

## Research Article

# A Fractional-Order Chaotic System with an Infinite Number of Equilibrium Points

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A new 4D fractional-order chaotic system, which has an infinite number of equilibrium points, is introduced. There is no-chaotic behavior for its corresponded integer-order system. We obtain that the largest Lyapunov exponent of this 4D fractional-order chaotic system is 0.8939 and yield the chaotic attractor. A chaotic synchronization scheme is presented for this 4D fractional-order chaotic system. Numerical simulations is verified the effectiveness of the proposed scheme.

## 1. Introduction

Because the chaotic (hyperchaotic) signal can be used in electrical engineering, telecommunications, information processing, material engineering, and so forth much attention has been paid to effectively generating chaotic and hyperchaotic systems. Many chaotic (hyperchaotic) systems and fractional-order chaotic (hyperchaotic) systems are reported in recent years [1–14], such as Lorenz chaotic (hyperchaotic) system and its corresponded fractional-order system, and integer-order and fractional-order Chen chaotic (hyperchaotic) system, integer-order and fractional-order Lü chaotic (hyperchaotic) system.

However, for all the previous integer-order and fractional-order chaotic (hyperchaotic), many systems have a finite number of equilibrium points. For example, some chaotic systems have one equilibrium point [15–17], some chaotic systems have two equilibrium points [18], and some chaotic systems have three equilibrium points [1, 2, 5, 6, 9, 10], so a natural and interesting question is can we construct a chaotic (hyperchaotic) system which has an infinite number of equilibrium points? Moreover, many fractional-order chaotic and hyperchaotic systems also possess chaotic attractor for its corresponded integer-order system, so the

other question is as follows: are the fractional-order chaotic and hyperchaotic systems no-chaotic behavior for its corresponded integer-order system? To the best of our knowledge, few results on the above mentioned two questions are reported.

Motivated by the above discussions, a new 4D fractional-order chaotic system is presented in this paper. This new 4D fractional-order chaotic system has an infinite number of equilibrium points, and no-chaotic behavior for its corresponded integer-order system. The largest Lyapunov exponent and chaotic attractor are yielded for the new 4D fractional-order chaotic system. A chaotic synchronization scheme is presented for this new 4D fractional-order chaotic system.

## 2. A New 4D Fractional-Order Chaotic System

Now, a new 4D fractional-order chaotic system is constructed, which is described as follows:

$$\frac{d^q x_1}{dt^q} = 10(x_2 - x_1) + x_4$$

$$\frac{d^q x_2}{dt^q} = 15x_1 - x_1x_3$$

$$\begin{aligned}\frac{d^q x_3}{dt^q} &= -2.5x_3 + 4x_1^2 \\ \frac{d^q x_4}{dt^q} &= -10x_2 - x_4,\end{aligned}\quad (1)$$

where  $q = 0.95$  is the fractional-order, and  $x_i$  ( $i = 1, 2, 3, 4$ ) are real state variables.

The real equilibrium points of system (1) is calculated by

$$\begin{aligned}10(x_2 - x_1) + x_4 &= 0 \\ 15x_1 - x_1x_3 &= 0 \\ -2.5x_3 + 4x_1^2 &= 0 \\ -10x_2 - x_4 &= 0.\end{aligned}\quad (2)$$

Obviously,  $(x_1, x_2, x_3, x_4) = (0, x_2, 0, -10x_2)$  is the real equilibrium points of system (1), where  $x_2$  is a any real numbers, so system (1) has an infinite number of real equilibrium points. To the best of our knowledge, this result is different from all the previous fractional-order chaotic and hyperchaotic systems. It implies that we yield a new 4D fractional-order system, which has an infinite number of real equilibrium points.

The Jacobian  $J$  at all equilibrium points is

$$J = \begin{pmatrix} -10 & 10 & 0 & 1 \\ 15 & 0 & 0 & 0 \\ 0 & 0 & -2.5 & 0 \\ 0 & -10 & 0 & -1 \end{pmatrix}\quad (3)$$

and its eigenvalues are  $\lambda_1 = -18.548$ ,  $\lambda_2 = -2.5$ ,  $\lambda_3 = 0$ , and  $\lambda_4 = 7.548$  for all  $x_2$ . Therefore, all the equilibrium points in system (1) are unstable.

The dynamical behaviors of system (1) for its corresponded integer-order system ( $q = 1$ ) can be characterized by its Lyapunov exponents. The Lyapunov exponents for its corresponded integer-order system are 0, 0,  $-0.779$ , and  $-12.724$ , respectively. Therefore, the fractional-order system (1) no-chaotic behaviors for  $q = 1$ , and which is periodic orbit for its corresponded integer-order system. Figure 1 shows the periodic orbit of fractional-order system (1) for its corresponded integer-order system ( $q = 1$ ).

Now, we discuss the numerical solution of fractional differential equations. It is well known that there are direct time domain approximation (the improved version of Adams-Bashforth-Moulton numerical algorithm) and frequency domain approximation for nonlinear fractional-order system [6]. However, frequency domain approximation may result in wrong consequences [19], so the direct time domain approximation [6] numerical simulation is used to solve the fractional-order system in this paper. Let  $h = T/N$ ,  $t_n = nh$  ( $n = 0, 1, 2, \dots, N$ ), and initial condition

$(x_1(0), x_2(0), x_3(0), x_4(0))$ , so the fractional-order chaotic system (1) can be discretized as follows:

$$\begin{aligned}x_1(n+1) &= x_1(0) + \frac{h^q}{\Gamma(q+2)} \\ &\times \left[ 10(x_2^p(n+1) - x_1^p(n+1)) + x_4^p(n+1) \right. \\ &\quad \left. + \sum_{j=0}^n \alpha_{1,j,n+1} \times (10(x_2(j) - x_1(j)) + x_4(j)) \right] \\ x_2(n+1) &= x_2(0) + \frac{h^q}{\Gamma(q+2)} \\ &\times \left[ 15x_1^p(n+1) - x_1^p(n+1)x_3^p(n+1) \right. \\ &\quad \left. + \sum_{j=0}^n \alpha_{2,j,n+1} (15x_1(j) - x_1(j)x_3(j)) \right] \\ x_3(n+1) &= x_3(0) + \frac{h^q}{\Gamma(q+2)} \\ &\times \left[ 4(x_1^p(n+1))^2 - 2.5x_3^p(n+1) \right. \\ &\quad \left. + \sum_{j=0}^n \alpha_{3,j,n+1} (4(x_1(j))^2 - 2.5x_3(j)) \right] \\ x_4(n+1) &= x_4(0) + \frac{h^q}{\Gamma(q+2)} \\ &\times \left[ -10x_2^p(n+1) - x_4^p(n+1) \right. \\ &\quad \left. + \sum_{j=0}^n \alpha_{4,j,n+1} \times (-10x_2(j) - x_4(j)) \right],\end{aligned}\quad (4)$$

where

$$\begin{aligned}x_1^p(n+1) &= x_1(0) + \frac{1}{\Gamma(q)} \\ &\times \sum_{j=0}^n b_{1,j,n+1} \times [10(x_2(j) - x_1(j)) + x_4(j)]\end{aligned}$$

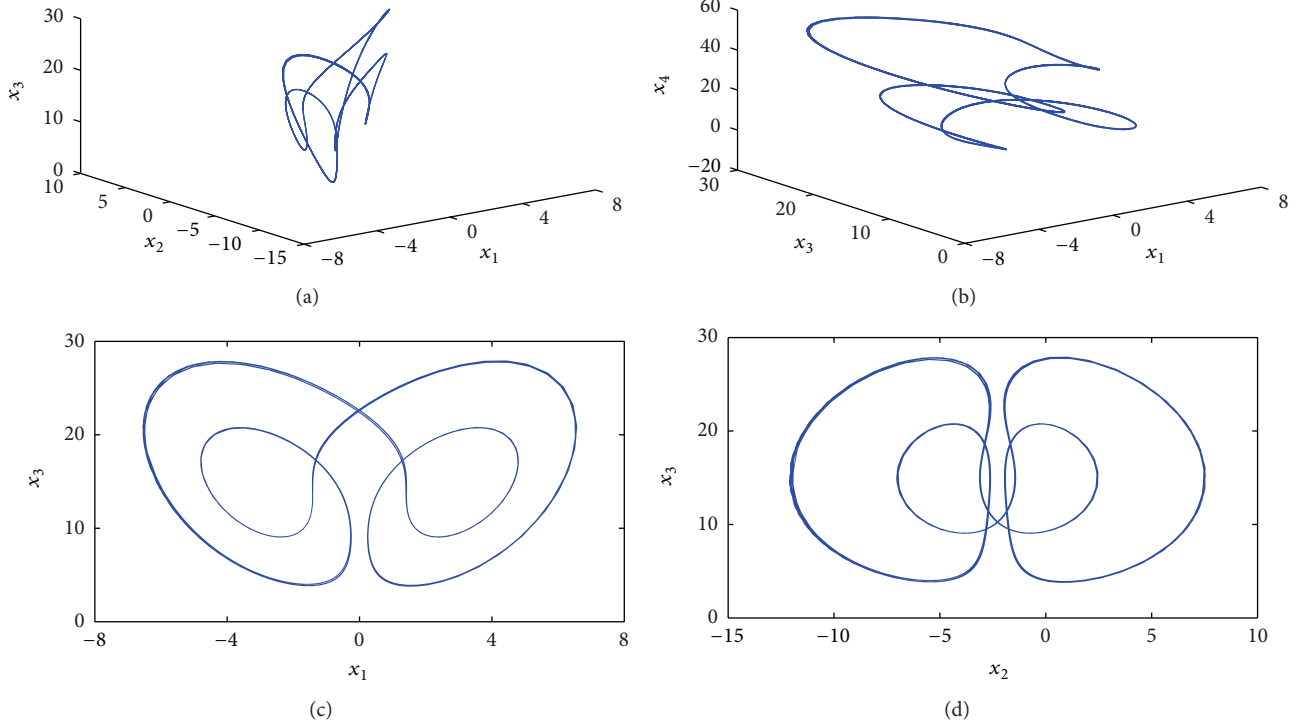


FIGURE 1: The periodic orbit of fractional-order system (1) for its corresponded integer-order system ( $q = 1$ ).

$$\begin{aligned}
 x_2^p(n+1) &= x_2(0) + \frac{1}{\Gamma(q)} \\
 &\quad \times \sum_{j=0}^n b_{2,j,n+1} [15x_1(j) - x_1(j)x_3(j)] \\
 x_3^p(n+1) &= x_3(0) + \frac{1}{\Gamma(q)} \\
 &\quad \times \sum_{j=0}^n b_{3,j,n+1} [4(x_1(j))^2 - 2.5x_3(j)] \\
 x_4^p(n+1) &= x_4(0) + \frac{1}{\Gamma(q)} \\
 &\quad \times \sum_{j=0}^n b_{4,j,n+1} \times [-10x_2(j) - x_4(j)],
 \end{aligned} \tag{5}$$

and for  $i = 1, 2, 3, 4$

$$\begin{aligned}
 &\alpha_{i,j,n+1} \\
 &= \begin{cases} n^{q+1} - (n-q)(n+1)^q, & j=0 \\ (n-j+2)^{q+1} + (n-j)^{q+1} - 2(n-j+1)^{q+1}, & 1 \leq j \leq n \\ 1, & j=n+1, \end{cases} \\
 &b_{i,j,n+1} = \frac{h^q}{q} [(n-j+1)^q - (n-j)^q], \quad 0 \leq j \leq n.
 \end{aligned} \tag{6}$$

The error of this approximation is described as follows:

$$|x_i(t_n) - x_i(n)| = o(h^p), \quad p = \min(2, 1+q). \tag{7}$$

The dynamical behaviors of 4D fractional-order system (1) can be characterized by its largest Lyapunov exponent. By computer simulation, we can obtain that the largest Lyapunov exponent of fractional-order system (1) is 0.8939, so the 4D fractional-order system (1) is chaotic. The chaotic attractor is shown in Figure 2.

According to the above mentioned, we obtain a new 4D fractional-order chaotic system, which has an infinite number of real equilibrium points. Moreover, the 4D fractional-order chaotic system is no-chaotic behaviors for its corresponded integer-order system ( $q = 1$ ). The result in our paper is different from all the previous fractional-order chaotic and hyperchaotic systems.

### 3. Chaotic Synchronization for the New 4D Fractional-Order Chaotic System

In this section, the chaotic synchronization for the new 4D fractional-order chaotic system (1) is considered. Based on the stability theory of nonlinear fractional-order systems [20–24], one synchronization scheme is proposed, and some numerical simulations are performed.

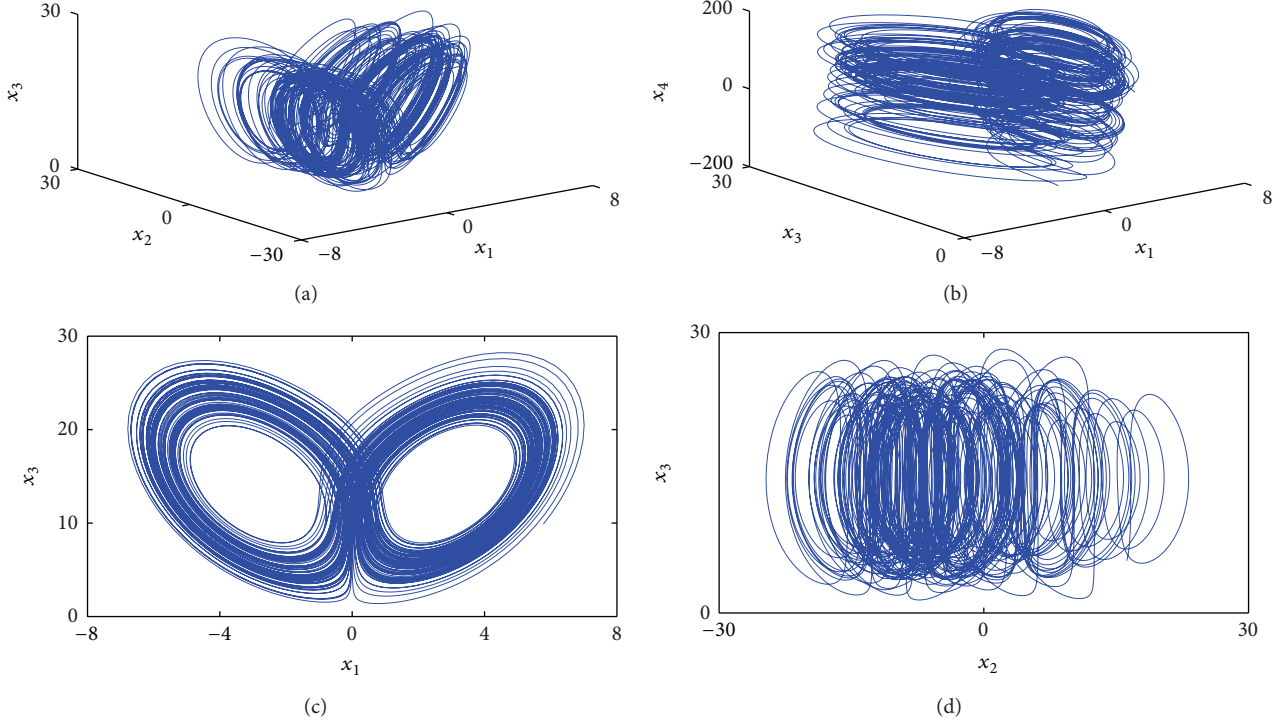


FIGURE 2: The chaotic attractor of 4D fractional-order system (1).

Now, the response fractional-order chaotic system is considered as

$$\begin{aligned}
 D^q y_1 &= 10(y_2 - y_1) + y_4 + u_1 \\
 D^q y_2 &= 15y_1 - y_1 y_3 + u_2 \\
 D^q y_3 &= -2.5y_3 + 4y_1^2 + u_3 \\
 D^q y_4 &= -10y_2 - y_4 + u_4,
 \end{aligned} \tag{8}$$

where  $u_i$  ( $i = 1, 2, 3, 4$ ) is the feedback controller, and  $y_i$  ( $i = 1, 2, 3, 4$ ) are real state variables. Our goal is to choose suitable  $u_i$  ( $i = 1, 2, 3, 4$ ) such that drive system (1) and response system (8) can be achieved with chaotic synchronization.

Definition the synchronization errors are  $e_i = y_i - x_i$  ( $i = 1, 2, 3, 4$ ). The following Theorem 1 is given in order to achieve the chaotic synchronization between the fractional-order chaotic system (1) and the fractional-order chaotic system (8).

**Theorem 1.** *If the feedback controllers are*

$$\begin{aligned}
 u_1 &= (y_3 - 25)e_2 - 4(y_1 + x_1)e_3 \\
 u_2 &= 10e_4 \\
 u_3 &= x_1 e_2 \\
 u_4 &= -e_1,
 \end{aligned} \tag{9}$$

then the chaotic synchronization between fractional-order chaotic system (1) and fractional-order chaotic system (8) can be arrived.

*Proof.* Combining the fractional-order chaotic system (1), fractional-order chaotic system (8), and the feedback controller (9), we can obtain the following error system

$$\begin{aligned}
 &\begin{pmatrix} D^q e_1 \\ D^q e_2 \\ D^q e_3 \\ D^q e_4 \end{pmatrix} \\
 &= \begin{pmatrix} -10 & -15 + y_3 & -4(y_1 + x_1) & 1 \\ 15 - y_3 & 0 & -x_1 & 10 \\ 4(y_1 + x_1) & x_1 & -2.5 & 0 \\ -1 & -10 & 0 & -1 \end{pmatrix} \\
 &\quad \times \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \\
 &\triangleq A(x, y) \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix},
 \end{aligned} \tag{10}$$

where matrix

$$A(x, y) \triangleq \begin{pmatrix} -10 & -15 + y_3 & -4(y_1 + x_1) & 1 \\ 15 - y_3 & 0 & -x_1 & 10 \\ 4(y_1 + x_1) & x_1 & -2.5 & 0 \\ -1 & -10 & 0 & -1 \end{pmatrix}, \tag{11}$$

$$x = (x_1, x_2, x_3, x_4)^T, y = (y_1, y_2, y_3, y_4)^T.$$

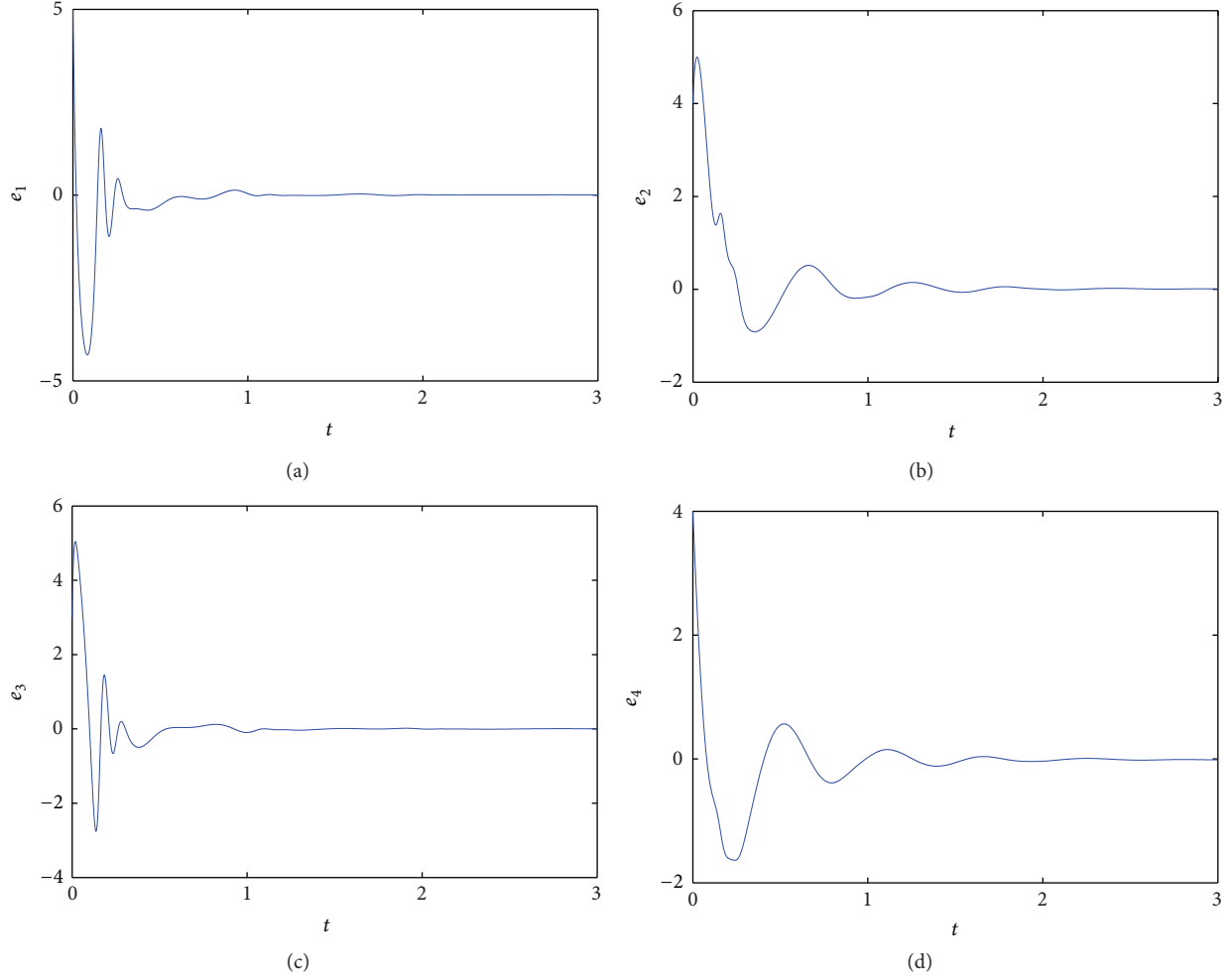


FIGURE 3: The synchronization errors between the drive systems (1) and the response system (8).

Let  $\lambda$  be one of the eigenvalues of  $A(x, y)$  and  $\psi$  is the corresponding nonzero eigenvector, so

$$\begin{aligned} A(x, y) \psi &= \lambda \psi, \\ \psi^H A(x, y)^H &= \bar{\lambda} \psi^H, \end{aligned} \quad (12)$$

where  $\mathbf{H}$  is conjugate transpose, and  $\bar{\lambda}$  is the conjugate for eigenvalues  $\lambda$ .

According to (12), one can obtain

$$\psi^H A(x, y) \psi + \psi^H A(x, y)^H \psi = \psi^H \lambda \psi + \bar{\lambda} \psi^H \psi. \quad (13)$$

Therefore

$$\lambda + \bar{\lambda} = \frac{\psi^H [\text{diag}(-20, 0, -5, -2)] \psi}{\psi^H \psi}, \quad (14)$$

so

$$\lambda + \bar{\lambda} \leq 0. \quad (15)$$

That is

$$\arg |\lambda| \geq \frac{\pi}{2} > \frac{q\pi}{2}. \quad (16)$$

Using the stability theory of nonlinear fractional-order systems, one can yield that the error system (10) is asymptotically stable, so

$$\lim_{t \rightarrow +\infty} e_i = 0 \quad (i = 1, 2, 3, 4). \quad (17)$$

Equation (17) indicates that the chaotic synchronization between fractional-order chaotic system (1) and fractional-order chaotic system (8) can be achieved. The proof is completed.  $\square$

Now, numerical simulations are considered. The numerical results are shown as Figure 3, in which the initial conditions are  $\mathbf{x} = (3, 3, 1, 2)^T$  for drive system (1), and  $\mathbf{y} = (8, 7, 4, 6)^T$  for response system (8), respectively.

#### 4. Conclusions

In this paper, we obtain a new 4D fractional-order chaotic system, which has an infinite number of equilibrium points and no-chaotic behavior for its corresponded integer-order system. We yield the largest Lyapunov exponent of the new

4D fractional-order system and the Lyapunov exponents for its corresponded integer-order system. The chaotic attractor for the new 4D fractional-order chaotic system and the periodic orbit for its corresponded integer-order system are given. Finally, we realize the chaotic synchronization for the new 4D fractional-order chaotic system, and some numerical simulations are performed.

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