alpha-1})\Gamma(\alpha)}, \quad 0 < \xi \leq s \leq t \leq 1, \\ g_3(t, s) &= \frac{[t(1-s)]^{\alpha-1} - at^{\alpha-1}(\xi-s)^{\alpha-1}}{(1-a\xi^{\alpha-1})\Gamma(\alpha)}, \quad 0 < t \leq s \leq \xi \leq 1, \\ g_4(t, s) &= \frac{[t(1-s)]^{\alpha-1}}{(1-a\xi^{\alpha-1})\Gamma(\alpha)}, \quad 0 < t \leq s \leq 1, \xi \leq s. \end{aligned} \tag{2.11}$$

We only need to prove $g_1(t, s) > 0$, $0 < s \leq t \leq 1$, $s \leq \xi$. Since

$$\begin{aligned} & [t(1-s)]^{\alpha-1} - at^{\alpha-1}(\xi-s)^{\alpha-1} - (t-s)^{\alpha-1}(1-a\xi^{\alpha-1}) \\ &= t^{\alpha-1}[(1-s)]^{\alpha-1} - a(\xi-s)^{\alpha-1} - \left(1 - \frac{s}{t}\right)^{\alpha-1}(1-a\xi^{\alpha-1}), \end{aligned} \tag{2.12}$$

set $g(t) = (1-s)^{\alpha-1} - a(\xi-s)^{\alpha-1} - (1-s/t)^{\alpha-1}(1-a\xi^{\alpha-1})$, we have

$$g'(t) = -(\alpha-1)\left(1 - \frac{s}{t}\right)^{\alpha-2} \frac{s}{t^2} (1-a\xi^{\alpha-1}) \leq 0, \quad \text{for } 0 < s < t \leq 1, s \leq \xi. \tag{2.13}$$

Then $g(t)$ is decreasing on $(0, 1)$. Meanwhile,

$$\begin{aligned} g(1) &= (1-s)^{\alpha-1} - a(\xi-s)^{\alpha-1} - (1-s)^{\alpha-1}(1-a\xi^{\alpha-1}) \\ &= a\xi^{\alpha-1} \left[(1-s)^{\alpha-1} - \left(1 - \frac{s}{\xi}\right)^{\alpha-1} \right] > 0, \quad 0 < s < t \leq 1, s \leq \xi. \end{aligned} \tag{2.14}$$

Therefore, $g_1(t, s) > 0$, for $0 < s < t \leq 1, s \leq \xi$. Clearly $g_1(t, s) > 0$, $t = s$, so $g_1(t, s) > 0$, $s, t \in (0, 1)$. It is easy to show that $g_2(t, s) > 0, g_3(t, s) > 0, g_4(t, s) > 0$. Hence, $G_1(t, s) > 0$, $s, t \in (0, 1)$.

Similarly, $G_2(t, s) > 0$, $s, t \in (0, 1)$. The proof of (1) is completed.

Let

$$g_2(t) = \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}(1-a\xi^{\alpha-1})}{(1-a\xi^{\alpha-1})\Gamma(\alpha)}, \quad 0 < \xi \leq s \leq t \leq 1, \tag{2.15}$$

then,

$$\begin{aligned}
 g_2'(t) &= \frac{(\alpha - 1)t^{\alpha-2}[(1-s)^{\alpha-1} - (1-s/t)^{\alpha-2}(1 - a\xi^{\alpha-1})]}{(1 - a\xi^{\alpha-1})\Gamma(\alpha)}, \quad 0 < \xi \leq s < t \leq 1, \\
 &[(1-s)^{\alpha-1} - (1-s/t)^{\alpha-2}(1 - a\xi^{\alpha-1})] \\
 &\leq [(1-s)^{\alpha-1} - (1-s)^{\alpha-2}(1 - a\xi^{\alpha-1})] \\
 &= [(1-s)^{\alpha-2}(a\xi^{\alpha-1} - s)] \leq 0, \quad 0 < \xi \leq s < t \leq 1,
 \end{aligned} \tag{2.16}$$

therefore,

$$g_2'(t) \leq 0, \quad 0 < \xi \leq s < t \leq 1. \tag{2.17}$$

So, $g_2(t, s)$ is decreasing with respect to t . Similarly, $g_1(t, s)$ is decreasing with respect to t . Also $g_3(t, s)$ and $g_4(t, s)$ are increasing with respect to t . We obtain that $G_1(t, s)$ is decreasing with respect to t for $s \leq t$ and increasing with respect to t for $t \leq s$.

With the use of the monotonicity of $G_1(t, s)$, we have

$$\max_{0 \leq t \leq 1} G_1(t, s) = G_1(s, s) = \begin{cases} \frac{[s(1-s)]^{\alpha-1} - a[s(\xi-s)]^{\alpha-1}}{\Gamma(\alpha)(1 - a\xi^{\alpha-1})}, & s \in (0, \xi], \\ \frac{[s(1-s)]^{\alpha-1}}{\Gamma(\alpha)(1 - a\xi^{\alpha-1})}, & s \in [\xi, 1). \end{cases} \tag{2.18}$$

Similarly,

$$\max_{0 \leq t \leq 1} G_2(t, s) = G_2(s, s) = \begin{cases} \frac{[s(1-s)]^{\beta-1} - b[s(\xi-s)]^{\beta-1}}{\Gamma(\beta)(1 - b\xi^{\beta-1})}, & s \in (0, \xi], \\ \frac{[s(1-s)]^{\beta-1}}{\Gamma(\beta)(1 - b\xi^{\beta-1})}, & s \in [\xi, 1). \end{cases} \tag{2.19}$$

The proof of (2) is completed. \square

3. Main Result

In this section, we will discuss the existence and uniqueness of positive solution for boundary value problem (1.1).

We define the space $X = \{u(t) \mid u(t) \in C[0, 1]\}$ endowed with $\|u\|_X = \max_{0 \leq t \leq 1} |u(t)|$, $Y = \{v(t) \mid v(t) \in C[0, 1]\}$ endowed with $\|v\|_Y = \max_{0 \leq t \leq 1} |v(t)|$.

For $(u, v) \in X \times Y$, let $\|(u, v)\|_{X \times Y} = \max\{\|u\|_X, \|v\|_Y\}$.

Define $P = \{(u, v) \in X \times Y \mid u(t) \geq 0, v(t) \geq 0\}$, then the cone $P \subset X \times Y$.

From Lemma 2.5 in Section 2, we can obtain the following lemma.

Lemma 3.1. *Suppose that $f(t, v)$ and $g(t, u)$ are continuous, then $(u, v) \in X \times Y$ is a solution of BVP (1.1) if and only if $(u, v) \in X \times Y$ is a solution of the integral equations*

$$\begin{aligned} u(t) &= \int_0^1 G_1(t, s) f(s, v(s)) ds, \\ v(t) &= \int_0^1 G_2(t, s) g(s, u(s)) ds. \end{aligned} \quad (3.1)$$

Let $T : X \times Y \rightarrow X \times Y$ be the operator defined as

$$\begin{aligned} T(u, v)(t) &= \left(\int_0^1 G_1(t, s) f(s, v(s)) ds, \int_0^1 G_2(t, s) g(s, u(s)) ds \right) \\ &=: (T_1 v(t), T_2 u(t)), \end{aligned} \quad (3.2)$$

then by Lemma 3.1, the fixed point of operator T coincides with the solution of system (1.1).

Lemma 3.2. *Let $f(t, v)$ and $g(t, u)$ be continuous on $[0, 1] \times [0, \infty) \rightarrow [0, \infty)$, then $T : P \rightarrow P$ defined by (3.2) is completely continuous.*

Proof. Let $(u, v) \in P$, in view of nonnegativeness and continuity of functions $G(t, s)$, f , and g , we conclude that $T : P \rightarrow P$ is continuous.

Let $\Omega \in P$ be bounded, that is, there exists a positive constant $h > 0$ such that $\|(u, v)\| \leq h$ for all $(u, v) \in \Omega$.

Let

$$\begin{aligned} M &= \max\{|f(t, v(t))| + 1 : 0 \leq t \leq 1, 0 \leq v \leq h\}, \\ N &= \max\{|g(t, u(t))| + 1 : 0 \leq t \leq 1, 0 \leq u \leq h\}, \end{aligned} \quad (3.3)$$

then we have

$$\begin{aligned} |T_1 v(t)| &= \left| \int_0^1 G_1(t, s) f(s, v(s)) ds \right| \leq M \int_0^1 G_1(s, s) ds, \\ |T_2 u(t)| &= \left| \int_0^1 G_2(t, s) g(s, u(s)) ds \right| \leq N \int_0^1 G_2(s, s) ds. \end{aligned} \quad (3.4)$$

Hence, $\|T(u, v)\| \leq \max\{M \int_0^1 G_1(s, s) ds, N \int_0^1 G_2(s, s) ds\}$. $T(\Omega)$ is uniformly bounded.

Since $G_1(t, s)$ is continuous on $[0, 1] \times [0, 1]$, it is uniformly continuous on $[0, 1] \times [0, 1]$. Thus, for fixed $s \in [0, 1]$ and for any $\varepsilon > 0$, there exists a constant $\delta > 0$, such that any $t_1, t_2 \in [0, 1]$ and $|t_1 - t_2| < \delta$,

$$|G_1(t_1, s) - G_1(t_2, s)| < \varepsilon/M. \quad (3.5)$$

Then

$$|T_1(v)(t_2) - T_1(v)(t_1)| \leq M \int_0^1 |G_1(t_2, s) - G_1(t_1, s)| ds < \varepsilon. \quad (3.6)$$

Similarly,

$$|T_2(u)(t_2) - T_2(u)(t_1)| \leq N \int_0^1 |G_2(t_2, s) - G_2(t_1, s)| ds < \varepsilon. \quad (3.7)$$

For the Euclidean distance d on R^2 , we have that if $t_1, t_2 \in [0, 1]$ are such that $|t_2 - t_1| < \delta$, then

$$d(T(u, v)(t_2), T(u, v)(t_1)) = \sqrt{(T_1v(t_2) - T_1v(t_1))^2 + (T_2u(t_2) - T_2u(t_1))^2} < \sqrt{2}\varepsilon. \quad (3.8)$$

That is to say, $T(P)$ is equicontinuous. By the means of the Arzela-Ascoli theorem, we have $T : P \rightarrow P$ is completely continuous. The proof is completed. \square

Theorem 3.3. Assume that $f(t, v)$ and $g(t, u)$ are continuous on $[0, 1] \times [0, \infty) \rightarrow [0, \infty)$, and there exist two positive functions $m(t), n(t)$ that satisfy

$$(H_1) \quad |f(t, v_2) - f(t, v_1)| \leq m(t)|v_2 - v_1|, \text{ for } t \in [0, 1], v_1, v_2 \in [0, \infty),$$

$$(H_2) \quad |g(t, u_2) - g(t, u_1)| \leq n(t)|u_2 - u_1|, \text{ for } t \in [0, 1], u_1, u_2 \in [0, \infty).$$

Then system (1.1) has a unique positive solution if

$$\rho = \int_0^1 G_1(s, s)m(s)ds < 1, \quad \theta = \int_0^1 G_2(s, s)n(s)ds < 1. \quad (3.9)$$

Proof. For all $(u, v) \in P$, by the nonnegativeness of $G(t, s)$ and $f(t, v), g(t, u)$, we have $T(u, v)(t) \geq 0$. Hence, $T(P) \subseteq P$.

$$\begin{aligned} \|T_1v_2 - T_1v_1\| &= \max_{t \in [0, 1]} |T_1v_2 - T_1v_1| \\ &= \max_{t \in [0, 1]} \left| \int_0^1 G_1(t, s) [f(s, v_2(s)) - f(s, v_1(s))] ds \right| \\ &\leq \int_0^1 G_1(s, s)m(s)ds \|v_2 - v_1\| \\ &\leq \rho \|v_2 - v_1\|. \end{aligned} \quad (3.10)$$

Similarly,

$$\|T_2u_2 - T_2u_1\| \leq \theta \|u_2 - u_1\|. \quad (3.11)$$

We have,

$$\|T(u_2, v_2) - T(u_1, v_1)\| \leq \max(\rho, \theta) \|(u_2, v_2) - (u_1, v_1)\|. \quad (3.12)$$

From Lemma 3.2, T is completely continuous, by Banach fixed point theorem, the operator T has a unique fixed point in P , which is the unique positive solution of system (1.1). This completes the proof. \square

Theorem 3.4. Assume that $f(t, v)$ and $g(t, u)$ are continuous on $[0, 1] \times [0, \infty) \rightarrow [0, \infty)$ and satisfy

$$(H_3) |f(t, v(t))| \leq a_1(t) + a_2(t)|v(t)|,$$

$$(H_4) |g(t, u(t))| \leq b_1(t) + b_2(t)|u(t)|,$$

$$(H_5) A_1 = \int_0^1 G_1(s, s) a_2(s) ds < 1, \quad 0 < B_1 = \int_0^1 G_1(s, s) a_1(s) ds < \infty,$$

$$(H_6) A_2 = \int_0^1 G_2(s, s) b_2(s) ds < 1, \quad 0 < B_2 = \int_0^1 G_2(s, s) b_1(s) ds < \infty.$$

Then the system (1.1) has at least one positive solution (u, v) in

$$C = \left\{ (u, v) \in P \mid \|(u, v)\| < \min\left(\frac{B_1}{1 - A_1}, \frac{B_2}{1 - A_2}\right) \right\}. \quad (3.13)$$

Proof. Let $C = \{(u, v) \in X \times Y : \|(u, v)\| < r\}$ with $r = \min(B_1/(1 - A_1), B_2/(1 - A_2))$, define the operator $T : C \rightarrow P$ as (3.2).

Let $(u, v) \in C$, that is, $\|(u, v)\| < r$. Then

$$\begin{aligned} \|T_1 v\| &= \max_{t \in [0, 1]} \left| \int_0^1 G_1(t, s) f(s, v(s)) ds \right| \\ &\leq \int_0^1 G_1(s, s) (a_1(s) + a_2(s)|v(s)|) ds \\ &\leq \int_0^1 G_1(s, s) a_1(s) ds + \int_0^1 G_1(s, s) a_2(s) ds \|v\| \\ &= B_1 + A_1 \|v\| \leq r. \end{aligned} \quad (3.14)$$

Similarly, $\|T_2 u\| \leq r$, so $\|T(u, v)\| \leq r$, $T(u, v) \subseteq \bar{C}$. From Lemma 3.2 $T : \bar{C} \rightarrow \bar{C}$ is completely continuous.

Consider the eigenvalue problem

$$(u, v) = \lambda T(u, v), \quad \lambda \in (0, 1). \quad (3.15)$$

Under the assumption that (u, v) is a solution of (3.15) for a $\lambda \in (0, 1)$, one obtains

$$\begin{aligned}
 \|u\| &= \|\lambda T_1 v\| \\
 &= \lambda \max_{t \in [0,1]} \left| \int_0^1 G_1(t, s) f(s, v(s)) ds \right| \\
 &< \int_0^1 G_1(s, s) (a_1(s) + a_2(s)|v(s)|) ds \\
 &= \int_0^1 G_1(s, s) a_1(s) ds + \int_0^1 G_1(s, s) a_2(s) ds \|v\| \\
 &= B_1 + A_1 \|v\| \leq r.
 \end{aligned} \tag{3.16}$$

Similarly, $\|v\| = \|\lambda T_2 u\| < r$, so $\|(u, v)\| < r$, which shows that $(u, v) \notin \partial C$. By Lemma 2.4, T has a fixed point in \bar{C} . We complete the proof of Theorem 3.4. \square

Example 3.5. Consider the problem

$$\begin{aligned}
 D^{7/4} u(t) + f(t, v(t)) &= 0, \quad 0 < t < 1, \\
 D^{3/2} v(t) + g(t, u(t)) &= 0, \quad 0 < t < 1, \\
 u(0) = 0, \quad u(1) &= \frac{1}{2} u\left(\frac{1}{2}\right), \\
 v(0) = 0, \quad v(1) &= \frac{3}{4} v\left(\frac{1}{2}\right),
 \end{aligned} \tag{3.17}$$

where

$$f(t, v(t)) = \frac{tv(t)}{(1+t)(1+v(t))}, \quad g(t, u(t)) = \arctan \frac{t}{1+t} |\sin u(t)|. \tag{3.18}$$

Set $v_1(t), v_2(t), u_1(t), u_2(t) \in [0, \infty)$ and $t \in [0, 1]$, then we have

$$\begin{aligned}
 |f(t, v_2(t)) - f(t, v_1(t))| &\leq \frac{t}{1+t} |v_2(t) - v_1(t)|, \\
 |g(t, u_2(t)) - g(t, u_1(t))| &\leq \arctan \frac{t}{1+t} |u_2(t) - u_1(t)|.
 \end{aligned} \tag{3.19}$$

Therefore,

$$\begin{aligned}
 \rho &= \int_0^1 G_1(s, s)m(s)ds \leq \int_0^1 G_1(s, s)ds \\
 &= \frac{1}{\Gamma(7/4)\left(1 - (1/2)^{7/4}\right)} \left\{ \int_0^{1/2} [s(1-s)]^{3/4} ds - \int_0^{1/2} \frac{1}{2} \left[s\left(\frac{1}{2}-s\right) \right]^{3/4} ds + \int_{1/2}^1 [s(1-s)]^{3/4} ds \right\} \\
 &= \frac{2\left(1 + (1/2)^{7/4}\right)}{5} \cdot \frac{\Gamma(3/4)}{\Gamma(1/2)} < \frac{4}{5} < 1, \\
 \\
 \theta &= \int_0^1 G_2(s, s)n(s)ds \leq \frac{\pi}{4} \int_0^1 G_2(s, s)ds \\
 &= \frac{\pi}{4} \frac{1}{\Gamma(3/2)\left(1 - (3/4)(1/2)^{1/2}\right)} \left\{ \int_0^{1/2} [s(1-s)]^{1/2} ds - \int_0^{1/2} \frac{3}{4} \left[s\left(\frac{1}{2}-s\right) \right]^{1/2} ds + \int_{1/2}^1 [s(1-s)]^{1/2} ds \right\} \\
 &= \frac{\pi}{4} \frac{\left[1 - (3/4)(1/2)^2\right]}{\left[1 - (3/4)(1/2)^{1/2}\right]} \cdot \frac{\Gamma(3/2)}{\Gamma(3)} \\
 &\approx 0.6018 < 1.
 \end{aligned} \tag{3.20}$$

With the use of Theorem 3.3, BVP (3.17) has a unique positive solution.

Example 3.6. Consider the problem

$$\begin{aligned}
 D^{7/4}u(t) + f(t, v(t)) &= 0, \quad 0 < t < 1, \\
 D^{3/2}v(t) + g(t, u(t)) &= 0, \quad 0 < t < 1, \\
 u(0) = 0, \quad u(1) &= \frac{1}{2}u\left(\frac{1}{2}\right), \\
 v(0) = 0, \quad v(1) &= \frac{3}{4}v\left(\frac{1}{2}\right),
 \end{aligned} \tag{3.21}$$

where

$$f(t, v(t)) = t^2 + \frac{t}{1+t} \ln(1 + v(t)), \quad g(t, u(t)) = 10 + \frac{t^2}{20} + u(t). \tag{3.22}$$

We have

$$|f(t, v(t))| t^2 + \frac{t}{1+t} \cdot |v(t)|, \quad |g(t, u(t))| \leq \left(10 + \frac{t^2}{20}\right) + |u(t)|. \tag{3.23}$$

Hence,

$$\begin{aligned}
 A_1 &= \int_0^1 G_1(s, s) a_2(s) ds \leq \int_0^1 G_1(s, s) ds = \frac{2(1 + (1/2)^{7/4})}{5} \cdot \frac{\Gamma(3/4)}{\Gamma(1/2)} < 1, \\
 B_1 &= \int_0^1 G_1(s, s) a_1(s) ds = \int_0^1 G_1(s, s) \cdot s^2 ds < \infty, \\
 A_2 &= \int_0^1 G_2(s, s) b_2(s) ds = \int_0^1 G_2(s, s) ds \approx 0.7666 < 1, \\
 B_2 &= \int_0^1 G_2(s, s) b_1(s) ds = \int_0^1 G_2(s, s) \left(10 + \frac{s^2}{20} \right) ds < \infty.
 \end{aligned} \tag{3.24}$$

By Theorem 3.4, BVP (3.21) has at least one positive solution in

$$C = \left\{ (u, v) \in P \mid \|u, v\| < \min \left(\frac{B_1}{1 - A_1}, \frac{B_2}{1 - A_2} \right) \right\}. \tag{3.25}$$

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