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Research Article

On Period of the Sequence of Fibonacci Polynomials Modulo m

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It is shown that the sequence obtained by reducing modulo m coefficient and exponent of each Fibonacci polynomials term is periodic. Also if p is prime, then sequences of Fibonacci polynomial are compared with Wall numbers of Fibonacci sequences according to modulo p. It is found that order of cyclic group generated with Q_2 matrix $\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}$ is equal to the period of these sequences.

1. Introduction

In modern science there is a huge interest in the theory and application of the Fibonacci numbers. The Fibonacci numbers F_n are the terms of the sequence $0, 1, 1, 2, 3, 5, \ldots$, where $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, with the initial values $F_0 = 0$ and $F_1 = 1$. Generalized Fibonacci sequences have been intensively studied for many years and have become an interesting topic in Applied Mathematics. Fibonacci sequences and their related higher-order sequences are generally studied as sequence of integer. Polynomials can also be defined by Fibonacci-like recurrence relations. Such polynomials, called Fibonacci polynomials, were studied in 1883 by the Belgian mathematician Eugene Charles Catalan and the German mathematician E. Jacobsthal. The polynomials $F_n(x)$ studied by Catalan are defined by the recurrence relation

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad n \geq 3, \tag{1}$$

where $F_1(x) = 1$, $F_2(x) = x$. The Fibonacci polynomials studied by Jocobstral are defined by

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x), \quad n \geq 3, \tag{2}$$

where $J_1(x) = J_2(x) = 1$. The Fibonacci polynomials studied by P. F. Byrd are defined by

$$\varphi_n(x) = 2x\varphi_{n-1}(x) + \varphi_{n-2}(x), \quad n \geq 2, \tag{3}$$

where $\varphi_0(x) = 0$, $\varphi_1(x) = 1$. The Lucas polynomials $L_n(x)$, originally studied in 1970 by Bicknell and they are defined by

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \quad n \geq 2, \tag{4}$$

where $L_0(x) = 2$, $L_1(x) = x$ [1].

Hoggatt and Bicknell introduced a generalized Fibonacci polynomials and their relationship to diagonals of Pascal's triangle [2]. Also after investigating the generalized Q-matrix, Ivie introduced a special case [3]. Nalli and Haukkanen introduced $h(x)$-Fibonacci polynomials that generalize both Catalan's Fibonacci polynomials and Byrd's Fibonacci Polynomials and the k-Fibonacci number. Also they provided properties for these $h(x)$-Fibonacci polynomials where $h(x)$ is a polynomial with real coefficients [1].

Definition 1. The Fibonacci polynomials are defined by the recurrence relation

$$F_n(x) = \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{if } n = 1, \\ xF_{n-1}(x) + F_{n-2}(x), & \text{if } n \geq 2, \end{cases} \tag{5}$$

that the Fibonacci polynomials are generated by a matrix Q_2,

$$Q_2 = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}, \qquad Q_2^n = \begin{pmatrix} F_{n+1}(x) & F_n(x) \\ F_n(x) & F_{n-1}(x) \end{pmatrix} \tag{6}$$

Table 1

Fibonacci polynomials	Coefficient array			
$F_0(x) = 0$	0			
$F_1(x) = 1$	1			
$F_2(x) = x$	1			
$F_3(x) = x^2 + 1$	1	1		
$F_4(x) = x^3 + 2x$	1	2		
$F_5(x) = x^4 + 3x^2 + 1$	1	3	1	
$F_6(x) = x^5 + 4x^3 + 3x$	1	4	3	
$F_7(x) = x^6 + 5x^4 + 6x^2 + 1$	1	5	6	1
⋮	⋮	⋮	⋮	⋮

can be verified quite easily by mathematical induction. The first few Fibonacci polynomials and the array of their coefficients are shown in Table 1 [2].

A sequence is *periodic* if, after a certain point, it consists of only repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the *period of the sequence*. For example, the sequence $a, b, c, d, e, b, c, d, e, b, c, d, e, \ldots$, is periodic after the initial element a and has period 4. A sequence is *simply* periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, a, b, c, d, a, b, c, d, \ldots$, is simply periodic with period 4 [4]. The minimum period length of $(F_i \bmod n)_{i=-\infty}^{\infty}$ sequence is stated by $k(n)$ and is named Wall number of n [5].

Theorem 2. *$k(n)$ is an even number for $n \geq 3$ [5].*

2. The Generalized Sequence of Fibonacci Polynomials Modulo m

Reducing the generalized sequence of coefficient and exponent of each Fibonacci polynomials term by a modulus m, we can get a repeating sequence, denoted by

$$\{F(x)^m\} = \{F_0(x)^m, F_1(x)^m, \ldots, F_n(x)^m, \ldots\}, \quad (7)$$

where $F_i(x)^m = F_n(x)(\bmod m)$. Let $hF(x)^m$ denote the smallest period of $\{F(x)^m\}$, called the period of the generalized Fibonacci polynomials modulo m.

Theorem 3. *$\{F(x)^m\}$ is a periodic sequence.*

Proof. Let $S_2 = \{(x_1, x_2) : 1 \leq x_i \leq 2\}$ where x_i is reduction coefficient and exponent of each term in $F_n(x)$ polynomials modulo m. Then, we have $|S_2| = (m^m)^2$ being finite, that is, for any $i > j$, there exist natural numbers i and j

$$F_{i+1}(x)^m = F_{j+1}(x)^m,$$
$$F_{i+2}(x)^m = F_{j+2}(x)^m, \ldots, F_{i+k}(x)^m = F_{j+k}(x)^m. \quad (8)$$

By definition of the generalized Fibonacci polynomials we have that $F_i(x)^m = xF_{i-1}(x)^m + F_{i-2}(x)^m$ and $F_j(x)^m = xF_{j-1}(x)^m + F_{j-2}(x)^m$. Hence, $F_i(x)^m = F_j(x)^m$, and then it follows that

$$F_{i-1}(x)^m = F_{j-1}(x)^m,$$
$$F_{i-2}(x)^m = F_{j-2}(x)^m, \ldots, F_{i-j}(x)^m = F_{j-j}(x)^m = F_0(x)^m \quad (9)$$

which implies that the $\{F(x)^m\}$ is a periodic sequence. □

Example 4. For $m = 2$, $\{F(x)^2\}$ sequence is $F_0(x)^2 = 0$, $F_1(x)^2 = 1$, $F_2(x)^2 = x$, $F_3(x)^2 = x^2 + 1 = x^0 + 1 = 2 = 0$, $F_4(x)^2 = 0x + x = x$, $F_5(x)^2 = x^2 + 0 = x^0 + 0 = 1$, $F_6(x)^2 = x + x = 2x = 0$, $F_7(x)^2 = 0x + 1 = 1$. We have $\{F(x)^2\} = \{0, 1, x, 0, x, 1, 0, 1, \ldots\}$, and then repeat. So, we get $hF(x)^2 = 6$.

Given a matrix $A = (h_{ij}(x))$ where $h_{ij}(x)$'s being polynomials with real coefficients, $A(\bmod m)$ means that every entry of A is modulo m, that is, $A(\bmod m) = (h_{ij}(x)(\bmod m))$. Let $\langle Q_2 \rangle_m = \{Q_2^i(\bmod m) \mid i \geq 0\}$ be a cyclic group and $|\langle Q_2 \rangle_m|$ denote the order of $\langle Q_2 \rangle_m$ where $Q_2^i(\bmod m)$ is reduction coefficient and exponent of each polynomial in Q_2^i matrix modulo m.

Theorem 5. *One has $hF(x)^m = |\langle Q_2 \rangle_m|$.*

Proof. Proof is completed if it is that $hF(x)^m$ is divisible by $|\langle Q_2 \rangle_m|$ and that $|\langle Q_2 \rangle_m|$ is divisible by $hF(x)^m$. Fibonacci polynomials are generated by a matrix Q_2,

$$Q_2 = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}, \qquad Q_2^n = \begin{pmatrix} F_{n+1}(x) & F_n(x) \\ F_n(x) & F_{n-1}(x) \end{pmatrix}. \quad (10)$$

Thus, it is clear that $|\langle Q_2 \rangle_m|$ is divisible by $hF(x)^m$. Then we need only to prove that $hF(x)^m$ is divisible by $|\langle Q_2 \rangle_m|$. Let $hF(x)^m = t$. It is seen that $Q_2^t = \begin{pmatrix} F_{t+1}(x) & F_t(x) \\ F_t(x) & F_{t-1}(x) \end{pmatrix}$. Hence $Q_2^t = I(\bmod m)$. We get that $|\langle Q_2 \rangle_m|$ is divisible by t. That is, $hF(x)^m$ is divisible by $|\langle Q_2 \rangle_m|$. So, we get $hF(x)^m = |\langle Q_2 \rangle_m|$. □

Theorem 6. *$hF(x)^p = pk(p)$ where p is a prime number.*

Proof. It is completed if it is that $hF(x)^p$ is divisible by $pk(p)$ and that $pk(p)$ divisible by $hF(x)^p$. From Theorem 5 $Q_2^n = \begin{pmatrix} F_{n+1}(x) & F_n(x) \\ F_n(x) & F_{n-1}(x) \end{pmatrix}$, $Q_2^{hF(x)^p} = I(\bmod p)$ for $Q_2 = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}$. Also, $Q_2^{pk(p)} = \begin{pmatrix} F_{pk(p)+1}(x) & F_{pk(p)}(x) \\ F_{pk(p)}(x) & F_{pk(p)-1}(x) \end{pmatrix}$. So, we get $Q_2^{pk(p)} = I(\bmod p)$. Thus $pk(p)$ is divisible by $hF(x)^p$. Moreover $pk(p)$ is divisible by $hF(x)^p$. Since $|\langle Q_2 \rangle_p| = hF(x)^p$, $hF(x)^p$ is divisible by $pk(p)$. Therefore $hF(x)^p = pk(p)$. □

Theorem 7. *$hF(x)^p$ is an even number where p is a prime number.*

Proof. It has been shown that $hF(x)^p = pk(p)$ in Theorem 6. If it is stated that $pk(p)$ is an even number then proof is

Table 2: Periods of the sequence of Fibonacci polynomials modulo p.

p	$k(p)$	$hF(x)^p$	Result
2	3	6	$hF(x)^2 = 2k(2)$
7	16	112	$hF(x)^7 = 7k(7)$
37	76	2812	$hF(x)^{37} = 37k(37)$
103	208	21424	$hF(x)^{103} = 103k(103)$
181	90	16290	$hF(x)^{181} = 181k(181)$
241	240	57840	$hF(x)^{241} = 241k(241)$
373	748	279004	$hF(x)^{373} = 373k(373)$
653	1308	854124	$hF(x)^{653} = 653k(653)$
853	1708	1456924	$hF(x)^{853} = 853k(853)$

completed. By Theorem 2, $k(p)$ is an even number and p is an even number for $p \geq 3$. Hence $pk(p)$ is always an even number. That is, $hF(x)^p$ is an even number.

Table 2 shows some periods of sequence of coefficient and exponent of Fibonacci polynomials modulo, which is a prime number, by using $k(p)$. □

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