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Research Article

Global Existence and Asymptotic Behavior of Solutions for a Class of Nonlinear Degenerate Wave Equations

Yaojun Ye

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This paper studies the existence of global solutions to the initial-boundary value problem for some nonlinear degenerate wave equations by means of compactness method and the potential well idea. Meanwhile, we investigate the decay estimate of the energy of the global solutions to this problem by using a difference inequality.

1. Introduction

We are concerned with the following nonlinear degenerate wave equation:

$$u_{tt} - \operatorname{div}\left(|Du|^{p-2}Du\right) + u_t = |u|^{m-2}u, \quad (x,t) \in \Omega \times [0,+\infty), \tag{1.1}$$

with the initial-boundary value conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega; \quad u(x,t)\big|_{\partial\Omega\times[0,+\infty)} = 0, \tag{1.2}$$

where Ω is a bounded open domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $m \geq 2$ is a nonnegative real number, and $\operatorname{div}(|Du|^{p-2}Du)$ is a divergence operator (degenerate Laplace operator) with $p > 2$ and $Du = (D_1u, D_2u, \ldots, D_nu)$, $D_i = \partial/\partial x_i$, $i = 1,2,\ldots,n$.

When $p = 2$, (1.1) is converted into the form

$$u_{tt} - \triangle u + u_t = |u|^{m-2}u. \tag{1.3}$$

The global existence, the decay property of weak solutions, and the blow up of solutions to the initial-boundary value problem for the semilinear wave equations related to (1.2)-(1.3), under suitable assumptive conditions, have been investigated by many people through various approaches [1–4]. However, little attention is paid to problem (1.1)-(1.2). Because the divergence operator $\operatorname{div}(|Du|^{p-2}Du)$ is a nonlinear operator, the reasonable proof and computation are greatly different from the Laplace operator; thus, the investigation of problem (1.1)-(1.2) becomes more complicated. In this paper, on the one hand, by a Galerkin approximation scheme [5], as well as combining it with the potential well method, we prove the global existence of solutions to problem (1.1)-(1.2). On the other hand, we obtain the asymptotic behavior of the global solutions to this problem by using a difference inequality.

For simplicity of notation, hereafter we denote by $\|\cdot\|_p$ the space $L^p(\Omega)$ norm. $\|\cdot\|$ denotes $L^2(\Omega)$ norm and we write equivalent norm $\|\nabla\cdot\|_p$ instead of $W_0^{1,p}(\Omega)$ norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$, C denotes various positive constants depending on the known constants and may be different at each appearance.

We define the functionals

$$J(u)=\frac{1}{p}\|\nabla u\|_p^p-\frac{1}{m}\|u\|_m^m,\quad K(u)=\|\nabla u\|_p^p-\|u\|_m^m,\quad u\in W_0^{1,p}(\Omega),\tag{1.4}$$

and according to [6] we put

$$d=\inf\left\{\sup_{\lambda\geq 0}J(\lambda u):u\in W_0^{1,p}(\Omega)/\{0\}\right\}.\tag{1.5}$$

Then, for problem (1.1)-(1.2) we are able to define the stable sets as follows:

$$W=\{u:u\in W_0^{1,p}(\Omega),\ K(u)>0,\ J(u)<d\}\cup\{0\}.\tag{1.6}$$

We denote the total energy related to (1.1) by

$$E(t)=E(u(t))=\frac{1}{2}\|u_t\|^2+\frac{1}{p}\|\nabla u\|_p^p-\frac{1}{m}\|u\|_m^m=\frac{1}{2}\|u_t\|^2+J(u(t)),\quad t\geq 0,\tag{1.7}$$

and $E(0)=(1/2)\|u_1\|^2+J(u_0)$ is the total energy of the initial data.

2. Some lemmas

We list up some useful lemmas here for the following discussion.

Lemma 2.1. *Let $a\geq 0$, $b\geq 0$ and $1/p+1/q=1$ for $1<p,q<+\infty$, then one has the inequality*

$$ab\leq\delta a^p+C(\delta)b^q,\tag{2.1}$$

where $\delta>0$ is an arbitrary constant and $C(\delta)>0$ is a positive constant depending on δ.

Lemma 2.2. *Let $u\in W_0^{1,p}(\Omega)$, then $u\in L^q(\Omega)$ and the inequality $\|u\|_q\leq C\|u\|_{W_0^{1,p}(\Omega)}$ holds, provided that (i) $1\leq q\leq np/(n-p)$ if $1<p<n$; (ii) $1\leq q<+\infty$ if $1\leq n\leq p$.*

Lemma 2.3. *Assume that $u\in W_0^{1,p}(\Omega)$ and (i) $p<m<np/(n-p)$ for $2<p<n$; (ii) $p<m<+\infty$ for $n\leq p$; then d is a positive real number.*

Proof. By Lemma 2.2, we have $\|u\|_m \leq C\|\nabla u\|_p$. Since

$$J(\lambda u) = \frac{\lambda^p}{p}\|\nabla u\|_p^p - \frac{\lambda^m}{m}\|u\|_m^m, \tag{2.2}$$

we get

$$\frac{d}{d\lambda}J(\lambda u) = \lambda^{p-1}\|\nabla u\|_p^p - \lambda^{m-1}\|u\|_m^m. \tag{2.3}$$

Let $(d/d\lambda)J(\lambda u) = 0$, which implies that

$$\lambda_1 = \left(\frac{\|\nabla u\|_p^p}{\|u\|_m^m}\right)^{1/(m-p)}. \tag{2.4}$$

An elementary calculation shows that

$$\left.\frac{d^2}{d\lambda^2}J(\lambda u)\right|_{\lambda=\lambda_1} < 0. \tag{2.5}$$

So, we have

$$\begin{aligned}\sup_{\lambda\geq 0}J(\lambda u) = J(\lambda_1 u) &= \frac{\lambda_1^p}{p}\|\nabla u\|_p^p - \frac{\lambda_1^m}{m}\|u\|_m^m \\ &= \frac{m-p}{mp}\left(\frac{\|\nabla u\|_p}{\|u\|_m}\right)^{mp/(m-p)} \geq \frac{m-p}{mp}C^{mp/(p-m)} > 0.\end{aligned} \tag{2.6}$$

Therefore

$$d = \inf\left\{\sup_{\lambda\geq 0}J(\lambda u) : u \in W_0^{1,p}(\Omega)/\{0\}\right\} > 0. \tag{2.7}$$

This completes the proof of Lemma 2.3. □

Lemma 2.4. *Provided that (i) $p < m < np/(n-p)$ for $2 < p < n$; (ii) $p < m < +\infty$ for $n \leq p$, then d is a finite real number and the set W is bounded in $W_0^{1,p}(\Omega)$.*

Proof. From the proof of Lemma 2.3 and the definition of d, we have for any $u \in W_0^{1,p}(\Omega)$ that

$$\begin{aligned}d \leq \sup_{\lambda\geq 0}J(\lambda u) = J(\lambda_1 u) &= \frac{\lambda_1^p}{p}\|\nabla u\|_p^p - \frac{\lambda_1^m}{m}\|u\|_m^m \\ &= \frac{m-p}{mp}\left(\frac{\|\nabla u\|_p}{\|u\|_m}\right)^{mp/(m-p)} = \frac{m-p}{mp}\lambda_1^p\|\nabla u\|_p^p < +\infty.\end{aligned} \tag{2.8}$$

So d is a finite real number.

Setting $u \in W$, then $\|\nabla u\|_p^p - \|u\|_m^m \geq 0$. Consequently,

$$d > J(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{m}\|u\|_m^m \geq \frac{m-p}{mp}\|\nabla u\|_p^p, \tag{2.9}$$

which yields

$$\|\nabla u\|_p^p \leq \frac{mp}{m-p} d < +\infty. \tag{2.10}$$

As a result, $u \in W_0^{1,p}(\Omega)$ and

$$W \subset \left\{u : u \in W_0^{1,p}(\Omega),\ \|\nabla u\|_p^p \leq \frac{mp}{m-p} d\right\}. \tag{2.11}$$

Thus the stable set W is bounded in $W_0^{1,p}(\Omega)$. □

Lemma 2.5 [7]. *Suppose that $\phi(t)$ is a nonincreasing nonnegative function on $[0,+\infty)$ and satisfies*

$$\phi(t)^{1+\alpha} \leq k\{\phi(t) - \phi(t+1)\} \tag{2.12}$$

for some constants $\alpha > 0$ and $k > 0$. Then $\phi(t)$ has the decay property

$$\phi(t) \leq \left\{\phi(0)^{-\alpha} + \alpha k^{-1}[t-1]^+\right\}^{-1/\alpha}, \quad t > 0, \tag{2.13}$$

where $[t-1]^+ = \max\{t-1, 0\}$.

Proof. Setting $\psi(t) = \phi(t)^{-\alpha}$, we see from (2.13) that

$$\begin{aligned} \psi(t+1) - \psi(t) &= \int_0^1 \frac{d}{d\theta}\{\theta\phi(t+1) + (1-\theta)\phi(t)\}^{-\alpha} d\theta \\ &= \alpha(\phi(t) - \phi(t+1)) \int_0^1 \{\theta\phi(t+1) + (1-\theta)\phi(t)\}^{-\alpha-1} d\theta \\ &\geq \alpha k^{-1}. \end{aligned} \tag{2.14}$$

Then we get

$$\psi(t) \geq \psi(0) + \alpha k^{-1} t \tag{2.15}$$

and the desired estimate (2.13). □

3. The global existence

Theorem 3.1. *Given that $p \leq m \leq np/(n-p)$, $p < n$, and $p < m < +\infty$, $n \leq p$, if $u_0 \in W$, $u_1 \in L^2(\Omega)$ and the initial data energy $E(0) < d$, then problem (1.1)-(1.2) admits a global solution $u(x,t)$ such that $u(x,t) \in W$ and*

$$u(x,t) \in L^\infty(0,T; W_0^{1,p}(\Omega)), \qquad u_t(x,t) \in L^\infty(0,T; L^2(\Omega)). \tag{3.1}$$

Proof. Let r be an integer for which $H_0^r(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$ is continuous. Then the eigenfunctions of $-\Delta^r \omega_j = \alpha_j \omega_j$ in $H_0^r(\Omega)$ yield a Gaerkin basis for both $H_0^r(\Omega) \subset W_0^{1,p}(\Omega)$ and $L^2(\Omega)$. We seek approximate solutions $u_N(t)$ to the problem (1.1)-(1.2) of the form

$$u_N(t) = \sum_{j=1}^{N} g_{jN}(t)\omega_j, \quad N = 1,2,\dots, \tag{3.2}$$

where the coefficients $g_{jN}(t)$ satisfy $g_{jN}(t) = (u_N(t), \omega_j)$ with

$$(u_N''(t), \omega_j) + \left(\operatorname{div}\left(|Du_N|^{p-2} Du_N\right), \omega_j\right) + (u_N'(t), \omega_j) = \left(|u_N|^{m-2} u_N, \omega_j\right), \tag{3.3}$$

$$u_N(0) = u_{0N}, \quad u_N'(0) = u_{1N}, \quad 1 \le j \le N. \tag{3.4}$$

Here $(u,v) = \int_\Omega u(x)v(x)dx$. Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ and $L^2(\Omega)$, we choose u_{0N}, $u_{1N} \in C_0^\infty(\Omega)$ such that $u_N(0) = u_{0N} \to u_0(x)$ in $W_0^{1,p}(\Omega)$ and $u_N'(0) = u_{1N} \to u_1(x)$ in $L^2(\Omega)$ as $N \to \infty$.

We observe that (3.3) is a system of ordinary differential equations in the variable t and has a local solution $u_N(t)$ in an interval $[0, t_m)$ by the existence theorem. In the next step, we obtain the a priori estimates for the solution $u_N(t)$ so that it can be extended to the whole interval $[0, T]$ according to the extension theorem.

Multiplying (3.3) by $g_{jN}'(t)$ and summing over j from 1 to N, and then integrating over $[0, t]$; we get

$$\frac{1}{2}\left\|u_N'(t)\right\|^2 + J(u_N(t)) + \int_0^t \left\|u_N'(\tau)\right\|^2 d\tau = \frac{1}{2}\left\|u_{1N}\right\|^2 + J(u_{0N}). \tag{3.5}$$

By using formula (3.5), we can obtain

$$u_N(t) \in W, \quad t \in [0, t_m). \tag{3.6}$$

In fact, suppose that (3.6) is false and let t_1 be the smallest time for $u_N(t_1) \notin W$. Then, by means of the continuity of $u_N(t)$, we see $u_N(t_1) \in \partial W$. From the definition of W and the continuity of $J(u(t))$ and $K(u(t))$ in t, we have either

$$J(u_N(t_1)) = d, \tag{3.7}$$

or

$$K(u_N(t_1)) = 0. \tag{3.8}$$

By (3.5) together with the condition $E(u(0)) < d$, we have

$$J(u_N(t_1)) \le \frac{1}{2}\left\|u_{1N}\right\|^2 + J(u_{0N}) = E(u_N(0)) < d. \tag{3.9}$$

So, case (3.7) is impossible.

Assume that (3.8) holds, then we obtain

$$\frac{d}{d\lambda}J(\lambda u_N(t_1)) = \lambda^{p-1}(1-\lambda^{m-p})\|\nabla u_N(t_1)\|_p^p. \tag{3.10}$$

Consequently,

$$\sup_{\lambda\geq 0} J(\lambda u_N(t_1)) = J(\lambda u_N(t_1))\,|_{\lambda=1} = J(u_N(t_1)) < d, \tag{3.11}$$

which contradicts the definition of d. Therefore, case (3.8) is impossible as well. Thus, we verify that $u_N(t) \in W$, $t \in [0, t_m)$.

From (3.5) and (3.6), we have

$$\frac{1}{2}\|u_N'\|^2 + \frac{m-p}{mp}\|\nabla u_N\|_p^p + \int_0^t \|u_N'(\tau)\|^2 d\tau \leq \frac{1}{2}\|u_{1N}\|^2 + d \leq C. \tag{3.12}$$

With this estimate, we can extend the approximate solutions $u_N(t)$ to the interval $[0,T]$ and we have

$$\{u_N\} \quad \text{is bounded in } L^\infty(0,T; W_0^{1,p}(\Omega)), \tag{3.13}$$

$$\{u_N'\} \quad \text{is bounded in } L^\infty(0,T; L^2(\Omega)), \tag{3.14}$$

$$\{u_N'\} \quad \text{is bounded in } L^2(0,T; L^2(\Omega)), \tag{3.15}$$

$$\operatorname{div}(|Du_N|^{p-2}Du_N) \quad \text{is bounded in } L^\infty(0,T; W_0^{-1,p/(p-1)}(\Omega)), \tag{3.16}$$

$$|u_N|^{m-2}u_N \quad \text{is bounded in } L^\infty(0,T; L^{m/(m-1)}(\Omega)). \tag{3.17}$$

Since our Galerkin basis was taken in the Hilbert space $H^r(\Omega) \subset W_0^{1,p}(\Omega)$, we can use the standard projection argument as described in [5]. Then from the approximate equation (3.3) and the estimates (3.13)–(3.17), we get

$$\{u_N''\} \quad \text{is bounded in } L^2(0,T; W^{-1,p/(p-1)}(\Omega)). \tag{3.18}$$

Now from (3.13)–(3.17) and the standard arguments of the approximate solutions, we conclude that after the extraction of suitable subsequence $\{u_\mu\}$ from $\{u_N\}$ if necessary, we have the following:

$$\{u_\mu\} \longrightarrow u \quad \text{weakly star in } L^\infty(0,T; W_0^{1,p}(\Omega)), \tag{3.19}$$

$$\{u_\mu'\} \longrightarrow u' \quad \text{weakly star in } L^\infty(0,T; L^2(\Omega)), \tag{3.20}$$

$$\{u_\mu'\} \longrightarrow u' \quad \text{weakly in } L^2(0,T; L^2(\Omega)), \tag{3.21}$$

$$\operatorname{div}(|Du_\mu|^{p-2}Du_\mu) \longrightarrow \chi_1 \quad \text{weakly star in } L^\infty(0,T; W_0^{-1,p/(p-1)}(\Omega)), \tag{3.22}$$

$$|u_\mu|^{m-2}u_\mu \longrightarrow \chi_2 \quad \text{weakly star in } L^\infty(0,T; L^{m/(m-1)}(\Omega)). \tag{3.23}$$

By applying the Lions-Aubin compactness lemma [5], we get that from (3.13) and (3.14),

$$\{u_\mu\} \longrightarrow u \quad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{3.24}$$

We receive that from (3.15) and (3.18)

$$\{u'_\mu\} \longrightarrow u' \quad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{3.25}$$

Using (3.13) and (3.24), we see that

$$\int_0^T \int_\Omega \left| \left| u_\mu \right|^{m-2} u_\mu \right|^{m/(m-1)} dx\,dt = \int_0^T ||u_\mu||_m^m dt \le C \int_0^T ||u_\mu||_{W_0^{1,p}}^m dt \le C, \tag{3.26}$$

and $|u_\mu|^{m-2}u_\mu \to |u|^{m-2}u$ almost everywhere in $(0,T)\times\Omega$. Therefore from [5, Lemma 1.3], we infer that

$$\left| u_\mu \right|^{m-2} u_\mu \longrightarrow |u|^{m-2}u \quad \text{weakly in } L^{m/(m-1)}(0,T;L^{m/(m-1)}(\Omega)). \tag{3.27}$$

We have from (3.23) and (3.27) that $\chi_2 = |u|^{m-2}u$. Finally, since we have the strong convergence (3.25), we can use a standard monotonicity argument as done by Lions in [5] or by Ye in [8] to show that $\chi_1 = \operatorname{div}(|Du|^{p-2}Du)$.

Multiplying both sides of (3.3) by $g(t) \in C^2[0,T]$ and letting $\mu = N \to \infty$, we get that $u(x,t)$ is a global solution of problem (1.1)-(1.2). This ends the proof of Theorem 3.1. □

4. The asymptotic behavior

Theorem 4.1. *Under the hypotheses of Theorem 3.1, the global solution $u(x,t)$ in W of problem (1.1)-(1.2) on $[0,+\infty)$ has the following decay property:*

$$E(t) \le E(0)\left\{1 + CE(0)I_0^{-2}[t-1]^+\right\}^{-1}, \quad t \in (0,+\infty), \tag{4.1}$$

where I_0 is some positive constant depending only on u_0 and u_1.

Proof. Multiplying (1.1) by u_t and integrating over $[t,t+1]\times\Omega$, $t>0$, we have

$$\int_t^{t+1} ||u_t(s)||^2 ds = E(t) - E(t+1) \equiv D(t)^2. \tag{4.2}$$

Thus, there exist $t_1 \in [t,t+1/4]$, $t_2 \in [t+3/4,t+1]$ such that

$$||u_t(t_i)|| \le 2D(t), \quad i = 1,2. \tag{4.3}$$

On the other hand, we multiply (1.1) by $u(t,x)$ and integrate over $[t_1,t_2]\times\Omega$, which yields

$$\begin{aligned}\int_{t_1}^{t_2} K(u(s))\,ds &= \int_{t_1}^{t_2} \|u_t(s)\|^2 ds \\ &\quad + (u_t(t_1),u(t_1)) - (u_t(t_2),u(t_2)) - \int_{t_1}^{t_2} (u_t(s),u(s))\,ds \\ &\le D(t)^2 + 5D(t)\sup_{t\le s\le t+1}\|u(s)\|.\end{aligned} \tag{4.4}$$

To estimate $\|u(t)\|$, we multiply (1.1) by $u(t,x)$ and integrate over $[0,t]\times\Omega$ to obtain

$$\frac{1}{2}\|u(t)\|^2 + \int_0^t K(u(s))\,ds = \frac{1}{2}\|u_0\|^2 + (u_1,u_0) + \int_0^t \|u_t(s)\|^2 ds - (u_t(t),u(t)). \tag{4.5}$$

Since $K(u(t))>0$, we derive that from Lemma 2.1,

$$\begin{aligned}\|u(t)\|^2 &\le \|u_0\|^2 + 2(u_1,u_0) + 2\|u\|\|u_t\| + 2\int_0^t \|u_t(s)\|^2 ds \\ &\le \|u_0\|^2 + 2(u_1,u_0) + 2\|u_t(t)\|^2 + \frac{1}{2}\|u(t)\|^2 + 2\int_0^t \|u_t(s)\|^2 ds.\end{aligned} \tag{4.6}$$

From (4.2), we get

$$\int_0^t \|u_t(s)\|^2 ds = E(0) - E(t) < E(0), \tag{4.7}$$

and hence we have from (4.6) that

$$\|u(t)\|^2 \le 2\{\|u_0\|^2 + 2(u_1,u_0) + 6E(0)\} \equiv I_0^2. \tag{4.8}$$

It follows from (4.4) and (4.8) that

$$\int_{t_1}^{t_2} K(u(s))\,ds \le D(t)^2 + 5I_0D(t). \tag{4.9}$$

Now, it follows from (4.2) and (4.9) that

$$\begin{aligned}E(t_2) &\le 2\int_{t_1}^{t_2} E(s)\,ds \le C\{D(t)^2 + I_0D(t)\}, \\ E(t_1) &= E(t_2) + \int_{t_1}^{t_2} \|u_t(s)\|^2 ds \le E(t_2) + \int_t^{t+1} \|u_t(s)\|^2 ds \\ &\le C\{D(t)^2 + I_0D(t)\} + D(t)^2 \le CI_0D(t),\end{aligned} \tag{4.10}$$

which implies by (4.2) that

$$\sup_{t\le s\le t+1} E(s)^2 \le CI_0^2D(t)^2 = CI_0^2\{E(t) - E(t+1)\}. \tag{4.11}$$

Thus, applying Lemma 2.5 to (4.11) and using the fact that $E(t) \le E(0) < d$, we derive the decay estimate

$$E(t) \le \{E(0)^{-1} + CI_0^{-2}[t-1]^+\}^{-1} = E(0)\{1 + CE(0)I_0^{-2}[t-1]^+\}^{-1}. \tag{4.12}$$

This completes the proof of Theorem 4.1. □

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Yaojun Ye: Department of Information and Computational Science, Henan Agricultural University, Zhengzhou 450002, China
Email address: yeyaojun2002@yahoo.com.cn