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Discretized Representations of Harmonic Variables by Bilateral Jacobi Operators

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Starting from a discrete Heisenberg algebra we solve several representation problems for a discretized quantum oscillator in a weighted sequence space. The Schrödinger operator for a discrete harmonic oscillator is derived. The representation problem for a q-oscillator algebra is studied in detail. The main result of the article is the fact that the energy representation for the discretized momentum operator can be interpreted as follows: It allows to calculate quantum properties of a large number of non-interacting harmonic oscillators at the same time. The results can be directly related to current research on squeezed laser states in quantum optics. They reveal and confirm the observation that discrete versions of continuum Schrödinger operators allow more structural freedom than their continuum analogs do.

Keywords: Schrödinger difference operators, 9-special functions

1 INTRODUCTION

During the last decades, lattice quantum field theories have shown up and solved a lot of interesting problems in physics. One basic effect is the regularizing influence of lattices that are artificially introduced into the theory. The quite complex lattice field theories often go back to original attempts in formulating quantum mechanics in a discretized phase space. A long time before quantum mechanics, Riemann was certainly one of the first mathematicians who thought about irregularities in the classical space, see [4]. But also in modern times the principal assumption that one should be aware of irregularities in the structure of space is broadly accepted. Let us mention in particular the work of Ord on fractal space time, see [13]. It can be regarded as one of the pioneer articles on research of noncontinuous space–time structures. Extending A. Einstein's theory with respect to quantum gravity is a further example for research on sophisticated space–time structures, see for example the contributions by Drechsler and Tuckey as well as by Breitenlohner *et al.* or Kleinert, compare [16–18]. There are also the celebrated approaches via string theory and duality by Seiberg and Witten [21,22], in addition by Sonnenschein *et al.*, Louis and Förger [23,24]. They altogether reveal the rich and deep

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character of non-homogeneous space and space–time structures.

Our approach to mathematical quantum models in irregular space, i.e. in lattice space structures, shall be given by solutions to discretized representation problems in lattice quantum mechanics. In this context, we will have to define first what the lattice shall be and then to discretize the involved operators. One knows that the lattices used by numerical approximations always must be finite. But one is also automatically confronted with the situation of an infinitely extended space axis in the case of a one-dimensional Schrödinger equation. Also in higher dimensions, this principal problem is still present. This may be one starting point for thinking of an infinite discretization of the space axis in order to obtain regularizing effects on the one hand but also to deal with the situation of an infinite position axis that is required within the Schrödinger theory on the other hand. Additionally, there arises a further problem: When dealing with a discretized space axis, what shall be the Fourier transform into momentum space? And having introduced a discrete space variable, does this automatically imply that momentum is also discretized?

To give a satisfactory answer to the stated questions one has to develop a consistent mathematical model for the discretization of the phase space that is free of some arbitrary input. The question however is how to find such a model.

Inspired by the research on quantum groups, J. Wess suggested a deformation of the conventional Heisenberg algebra in 1991, see [2,7], where the deformation itself shall cause a discretization of the phase space. We will refer to the discretized Heisenberg algebra as the q-Heisenberg algebra throughout this work. The basic idea when introducing the q-Heisenberg algebra is to deform the conventional algebra by a real number $q>1$ that shall play the role of a lattice parameter. Actually it turns out that the chosen q-deformation of the Heisenberg algebra leads to a discretization of both, the deformed momentum and the deformed space operator. This formalism fits into the more general framework of quantization by noncommutative structures, outlined for example in the work by Connes and Jaffe, see [19,20].

The organization of this article shall be as follows: Our aim is first to revise the relations of the q-Heisenberg algebra and of the Biedenharn–MacFarlane q-oscillator algebra. In the second section we introduce the q-discrete harmonic oscillator. A representation theorem for a deformed oscillator algebra will be stated in Section 3.

The main result of this article shall be presented in Section 4: There, we will investigate the action of the discretized momentum operator on the eigenfunctions of the q-deformed oscillator. We obtain the fact that the situation totally differs from the conventional continuum analog in quantum mechanics. The main difference can be interpreted as follows: It allows to calculate quantum properties of a large number of non-interacting harmonic oscillators at the same time. This can be directly related to research on squeezed laser states in quantum optics, where a great advance has been provided by a recent article of Penson and Solomon, see [14]. Prosecuting the research into this direction, new advances – not only with respect to mathematics and theoretical physics – but also with respect to technological needs in modern society can be expected.

To provide the basics for this article, we first cite several results from [9,11,12,15]. The relations of the q-Heisenberg algebra are given by

$$p\xi - q\xi p = -iq^{3/2}u, \tag{1.1}$$

$$\xi p - qp\xi = iq^{3/2}u^{-1}, \tag{1.2}$$

$$up = qpu \quad \xi u = qu\xi \qquad q>1 \tag{1.3}$$

and we refer to q-oscillator relations (i.e. deformed oscillator relations) of the type

$$aa^{+} - q^{-2}a^{+}a = 1, \tag{1.4}$$

which were already known to Heisenberg, see [5]. As pointed out in several publications [3,10–12], the cited q-Heisenberg algebra is a discretized analog of the quantum mechanical Heisenberg algebra

$$p_x x - x p_x = -i, \tag{1.5}$$

whereas the q-oscillator algebra is a q-deformed analog of the quantum oscillator algebra

$$aa^+ - a^+ a = 1. \tag{1.6}$$

In spite of the fact that the operators of the quantum oscillator algebra (1.6) can be easily expressed in terms of the Heisenberg variables (1.5), namely

$$a := \frac{1}{\sqrt{2}}(p_x - ix) \quad a^+ := \frac{1}{\sqrt{2}}(p_x + ix), \tag{1.7}$$

the same task in the q-case, i.e. finding

$$a = a(p,\xi,u), \quad a^+ = a^+(p,\xi,u), \quad \text{such that } aa^+ - q^{-2}a^+a = 1 \tag{1.8}$$

is highly non-trivial as there is no simple linear transform that tells us to do so. We want to focus on this problem in the next section and find suitable tools that allow us to classify at least one family of solutions to the problem (1.8); – analytically, we have to add more structure. We want to represent the variables p, ξ, u, a, a^+ as operators in a suitable Hilbert space and refer to a Hilbert space H which is fixed by the scalar product $(*,*)$ of the orthogonal basis vectors $e_n^\sigma, e_m^\tau, m, n \in \mathbb{Z}, \sigma, \tau \in \{+1,-1\}$

$$(e_n^\sigma, e_m^\tau) = (q-1)^{-(1/2)} \frac{1}{q^n} \delta_{\sigma\tau}\delta_{mn}. \tag{1.9}$$

The Hilbert space itself is canonically spanned by the vectors e_n^σ, i.e. it is a weighted sequence space

$$H = \left\{ f = (q-1) \sum_{n=-\infty}^{\infty} \sum_\sigma q^n c_{\sigma n} e_n^\sigma \,|\, (f,f) = (q-1) \sum_{n=-\infty}^{\infty} \sum_\sigma q^n |c_{\sigma n}|^2 < \infty \right\}. \tag{1.10}$$

The momentum operator p in the relations above can be chosen as diagonal in the e_n^σ-basis [9] and, according to the q-Heisenberg relations, shows an exponential spectrum

$$pe_n^\sigma = \sigma q^n e_n^\sigma. \tag{1.11}$$

We will also make use of the Hilbert subspaces H_+, H_-:

$$H_\sigma := \left\{ f \in H \,|\, f = \sum_{n=-\infty}^{\infty} q^n c_{\sigma,n} e_n^\sigma \right\} \quad \sigma \in \{+1,-1\}. \tag{1.12}$$

The actions of the formally symmetric operator ξ, densely defined in H, and the unitary operators $u, u^+ = u^{-1}$ are respectively given on the e_n^σ by

$$\xi e_n^\sigma = i\sigma \frac{1}{1-(1/q^2)} \frac{1}{q^n}(e_{n-1}^\sigma - e_{n+1}^\sigma), \tag{1.13}$$

$$ue_n^\sigma = q^{-1/2} e_{n-1}^\sigma \quad u^{-1} e_n^\sigma = q^{+1/2} e_{n+1}^\sigma. \tag{1.14}$$

Note finally that the definition ranges $D(\xi), D(p)$, $D(u)$ and $D(u^{-1})$ of the operators ξ, p, u, u^{-1} contain invariant subspaces

$$\xi(D(\xi) \cap H_\sigma) \subseteq H_\sigma, \quad p(D(p) \cap H_\sigma) \subseteq H_\sigma \quad \sigma \in \{+1, -1\} \tag{1.15}$$

$$u(D(u) \cap H_\sigma) \subseteq H_\sigma, \quad u^{-1}(D(u^{-1}) \cap H_\sigma) \subseteq H_\sigma \quad \sigma \in \{+1, -1\}. \tag{1.16}$$

For the well-defined continuum limit $q \to 1$, see [2,10].

2 DISCRETE HARMONIC OSCILLATORS

We first revise several results which were stated by the author in [15].

To find a suitable approach to the stated problem (1.8), we address to the following situation in the continuum case:

The operator

$$a^+a = (p_x + ix)(p_x - ix)$$

and the classical Heisenberg algebra $p_x x - x p_x = -i$ imply the relations

$$a\psi_0 = 0 \quad \text{with scalar product} \quad (\psi_0, \psi_0)_{\mathcal{L}^2(\mathbb{R})} < \infty \quad \text{and} \tag{2.1}$$

$$a^+ a a^+ \psi_0 = 2a^+\psi_0, \tag{2.2}$$

$(*, *)_{\mathcal{L}^2(\mathbb{R})}$ denoting the standard scalar product in $\mathcal{L}^2(\mathbb{R})$.

In the following we will address to operators a^+a which fulfill relations of type (2.1) and (2.2) with respect to the scalar product $(*, *)_H$ in the Hilbert space H.

Generalizing the ansatz (2.1) and (2.2) we thus want to know how an operator a^+a must look like in terms of the q-Heisenberg variables p, ξ, u in order to generalize relations (2.1) and (2.2). We start from the ansatz

$$a := u^2 h(p) - iu\xi =: u^2 h - iu\xi, \tag{2.3}$$

$$a^+ := h(p)u^{-2} + i\xi u^{-1}, \tag{2.4}$$

where

$$(ae_n^\sigma, e_m^\tau) = (e_n^\sigma, a^+ e_m^\tau) \quad \sigma, \tau \in \{+1, -1\} \tag{2.5}$$

and

$$h(p)e_n^\sigma := h(\sigma q^n)e_n^\sigma, \quad \sigma \in \{+1, -1\} \tag{2.6}$$

with real valued $h(\sigma q^n)$. Moreover we give the following:

DEFINITION a^+a, a^+, a, p *and* ξ *shall be called harmonic variables. In detail*, a^+a *shall be called discrete harmonic oscillators.*

Note that the action of the harmonic variables on the basis vectors $e_n^\sigma, n \in \mathbb{Z}$ reveals the fact that they are bilateral Jacobi operators. For basic facts on bilateral Jacobi operators, see for example [25], for basic facts on monolateral Jacobi operators, compare for instance [8].

We allow a and a^+ to have maximal definition ranges, $D_{\max}(a)$ resp. $D_{\max}(a^+)$ in H. These definition ranges are given as usual by

$$D_{\max}(a) := \{\varphi \in H \,|\, (a\varphi, a\varphi) < \infty\},$$
$$D_{\max}(a^+) := \{\varphi \in H \,|\, (a^+\varphi, a^+\varphi) < \infty\}.$$

Let us continue with the action of the operator a^+ on the basis vectors e_n^+ of H_+. It is given by the following bilateral Jacobi operator:

$$\begin{aligned} a^+ e_n^+ &= (hu^{-2} + i\xi u^{-1})e_n^+ \\ &= qh(q^{n+2})e_{n+2}^+ + q^{-1}u^{-1}i\xi e_n^+ \\ &= \alpha_{n+2}e_{n+2}^+ + \beta_n e_n^+, \end{aligned} \tag{2.7}$$

while we make use of the following abbreviations:

$$\begin{aligned} \alpha_{n+2} &:= qh(q^{n+2}) + q^{-n-(1/2)}(1 - q^{-2})^{-1} \quad \text{and} \\ \beta_n &:= -(1 - q^{-2})^{-1} q^{-n-(1/2)}. \end{aligned} \tag{2.8}$$

Similarly, we obtain a as a bilateral Jacobi operator via

$$\begin{aligned} ae_n^+ &= (u^2 h - iu\xi)e_n^+ \\ &= q^{-2}\alpha_n e_{n-2}^+ + \beta_n e_n^+ \end{aligned} \tag{2.9}$$

and thus

$$\begin{aligned} aa^+ e_n^+ &= (q^{-2}\alpha_{n+2}^2 + \beta_n^2)e_n^+ + \alpha_{n+2}\beta_{n+2}e_{n+2}^+ \\ &\quad + q^{-2}\alpha_n\beta_n e_{n-2}^+. \end{aligned} \tag{2.10}$$

Note the following invariance properties for the single operators a, a^+ and their composites

$$a(D(a) \cap H_+) \subseteq H_+, \tag{2.11}$$
$$a^+(D(a^+) \cap H_+) \subseteq H_+, \tag{2.12}$$
$$a^+a(D(a^+a) \cap H_+) \subseteq H_+, \tag{2.13}$$
$$aa^+(D(aa^+) \cap H_+) \subseteq H_+. \tag{2.14}$$

The same holds analogously for the Hilbert space

$$H_- := \left\{ f \in H \,|\, f = \sum_{n=-\infty}^{\infty} q^n c_n e_n^{-1} \right\}. \tag{2.15}$$

Equations (2.11)–(2.14) imply that a, a^+, a^+a, aa^+ have invariant subspaces. Without loss of generality we thus can restrict first to discrete harmonic oscillators in one of the Hilbert spaces H_+, H_-. To do so, we consider now the following two equations which generalize (2.1) and (2.2):

$$aa^+ \sum_{n=-\infty}^{\infty} c_n e_n^+ = 2 \sum_{n=-\infty}^{\infty} c_n e_n^+, \tag{2.16}$$

$$a\psi_0 = a \sum_{n=-\infty}^{\infty} c_n e_n^+ = 0. \tag{2.17}$$

Suppose that they can be simultaneously solved in the Hilbert space H_+. Then it follows that $\sum_{n=-\infty}^{\infty} c_n e_n^+$ is a further eigenvector of a^+a in H_+. This easily can be seen: From the left-hand side of (2.16) we conclude that $a^+ \sum_{n=-\infty}^{\infty} c_n e_n^+ \in D(a) \subseteq H_+$, thus

$$(a^+a)a^+ \sum_{n=-\infty}^{\infty} c_n e_n^+ = 2a^+ \sum_{n=-\infty}^{\infty} c_n e_n^+. \tag{2.18}$$

Therefore, a common solution $\sum_{n=-\infty}^{\infty} c_n e_n^+$ of (2.16) and (2.17) immediately implies the existence of a discrete harmonic oscillator a^+a.

Next we want to decide explicitly for which functions h in (2.3) and (2.4) the operator a^+a can indeed become a discrete harmonic oscillator.

LEMMA *There exists a necessary condition on h such that $a = u^2h - iu\xi$ and $a^+ = hu^{-2} + i\xi u^{-1}$ constitute a discrete harmonic oscillator a^+a in H_+, namely*

$$\begin{aligned}(qh(q^n) + (1 - q^{-2})^{-1} q^{-n+(3/2)})^2 \\ = 2q^4(q^2-1)^{-1} + (1-q^{-2})^{-2} q^{-2n+3} \\ + (\gamma_1 \delta_{(-1)^n,1} + \gamma_2 \delta_{(-1)^n,-1}) q^{-n}\end{aligned} \tag{2.19}$$

for all $n \in \mathbb{Z}$ with suitable and fixed parameters $\gamma_1, \gamma_2 \in \mathbb{R}$.

Proof On the one hand, the eigenvalue problem

$$aa^+ \sum_{n=-\infty}^{\infty} c_n e_n^+ = 2 \sum_{n=-\infty}^{\infty} c_n e_n^+ \tag{2.20}$$

leads to the following recursion relation between the coefficients c_n:

$$\begin{aligned}c_n(q^{-2}\alpha_{n+2}^2 + \beta_n^2 - 2) + q^{-2}\alpha_{n+2}\beta_{n+2}c_{n+2} \\ + \alpha_n\beta_n c_{n-2} = 0.\end{aligned} \tag{2.21}$$

On the other hand, the equation

$$a\psi_0 = a \sum_{n=-\infty}^{\infty} c_n e_n^+ = 0 \tag{2.22}$$

yields a second recursion relation

$$c_{n-2} = c_n(q^{-(5/2)}(1-q^{-2})q^n h(q^n) + q^{-2}). \tag{2.23}$$

One easily verifies that

$$\alpha_n = (q^{n-(1/2)}(1-q^{-2})h(q^n) + 1)(-\beta_{n-2}), \tag{2.24}$$

see (2.8). Inserting (2.24) into (2.23), we obtain

$$q^4 \beta_n c_{n-2} = -c_n \alpha_n. \tag{2.25}$$

This last equation now serves to eliminate one of the coefficients in (2.21). The result is

$$\begin{aligned}c_n(q^{-2}\alpha_{n+2}^2 + \beta_n^2 - 2 - q^{-4}\alpha_n^2) \\ + q^{-2}\alpha_{n+2}\beta_{n+2}c_{n+2} = 0.\end{aligned} \tag{2.26}$$

Next we substitute $n \to n+2$ in (2.25) and insert (2.25) into relation (2.26). Thus we receive

$$q^{-2}\alpha_{n+2}^2 + \beta_n^2 - 2 - q^{-4}\alpha_n^2 = q^2\beta_{n+2}^2 \quad \text{or} \tag{2.27}$$

$$q^2\alpha_{n+2}^2 - \alpha_n^2 = (q^2 - q^4)\beta_n^2 + 2q^4. \tag{2.28}$$

The last equation determines the numbers α_n and hence also $h(q^n)$ (because of (2.8)). One solution of (2.28) is

$$\alpha_n^2 = (1-q^{-2})^{-2} q^{3-2n} + 2(q^2-1)^{-1} q^4. \tag{2.29}$$

Our next aim is to find all solutions of (2.28). Thus let a_n be any other solution of (2.28). We then receive

$$q^2(a_{n+2}^2 - \alpha_{n+2}^2) - (a_n^2 - \alpha_n^2) = 0. \tag{2.30}$$

With the abbreviation $\delta_n := q^n(a_n^2 - \alpha_n^2)$ finally follows $\delta_n = \text{const.}$ because of $q^n \neq 0$. We therefore obtain the general solution of (2.28) by adding any real number times q^{-n} to the special solution α_n^2. This however requires two restrictions. First we have to distinguish between even and odd n: The general solution of (2.28) then reads

$$\alpha_n^2 = (1-q^{-2})^{-2}q^{3-2n} + 2(q^2-1)^{-1}q^4 + (\gamma_1\delta_{(-1)^n,1} + \gamma_2\delta_{(-1)^n,-1})q^{-n} \tag{2.31}$$

Secondly, the constants γ_1 and γ_2 must be chosen in a way such that indeed $\alpha_n^2 > 0$ for all $n \in \mathbb{Z}$. We will consider this choice in more detail in the next section.

As α_n^2 contains all information about h via (2.8), Eq. (2.31) yields a necessary criterion for the existence of discrete harmonic oscillators a^+a in the Hilbert space H_+. This completes the proof of the Lemma.

It is not yet clear whether the criterion (2.31) already implies the existence of a solution for (2.16) and (2.17) in the Hilbert space H_+, i.e. whether

$$\psi_0 = \sum_{n=-\infty}^{\infty} c_n e_n^+ \in H_+. \tag{2.32}$$

In the sequel we will give an answer to this question by Theorem 1.

3 A REPRESENTATION THEOREM

By direct calculations one verifies that the solutions of Eq. (2.28) lead to the following action of the operators aa^+ and a^+a on the basis vectors e_n^+:

$$aa^+e_n^+ = (q^{-2}\alpha_{n+2}^2 + \beta_n^2)e_n^+ + \alpha_{n+2}\beta_{n+2}e_{n+2}^+ + q^{-2}\alpha_n\beta_n e_{n-2}^+, \tag{3.1}$$

$$a^+ae_n^+ = (q^{-2}\alpha_n^2 + \beta_n^2)e_n^+ + \alpha_{n+2}\beta_n e_{n+2}^+ + q^{-2}\alpha_n\beta_{n-2}e_{n-2}^+. \tag{3.2}$$

With the help of (2.8) we see that the operator $C := aa^+ - q^{-2}a^+a$ is diagonal in the basis $\{e_n^+ \mid n \in \mathbb{Z}\}$ of the p-eigenvectors in H_+, namely

$$(aa^+ - q^{-2}a^+a)e_n^+ = (q^{-4}(q^2\alpha_{n+2}^2 - \alpha_n^2) + (1-q^{-2})\beta_n^2)e_n^+. \tag{3.3}$$

Comparing with (2.26) yields the well known q-oscillator relations (see also [1,5,6])

$$\begin{aligned}(aa^+ - q^{-2}a^+a)e_n^+ &= 2e_n^+\\ \Leftrightarrow (bb^+ - q^{-2}b^+b)e_n^+ &= e_n^+\\ \text{where } a &= \sqrt{2}b\end{aligned} \tag{3.4}$$

Vice versa: Let us consider the ansatz

$$a = (u^2h - iu\xi) \quad a^+ = (hu^{-2} + i\xi u^{-1}) \tag{3.5}$$

with real valued h such that a, a^+ fulfill the relations (3.4): The choice of h now defines the coefficients $\alpha_n := qh(q^n) + q^{-n-(5/2)}(1-q^{-2})^{-1}$, as described above, and they fulfill relation (2.28). This can be checked by calculating them explicitly. These observations lead us to the following:

THEOREM 1 *Let $a = (u^2h - iu\xi)$ and $a^+ = (hu^{-2} + i\xi u^{-1})$ be defined as in (2.3), (2.4), (2.7) and (2.9) with a real valued function h on the lattice $\{q^n \mid n \in \mathbb{Z}\}$. Under these assertions the operators a and a^+ satisfy the commutation relations*

$$(aa^+ - q^{-2}a^+a)e_n^+ = 2e_n^+, \tag{3.6}$$

if and only if there are real constants γ_1, γ_2, such that for all $n \in \mathbb{Z}$:

$$(qh(q^n) + (1-q^{-2})^{-1}q^{-n+(3/2)})^2 = 2q^4(q^2-1)^{-1} + (1-q^{-2})^{-2}q^{-2n+3} + (\gamma_1\delta_{(-1)^n,1} + \gamma_2\delta_{(-1)^n,-1})q^{-n}. \tag{3.7}$$

Among all these (γ_1, γ_2) exists at least one couple for which the kernel of a is non-trivial and in H_+, namely $(\gamma_1, \gamma_2) = (0, 0)$.

We are going to verify the last statement of Theorem 1 during the next steps; – up to now we have solved the original representation problem (1.8) for a two parameter family of operators a, a^+

in the Hilbert space H_+ with γ_1, γ_2 being the parameters. We can also extend our result to the whole Hilbert space $H = H_+ \oplus H_-$ because of the following observation:

Introducing a parity like operator $\prod$ by

$$\prod e_n^\sigma := e_n^{-\sigma}, \tag{3.8}$$

where $n \in \mathbb{Z}$, $\sigma \in \{+1, -1\}$ we see that $\prod$ obviously converts H_+ into H_- and vice versa. If $(\gamma_1, \gamma_2) = (0, 0)$ then a commutes with $\prod$ in the following sense:

$$\prod a_{(\gamma_1,\gamma_2)} e_n^\sigma = a_{(\gamma_1,\gamma_2)} \prod e_n^\sigma. \tag{3.9}$$

If $(\gamma_1, \gamma_2) = (0, 0)$ the stated equation

$$\begin{aligned}\forall n \in \mathbb{Z}\colon\ & (qh(q^n) + (1 - q^{-2})^{-1} q^{-n+(3/2)})^2 \\ &= 2q^4(q^2 - 1)^{-1} + (1 - q^{-2})^{-2} q^{-2n+3} \\ &\quad + (\gamma_1 \delta_{(-1)^n,1} + \gamma_2 \delta_{(-1)^n,-1}) q^{-n}\end{aligned} \tag{3.10}$$

provides two solutions for $h(q^n)$. One of them yields the proper continuum limit to quantum mechanics by sending $q \to 1$. It is given by

$$h(q^n) = -h(-q^n) = \frac{\sqrt{1 + 2q^{2n-1}(1 - (1/q^2))} - 1}{q^{n-(1/2)}(1 - (1/q^2))}, \tag{3.11}$$

where the equality $h(q^n) = -h(-q^n)$ is implied by (3.8) and (3.9). As a result, we have explicit formulas for a and a^+

$$a = u^2 \frac{\sqrt{1 + 2q^{-1}(1 - q^{-2})p^2} - 1}{q^{-(1/2)}(1 - q^{-2})p} - iu\xi \tag{3.12}$$

and

$$a^+ = \frac{\sqrt{1 + 2q^{-1}(1 - q^{-2})p^2} - 1}{q^{-(1/2)}(1 - q^{-2})p} u^{-2} + i\xi u^{-1}. \tag{3.13}$$

a and a^+ solve the representation problem (1.8) on the common maximal domain $M := D_{\max}(a) \cap D_{\max}(a^+)$, $M \subseteq H$.

By direct calculation one recognizes that a vector $\psi_0 \in H = H_+ \oplus H_-$ which satisfies

$$\begin{aligned} a\psi_0 &= (u^2 h(p) - iu\xi)\psi_0 \\ &= a \sum_{n=-\infty}^{\infty} c_n (e_n^+ + e_n^-) \stackrel{!}{=} 0 \end{aligned} \tag{3.14}$$

must fulfill the following condition for the recursion coefficients c_n:

$$c_{n-2} = c_n q^{-2} \sqrt{1 + 2q^{2n-1}(1 - q^{-2})}. \tag{3.15}$$

This recursion implies that ψ_0 is indeed an element of the Hilbert space H:

$$(\psi_0, \psi_0) < \infty. \tag{3.16}$$

With these tools we state now

THEOREM 2 *All vectors $(a^+)^n \psi_0$ are well-defined in H and satisfy the following recurrence relation*:

$$\begin{aligned} &(a^+)^{n+1}\psi_0 - 2q^{-(3/2)} q^{-2n} p (a^+)^n \psi_0 \\ &\quad + 2q^{-2}[n](a^+)^{n-1}\psi_0 = 0, \end{aligned} \tag{3.17}$$

where

$$[n] := \frac{q^{-2n} - 1}{q^{-2} - 1}, \quad n \in \mathbb{N}_0. \tag{3.18}$$

Proof We first investigate the action of a^+ on ψ_0. To do so we refer to the abbreviation

$$\psi_0^k := \sum_{n=-k}^{k} c_n (e_n^+ + e_n^-) \tag{3.19}$$

$$\begin{aligned} \Rightarrow\quad a^+ \psi_0^k = \sum_{n=-k}^{k} c_n(&\alpha_{n+2}(e_{n+2}^+ + e_{n+2}^-) \\ &+ \beta_n (e_n^+ + e_n^-)). \end{aligned} \tag{3.20}$$

Inserting $\alpha_n^2 = (q^{n-(1/2)} h(q^n)(1 - q^{-2}) + 1)^2 (-\beta_{n-2})^2$ yields

$$\beta_n - \alpha_n^2 \beta_n^{-1} q^{-4} = 2q^{-(3/2)} q^n \tag{3.21}$$

$$\begin{aligned} &\Rightarrow\quad (e_n^+ + e_n^-, a^+ \psi_0^k) = 2q^{-(3/2)} q^n c_n \\ &\quad \text{for } |n| < k - 2,\ \ k > 2 \end{aligned} \tag{3.22}$$

$$\Rightarrow \quad a^+\psi_0 = \lim_{k\to\infty} a^+\psi_0^k,$$
$$\text{i.e. } (a^+\psi_0, a^+\psi_0) < \infty. \tag{3.23}$$

It is remarkable that a^+ acts on ψ_0 in the same way the operator $2q^{-(3/2)}p$ does, see (3.22). Thus we have

$$a\psi_0 = 0 \quad a^+\psi_0 = 2q^{-(3/2)}p\psi_0. \tag{3.24}$$

Making use of the commutation relation

$$a^+pe_n^\sigma = \left(q^{-2}pa^+ - \frac{1}{\sqrt{q}}\right)e_n^\sigma, \quad \sigma \in \{+1, -1\} \tag{3.25}$$

and of the q-Heisenberg algebra relation

$$(p\xi - q\xi p)e_n^\sigma = -iq^{(3/2)}ue_n^\sigma \tag{3.26}$$

one concludes

$$(a^+)^2\psi_0 - 2q^{-(3/2)}\left(q^{-2}pa^+ - \frac{1}{\sqrt{q}}\right)\psi_0 = 0, \tag{3.27}$$

where $(q^{-2}pa^+ - (1/\sqrt{q}))\psi_0$ is well defined in H. Therefore $(a^+)^2\psi_0$ is also an element of H. By induction one finds in complete analogy to (3.27) the recurrence relation

$$\begin{aligned}(a^+)^{n+1}\psi_0 - 2q^{-(3/2)}q^{-2n}p(a^+)^n\psi_0 \\ + 2q^{-2}[n](a^+)^{n-1}\psi_0 = 0,\end{aligned} \tag{3.28}$$

where again $[n] = ((q^{-2n} - 1)/(q^{-2} - 1))$, $n \in \mathbb{N}_0$.

Similarly one concludes by induction that $(a^+)^n\psi_0$ have finite norms for $n \in \mathbb{N}_0$. This proves Theorem 2.

4 DISCRETE ENERGY REPRESENTATION

We first define a subspace of H, being spanned by eigenfunctions of a^+

$$HS' := \left\{\psi = \sum_{n=0}^{\infty} c_n(a^+)^n\psi_0 \,|\, (\psi, \psi) < \infty,\, c_n \in \mathbb{C}\right\} \tag{4.1}$$

The momentum operator p maps the intersection of its domain with HS', i.e. $D(p) \cap HS'$ to HS'. This allows to construct an energy representation of p in HS'. Like in conventional quantum mechanics, this shall be the action of p on the eigenstates of a^+a. The recurrence relation (3.28) for $\gamma_1 = \gamma_2 = 0$ is

$$\begin{aligned}(a^+)^{n+1}\psi_0 - 2q^{-(3/2)}q^{-2n}p(a^+)^n\psi_0 \\ + 2q^{-2}[n]_{q^{-2}}(a^+)^{n-1}\psi_0 = 0 \\ (n \in \mathbb{N}_0).\end{aligned} \tag{4.2}$$

The $(a^+)^n\psi_0$ are pairwise orthogonal but not yet orthonormal. We next determine their normalization. To do so, we start from the ansatz

$$(a^+)^n\psi_0 = \nu_n\psi_n, \quad (\psi_n, \psi_n) = 1. \tag{4.3}$$

The recurrence relation then can be rewritten as follows:

$$\begin{aligned}\nu_{n+1}\psi_{n+1} - 2q^{-(3/2)}q^{-2n}p\nu_n\psi_n \\ + 2q^{-2}[n]_{q^{-2}}\nu_{n-1}\psi_{n-1} = 0\end{aligned} \tag{4.4}$$

which is equivalent to

$$\begin{aligned}\nu_{n+1}\nu_n^{-1}(2q^{-(3/2)}q^{-2n})^{-1}\psi_{n+1} - p\psi_n \\ + q^{2n}[n]_{q^{-2}}q^{-(1/2)}\nu_{n-1}\nu_n^{-1}\psi_{n-1} = 0,\end{aligned} \tag{4.5}$$

i.e.

$$p\psi_n = b_n\psi_{n+1} + a_{n-1}\psi_{n-1}, \tag{4.6}$$

where b_n and a_n are the coefficients in (4.5).

As p is a symmetric operator, one finds

$$b_{n-1} = a_{n-1}, \quad a_{-1} = 0, \quad n = 0, 1, \ldots \tag{4.7}$$

and hence

$$\begin{aligned}a_n = (2)^{-(1/2)}q^{(3/2)}\sqrt{[n+1]_{q^{-2}}}q^{2n}, \\ n = 0, 1, \ldots\end{aligned} \tag{4.8}$$

The normalization coefficients are related as follows:

$$\begin{aligned}\nu_0 \stackrel{!}{=} 1 \quad \nu_{n+1} = \sqrt{2[n+1]_{q^{-2}}}\nu_n, \\ n = 0, 1, \ldots\end{aligned} \tag{4.9}$$

These informations are useful to determine whether p is an essentially self-adjoint operator with respect to the energy representation given by (4.6), (4.7) and (4.8). To investigate this topic, we have to apply results from the theory of infinite monolateral Jacobi matrices [8], p. 522. Let us reformulate these results in some generality in order to apply them to the operator p.

The action of the operator $p = A$ on the energy eigenstates looks as follows:

$$A\psi_n = b_{n-1}\psi_{n-1} + b_n\psi_{n+1} \tag{4.10}$$

By direct calculation, one verifies that the energy eigenstates ψ_n have a polynomial representation of the following type:

$$\psi_n = P_n(A)\psi_0, \tag{4.11}$$

where the $P_n(A)$ are in general $\mathbb{C}$-valued polynomials that satisfy the relations

$$\lambda P_n(\lambda) = b_n P_{n+1}(\lambda) + b_{n-1}P_{n-1}(\lambda) \quad (n \in \mathbb{N}) \tag{4.12}$$

$$P_{-1}(\lambda) = P_0(\lambda) - 1 = 0 \tag{4.13}$$

The following theorems characterize the property of essential self-adjointness in the case of the operator A, see also [8], p. 522.

THEOREM 3 *A is not essentially self-adjoint if the series $\sum_{k=0}^{\infty} |P_k(i)|^2$ converges.*

Proof We denote the scalar product of $x, y \in HS'$ again with (x, y). If the series in Theorem 3 is guaranteed to converge, there exists an $x \in HS'$ such that

$$(\psi_k, x) = P_k(i). \tag{4.14}$$

Taking into account the relations (4.10) and (4.12) we obtain

$$(A\psi_k, x) = b_{k-1}\overline{P_{k-1}(i)} + b_k\overline{P_{k+1}(i)} = i\overline{P_k(i)}, \tag{4.15}$$

where the overline denotes complex conjugation. Because of (4.14) one receives

$$(A\psi_k, x) = (\psi_k, ix). \tag{4.16}$$

As the operator A as well as the scalar product are distributive, one gets in the case of any finite element

$$y = \sum_{j=0}^{k} y_j\psi_j \tag{4.17}$$

the following relation:

$$(Ay, x) = (y, ix). \tag{4.18}$$

Applying the theory of monolateral Jacobi matrices, [8], p. 521, one knows that (4.18) is valid for any $y \in D(A)$. Therefore, the following result holds:

$$x \in D(A^*) \quad \text{and} \quad A^*x = ix. \tag{4.19}$$

This immediately implies the statement of Theorem 3.

THEOREM 4 *If the series $\sum_{k=0}^{\infty} |P_k(i)|^2$ diverges, the operator A is essentially self-adjoint.*

Proof We have to show that A^* has definitely not $+i$, $-i$ as eigenvalues. If

$$A^*x = ix \quad \text{with } (x, x) \neq 0, \ (x, x) < \infty, \tag{4.20}$$

then one concludes because of the definition for A^* and because of $\psi_k \in D(A)$ that

$$(A\psi_k, x) = (\psi_k, ix) \tag{4.21}$$

or

$$(x, A\psi_k) = i(x, \psi_k) = ix_k, \tag{4.22}$$

$$x_k := (\psi_k, x), \tag{4.23}$$

thus in total

$$(x, b_{k-1}, \psi_{k-1} + b_k\psi_{k+1}) = ix_k. \tag{4.24}$$

The scalar product yields

$$b_{k-1}x_{k-1} + b_k x_{k+1} = ix_k. \tag{4.25}$$

Because of the polynomial relation (4.12) and by means of induction we receive

$$x_k = P_k(i)x_0 \quad \text{and} \quad x_0 \neq 0. \tag{4.26}$$

This is in obvious contradiction to the required divergence of the series $\sum_{k=0}^{\infty} |P_k(i)|^2$. Replacing in this series i by $-i$ one obtains again a divergent series because of

$$P_k(-i) = \overline{P_k(i)}. \tag{4.27}$$

Thus, as for A^*, i and $-i$ cannot be eigenvalues. Therefore, Theorem 4 holds.

With analogous methods one proves:

Theorem 5 *Only one of the following cases can be true: For any non-real $z \in \mathbb{C}$ the function*

$$F(z) := \sum_{k=0}^{\infty} |P_k(z)|^2 \tag{4.28}$$

becomes divergent or for any non-real $z \in \mathbb{C}$ the function F converges. A is essentially self-adjoint in the first case. In the second case, A is not essentially self-adjoint.

If A is not essentially self-adjoint, i.e.

$$\sum_{k=0}^{\infty} |P_k(i)|^2 < \infty, \tag{4.29}$$

one verifies by the proof of Theorem 4 that $x_k = P_k(i)x_0$ $(k = 1, 2, \ldots)$. Note that the choice of $x_0 \neq 0$ is arbitrary.

This however means that the eigenspace that belongs to i is one-dimensional. The same holds when substituting $i \to -i$ where the x_k are now replaced by $\overline{x_k}$. Therefore, because of

$$\dim Eig(A, +i) = \dim Eig(A, -i) = 1, \tag{4.30}$$

the deficiency indices are equal and thus A has self-adjoint extensions.

By denoting

$$x_i := \sum_{k=0}^{\infty} P_k(i)\psi_k, \tag{4.31}$$

$$x_{-i} := \sum_{k=0}^{\infty} P_k(-i)\psi_k, \tag{4.32}$$

one sees that the elements of the domain $D(A_\theta) \subset HS$ of a special self-adjoint extension A_θ of A are uniquely determined by

$$v := x_A + \alpha x_\theta, \tag{4.33}$$

where

$$x_A \in D(A) \quad \text{and} \quad \alpha \in \mathbb{C}, \tag{4.34}$$

as well as

$$x_\theta := i(e^{(i\theta/2)}x_i + e^{(-i\theta/2)}x_{-i}) \quad \text{with } 0 \leq \theta < 2\pi. \tag{4.35}$$

Let $\rho(\lambda)$ be the spectral density of A (if A is essentially self-adjoint) or of A_θ (if A is not essentially self-adjoint). We then obtain in analogy to [8], p. 524:

Result 1 The polynomials $P_k(\lambda)$ constitute a complete orthonormal set with respect to $\rho(\lambda)$, i.e.

$$\int_{-\infty}^{\infty} P_i(\lambda)P_j(\lambda)\mathrm{d}(\rho(\lambda)) = \delta_{ij}. \tag{4.36}$$

We now are going to decide whether the operator

$$A : D(p) \cap HS' \to HS', \quad x \mapsto Ax := px$$

is self-adjoint or whether it has self-adjoint extensions. Following the outlined theorems, only one of these two cases will be true.

Let $\lambda \in \mathbb{C}$. The ansatz

$$p\sum_{n=0}^{\infty} c_n\psi_n = \lambda \sum_{n=0}^{\infty} c_n\psi_n \tag{4.37}$$

then leads to the three-term recurrence relation

$$c_{n+1} = \lambda a_n^{-1}c_n - a_{n-1}a_n^{-1}c_{n-1}, \tag{4.38}$$

where the a_n are given by (4.8). Equality (4.38) yields

$$c_n = \lambda a_{n-1}^{-1}c_{n-1} - a_{n-2}a_{n-1}^{-1}c_{n-2} \tag{4.39}$$

and therefore

$$c_{n+1} = (\lambda^2 a_n^{-1} a_{n-1}^{-1} - a_{n-1} a_n^{-1}) c_{n-1} - \lambda a_{n-2} a_{n-1}^{-1} a_n^{-1} c_{n-2}. \tag{4.40}$$

Looking simultaneously at (4.39) and (4.40), one obtains a system of linear equations as follows:

$$\begin{aligned} c_{n+1} &= \alpha_n c_{n-1} + \beta_n c_{n-2}, \\ c_n &= \gamma_n c_{n-1} + \delta_n c_{n-2}. \end{aligned} \tag{4.41}$$

Note that the coefficients $\alpha_n, \beta_n, \gamma_n, \delta_n$ are fixed by (4.39) and (4.40). We define the vectors $v_n \in \mathbb{C}^2$ by

$$v_n := (c_{n+1}, c_n)^{\mathrm{T}} \quad n = 0, 2, 4, \ldots \tag{4.42}$$

and the sequence of matrices $A_n \in \mathbb{C}^{2\times 2}$ with components $\alpha_n, \beta_n, \gamma_n, \delta_n$. They are related by

$$v_{n+2} = A_n v_n \tag{4.43}$$

and by inserting a_n from (4.8) one finds for any fix element $x \in \mathbb{C}^2$

$$\lim_{n\to\infty} A_n x = -q^{-2} x. \tag{4.44}$$

Thus, A_n converges pointwise and the limit is $-q^{-2}E$ where $E \in \mathbb{C}^{2\times 2}$ denotes the identity matrix.

Using the norm

$$\|x\| := \sqrt{x_1 \overline{x_1} + x_2 \overline{x_2}} \tag{4.45}$$

for an element $x = (x_1, x_2) \in \mathbb{C}^2$, we know that $\mathbb{C}^2$, established with $\|*\|$, becomes a Banach space. Thus we can conclude that the sequence of norms $(\|A_n\|)$ is bounded, where

$$\|A_n\| := \sup_{\|x\|=1} \|A_n x\|. \tag{4.46}$$

One finally perceives that $\|A_n\| \to q^{-2}$ as $\mathbb{C}^2$ is finite-dimensional. Consequently, to any $\epsilon > 0$ there exists an index $N(\epsilon) \in \mathbb{N}$ such that for all $n \geq N(\epsilon)$:

$$|\|A_n\| - q^{-2}| < \epsilon \;\Rightarrow\; \|A_n\| < q^{-2} + \epsilon. \tag{4.47}$$

Choose $\epsilon > 0$ such that $\rho := q^{-2} + 2\epsilon < 1$. With the help of (4.43) we then obtain

$$\|v_{n+2}\| \leq \|A_n\| \|v_n\| \;\Rightarrow \tag{4.48}$$

$$\|v_{n+2}\| < (q^{-2} + \epsilon) \|v_n\| \;\Rightarrow \tag{4.49}$$

$$\|v_{n+2}\| < \rho \|v_n\|, \quad 0 < \rho < 1 \quad \text{from a certain index } N \text{ on.} \tag{4.50}$$

In detail, one has $a_{-1} = 0$ and therefore, because of (4.38), the sequence of the c_n is uniquely fixed after c_0 is chosen. As a consequence, any c_n can be uniquely represented as a polynomial χ_n in c_0

$$c_n = \chi_n(c_0) \quad n = 1, 2, \ldots \tag{4.51}$$

Choosing $\lambda = +i$ resp. $\lambda = -i$, we assume that the series

$$\sum_{n=0}^{\infty} c_n \psi_n \tag{4.52}$$

diverges for any $c_0 \in \mathbb{C}$.

Because of (4.50), one receives

$$|c_{n+2}^2| + |c_{n+1}^2| < \rho(|c_n^2| + |c_{n-1}^2|). \tag{4.53}$$

Thus we find for $x_n := |c_n^2| + |c_{n+1}^2|$

$$\sum_{n=0}^{\infty} x_n < \infty. \tag{4.54}$$

Therefore, the series (4.52) always converges.

Applying now the general facts from the stated theorems, we finally obtain

Result 2 Let $\mathbf{p}$ be the restriction p on HS'. The operator $\mathbf{p}$ is not self-adjoint for $q > 1$. However, there exist self-adjoint extensions $\mathbf{p}_\theta$ via

$$\begin{aligned} \mathbf{p}_\theta &: D(\mathbf{p}) \cup \{\psi = \phi + \alpha\theta_+ \\ &\quad + \beta\theta_- \mid \phi \in D(\mathbf{p})\} \to HS', \\ \mathbf{p}_\theta \phi &:= \mathbf{p}\phi \quad \mathbf{p}_\theta \theta_{+/-} := \mathbf{p}^* \theta_{+/-}, \end{aligned} \tag{4.55}$$

$$\theta_{+/-} := i(\mathrm{e}^{(\mathrm{i}\vartheta_{+/-}/2)} x_i^{+/-} + \mathrm{e}^{(-\mathrm{i}\vartheta_{+/-}/2)} x_{-i}^{+/-}), \tag{4.56}$$

$$0 \leq \vartheta_{+/-} < 2\pi, \tag{4.57}$$

$$\mathbf{p}^* x_i^{+/-} = i x_i^{+/-} \quad \mathbf{p}^* x_{-i}^{+/-} = -i x_{-i}^{+/-}, \tag{4.58}$$

$$x_i^{+/-} := \sum_{k=0}^{\infty} P_k(i) \psi_{k+/-},$$
$$x_{-i}^{+/-} := \sum_{k=0}^{\infty} P_k(-i) \psi_{k+/-}. \tag{4.59}$$

In total, this result reveals the fact that the energy representation of the discrete q-momentum operator in terms of a^+a-eigenfunctions is completely different from the analogous situation in the continuum, when $q=1$. In the continuum situation, the operator p is essentially self-adjoint in the basis of $\mathcal{L}^2(\mathbb{R})$ that is provided by the classical Hermite functions. As stated in the introduction, this observation is directly related to the interpretation of q-oscillator algebras in the context of squeezed laser state research, see [14]. A fascinating development on this area can be expected and a lot of mathematical investigation concerning related spectral theoretical results still has to be performed. This article gives one more contribution into this direction.

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References

[1] L. Biedenharn, *J. Phys. A* **22** (1989) L873.
[2] M. Fichtmüller, Diplomarbeit "Der Grenzübergang $q \to 1$ der q-Fouriertransformation", LMU München, 1994.
[3] A. Hebecker *et al.* Representations of a q-deformed Heisenberg algebra, *Z. Phys.* **C64**: 355–360, 1994.
[4] B. Riemann, On the hypotheses which lie at the bases of geometry (transl. to English by William Kingdon Clifford), *Nature*, Vol. 8, pp. 14–17, 36, 37 based on Riemann, B. Ueber die Hypothesen, welche der Geometrie zu Grunde liegen, Abhandlungen der Königlichen, Gesellschaft der Wissenschaften zu Göttingen, Vol. 13, 1867.
[5] M. Chaichian, H. Grosse and P. Prešnajder, Unitary representations of the q-oscillator algebra, *J. Phys. A: Math. Gen.* **27**: 2045–2051, 1994.
[6] A. Macfarlane, *J. Phys. A* **22**: 4581, 1989.
[7] J. Wess and B. Zumino, Covariant differential calculus on the quantum hyperplane, *Nucl. Phys. B.* (*Proc. Suppl.*) **18B**: 302, 1990.
[8] W.I. Smirnow, Lehrgang der Höheren Mathematik, Teil 5, VEB Deutscher Verlag der Wissenschaften, Berlin, 1988.
[9] A. Ruffing, Dissertation, Universität München, 1996.
[10] M. Fichtmüller, A. Lorek and J. Wess, q-Deformed phase space and its lattice structure, *Zeitschrift für Physik*, **C71**: 533–538, 1996.
[11] A. Lorek, A. Ruffing and J. Wess, A q-deformation of the harmonic oscillator, *Zeitschrift für Physik*, **C74**: 369–377 (1997).
[12] A. Ruffing and M. Witt, On the integrability of q-oscillators based on invariants of discrete fourier transforms, *Lett. Math. Phys.* October (II) **42**(2): 1997.
[13] G.N. Ord, Fractal space–time: A geometric analog of relativistic quantum mechanics, *J. Phys A.* **16**: 1869–1884, 1983.
[14] K.A. Penson and A.I. Solomon, New generalized coherent states, *J. Math. Phys.* (to appear)
[15] A. Ruffing, On Schrödinger–Hermite operators in lattice quantum mechanics, *Lett. Math. Phys.* **47**: 197–214, 1999.
[16] W. Drechsler and P.A. Tuckey, On quantum and parallel transport in a Hilbert bundle over space–time, *Class. Quant. Grav.* **13**: 611–632, 1996.
[17] P. Breitenlohner, H. Hollmann and D. Maison, Quantization of the Reissner–Nordstrom black hole, *Phys. Lett.* **B432**: 293–297, 1998.
[18] H. Kleinert, Universality principle for orbital angular momentum and spin in gravity with torsion, gr-qc/9807021, 1998.
[19] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego, 1995.
[20] A. Jaffe, Quantum harmonic analysis and geometric invariants, 1997, Advances in Mathematics (to appear)
[21] N. Seiberg and E. Witten, String theory and noncommutative geometry, hepth/9908142, 1999.
[22] E. Witten, Duality, Space–time and quantum mechanics, *Phys. Today* **50**(5): 28–33, 1997.
[23] J. Sonnenschein, S. Theisen and S. Yankielowicz, On the relation between the holomorphic prepotential and the quantum moduli in Susy Gauge theories, *Phys. Lett.* **B367**: 145–150, 1996.
[24] J. Louis and K. Förger, Holomorphic couplings in string theory, *Nucl. Phys. Proc. Suppl.* **55B**: 33–64, 1997.
[25] D.R. Masson and J. Repka, Spectral theory of Jacobi matrices in $l^2(\mathbb{Z})$ and the su(1, 1) Lie algebra, *SIAM J. Math. Anal.* **22**: 1131–1146, 1991.