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Research Article

Bifurcation and Chaotic Behavior of a Discrete-Time SIS Model

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The discrete-time epidemic model is investigated, which is obtained using the Euler method. It is verified that there exist some dynamical behaviors in this model, such as transcritical bifurcation, flip bifurcation, Hopf bifurcation, and chaos. The numerical simulations, including bifurcation diagrams and computation of Lyapunov exponents, not only show the consistence with the theoretical analysis but also exhibit the rich and complex dynamical behaviors.

1. Introduction

Epidemic models have been widely used in different forms for studying epidemiological processes such as the spread of HIV [1], SARS [2], and influenza [3]. It is well known that dynamical systems with simple dynamical behavior in the constant parameter case display very complex behaviors including chaos when they are periodically perturbed [4, 5]. The continuous-time epidemic models have been widely investigated in many articles (e.g., [6–10] and the references cited therein). In recent years, we have found that more attention is paid to the discrete-time epidemic models (see [11–15] and the references cited therein). The reasons are as follows: first, difference models are more realistic than continuous differential ones because the epidemic statistics are compiled from given time intervals and are discontinuous. Second, the discrete-time models can provide natural simulators for the continuous cases. One can thus not only study the behaviors of the continuous-time model with good accuracy, but also assess the effect of lager time steps. At last, the use of discrete -time models makes it possible to use the entire arsenal of methods recently developed for the study of mappings and lattice equations, either from the integrability and/or chaos points of view.

On the other hand, daily treatments are frequently done for some infections, such as the group of those being responsible for the common cold, which do not confer any long lasting immunity. Such infections do not have a recovered state and individuals become susceptible again after infection. For such reasons, according to [16], we firstly consider the SIS epidemic model with nonlinear incidence rate:

$$\begin{aligned} \frac{dS}{dt} &= bS\left(1-\frac{S}{K}\right)-\beta S^2 I+\gamma I, \\ \frac{dI}{dt} &= \beta S^2 I - dI - \gamma I, \end{aligned} \tag{1}$$

where S denotes the susceptible population, I is the infected population, and b is the intrinsic birth rate constant. K, β are the carrying capacity and the infection rate, respectively. $\gamma \geq 0$ is the recovery rate constant ($1/\gamma$ is the average infective time).

Let

$$S=\sqrt{\frac{d+\gamma}{\beta}}x, \qquad I=\sqrt{\frac{d+\gamma}{\beta}}y, \qquad d\tau=(d+\gamma)\,dt. \tag{2}$$

We obtain the following system analogous to (1):

$$\begin{aligned} \frac{dx}{d\tau} &= a_0 x - a_1 x^2 - x^2 y + a_2 y, \\ \frac{dy}{d\tau} &= x^2 y - y, \end{aligned} \tag{3}$$

where

$$a_0=\frac{b}{d+\gamma}, \qquad a_1=\frac{b}{K\sqrt{\beta(d+\gamma)}}, \qquad a_2=\frac{\gamma}{d+\gamma}. \tag{4}$$

Applying Euler scheme to (3), we obtain the following equation:

$$x_{n+1} = (a_0 + 1) x_n - a_1 x_n^2 - x_n^2 y_n + a_2 y_n, \quad y_{n+1} = x_n^2 y_n. \tag{5}$$

This paper is organized as follows. In Section 2, we give sufficient conditions of existence for transcritical bifurcation, flip bifurcation, and Hopf bifurcation. In Section 3, a series of numerical simulations show that there are bifurcation and chaos in the discrete-time epidemic model. Finally, we give remarks to conclude this paper in Section 4.

2. Bifurcations

It is easy to visualize that system (5) has three fixed points $P_0(0,0)$, $P_1(a_0/a_1, 0)$, and $P_2(1, (a_0 - a_1)/(1 - a_2))$ when a_1, a_0 are fixed. We can see that the fixed point $P_0(0,0)$ is a saddle. In the following paper, we focus on investigating the bifurcations of P_1, P_2.

Theorem 1. *If $a_0 = a_1$ and $a_0 \neq 2$, (5) undergo a transcritical bifurcation at P_1. Furthermore, the system has three fixed points when $a_0 > a_1$ and has two fixed points when $a_0 \leq a_1$.*

Proof. The Jacobian matrix of (5) at P_1 takes the form

$$J(P_1) = \begin{pmatrix} 1 - a_0 & a_2 - \dfrac{a_0^2}{a_1^2} \\ 0 & \dfrac{a_0^2}{a_1^2} \end{pmatrix}. \tag{6}$$

$J(P_1)$ has eigenvalues $\lambda_1 = 1 - a_0$, $\lambda_2 = a_0^2/a_1^2$. And $a_0 \neq 2$ implies that $|\lambda_1| \neq 1$.

Let

$$\xi_n = x_n - \frac{a_0}{a_1}, \qquad \eta_n = y_n, \qquad c_n = a_0 - a_1, \tag{7}$$

equation (5) becomes

$$\begin{aligned} \xi_{n+1} &= (1 - a_1)\xi_n + (a_2 - 1)\eta_n + F_1(\xi_n, \eta_n, c_n), \\ \eta_{n+1} &= \eta_n + F_2(\xi_n, \eta_n, c_n), \\ c_{n+1} &= c_n, \end{aligned} \tag{8}$$

where

$$\begin{aligned} F_1(\xi_n, \eta_n, c_n) = &-(a_1 + \eta_n)\xi_n^2 - \frac{2\xi_n\eta_n c_n}{a_1} \\ &- (2\eta_n + c_n)\xi_n - \frac{\eta_n c_n^2}{a_1^2} - \frac{2c_n\eta_n}{a_1}, \\ F_2(\xi_n, \eta_n, c_n) = &\, 2\xi_n\eta_n + \frac{2c_n\eta_n}{a_1} + \xi_n^2\eta_n + \frac{\eta_n c_n^2}{a_1^2} + \frac{2\xi_n\eta_n c_n}{a_1}. \end{aligned} \tag{9}$$

By the following transformation:

$$\begin{pmatrix} \xi_n \\ \eta_n \\ c_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \dfrac{a_1}{a_2 - 1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_n \\ q_n \\ \sigma_n \end{pmatrix}, \tag{10}$$

equation (8) becomes

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \\ \sigma_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - a_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_n \\ q_n \\ \sigma_n \end{pmatrix} + \begin{pmatrix} f_1(p_n, q_n, \sigma_n) \\ f_2(p_n, q_n, \sigma_n) \\ 0 \end{pmatrix}, \tag{11}$$

where

$$\begin{aligned} f_1(p_n, q_n, \sigma_n) &= \left(\frac{a_1 + a_2 - 1}{1 - a_2} q_n - a_1\right) p_n^2 \\ &\quad + \left[\frac{2(a_1 + a_2 - 1)(\sigma_n + 1)(q_n + q_n^2)}{a_1(1 - a_2)} - \sigma_n - 2a_1 q_n\right] p_n \\ &\quad + \frac{2(a_1 + a_2 - 1) q_n^3}{a_1(1 - a_2)} + \left[\frac{2(a_1 + a_2 - 1)(\sigma_n + a_1)}{a_1(1 - a_2)} - a_1\right] q_n^2 \\ &\quad + \left[\frac{(a_1 a_2 + 2a_2 - 2)\sigma_n}{a_1(1 - a_2)} + \frac{2(a_1 + a_2 - 1)\sigma_n^2}{a_1^3(1 - a_2)}\right] q_n, \\ f_2(p_n, q_n, \sigma_n) &= q_n^3 + 2\left(1 + p_n + \frac{\sigma_n}{a_1}\right) q_n^2 \\ &\quad + 2p_n q_n + \frac{2q_n\sigma_n}{a_1} + \frac{(a_1 p_n + \sigma_n)^2 q_n}{a_1^2}. \end{aligned} \tag{12}$$

Then, we can consider

$$p_n = g(q_n, \sigma_n) = \varepsilon_1 q_n^2 + \varepsilon_2 q_n \sigma_n + \varepsilon_3 \sigma_n^2 + o\left((|p_n| + |q_n|)^3\right), \tag{13}$$

which must satisfy

$$\begin{aligned} g(q_{n+1}, \sigma_{n+1}) &= g(q_n + f_2(g(q_n, \sigma_n), q_n, \sigma_n), \sigma_{n+1}) \\ &= (1 - a_1) g(q_n, \sigma_n) + f_1(g(q_n, \sigma_n), q_n, \sigma_n). \end{aligned} \tag{14}$$

Thus, we have

$$\varepsilon_1 = \frac{a_1 a_2 + 2a_2 - 2}{a_1(1 - a_2)}, \qquad \varepsilon_2 = \frac{a_1 a_2 + 2a_2 - 2}{a_1^2(1 - a_2)}, \qquad \varepsilon_3 = 0. \tag{15}$$

And (8) is restricted to the center manifold, which is given by

$$\begin{aligned} f: q_{n+1} = &\frac{2\left(a_1 a_2 + 2a_2 - 2\right)}{a_1\left(1-a_2\right)}\left(q_n^3 + \frac{q_n^2 \sigma_n}{a_1}\right) \\ &+ q_n^3 + 2q_n^2 + q_n + \frac{2q_n^2\sigma_n}{a_1} + \frac{2q_n\sigma_n}{a_1} \\ &+ o\left(\left(\left|q_n\right| + \left|\sigma_n\right|\right)^4\right). \end{aligned} \tag{16}$$

Since

$$\begin{gathered} f\left(0,\sigma_n\right) = 0, \qquad \left.\frac{\partial f}{\partial q}\right|_{(0,0)} = 1, \qquad \left.\frac{\partial^2 f}{\partial q^2}\right|_{(0,0)} = 4, \\ \left.\frac{\partial^2 f}{\partial q \partial \sigma}\right|_{(0,0)} = -\frac{2}{a_1} \neq 0, \end{gathered} \tag{17}$$

system (5) undergoes a transcritical bifurcation at P_1. This proves the theorem. □

Theorem 2. *Equation* (5) *undergoes a flip bifurcation at* P_1 *when* $a_0 = 2$, $a_1 \neq a_0$. *Furthermore, the stable periodic-2 point bifurcates from this fixed point.*

Proof. Let

$$\xi_n = x_n - \frac{a_0}{a_1}, \qquad \rho_n = y_n, \qquad c_n = a_0 - 2, \tag{18}$$

system (5) becomes

$$\begin{aligned} \xi_{n+1} &= -\xi_n + \left(a_2 - \frac{4}{a_1^2}\right)\rho_n + F_1\left(\xi_n, c_n, \rho_n\right), \\ c_{n+1} &= -c_n, \\ \rho_{n+1} &= \frac{4}{a_1^2}\rho_n + F_2\left(\xi_n, c_n, \rho_n\right), \end{aligned} \tag{19}$$

where

$$\begin{aligned} F_1\left(\xi_n, c_n, \rho_n\right) = &-\left(a_1 + \rho_n\right)\xi_n^2 \\ &- \left(c_n + \frac{2\left(c_n+2\right)\rho_n}{a_1}\right)\xi_n - \frac{4c_n + c_n^2}{a_1^2}\rho_n, \\ F_2\left(\xi_n, c_n, \rho_n\right) = &\rho_n \xi_n^2 + \frac{4c_n + c_n^2}{a_1^2}\rho_n + \frac{2\left(c_n+2\right)\rho_n}{a_1}\xi_n. \end{aligned} \tag{20}$$

By the following transformation:

$$\begin{pmatrix} \xi_n \\ c_n \\ \rho_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{\left(a_1^2+4\right)}{\left(a_2 a_1^2 - 4\right)} \end{pmatrix} \begin{pmatrix} \phi_n \\ u_n \\ v_n \end{pmatrix}, \tag{21}$$

equation (19) becomes

$$\begin{pmatrix} \phi_{n+1} \\ u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \dfrac{4}{a_1^2} \end{pmatrix} \begin{pmatrix} \phi_n \\ u_n \\ v_n \end{pmatrix} + \begin{pmatrix} f_1\left(\phi_n, u_n, v_n\right) \\ 0 \\ f_2\left(\phi_n, u_n, v_n\right) \end{pmatrix}, \tag{22}$$

where

$$\begin{aligned} &f_1\left(\phi_n, u_n, v_n\right) \\ &\quad = \left(\frac{a_1^2 + a_2 a_1^2}{4 - a_2 a_1^2} v_n - a_1\right)\phi_n^2 \\ &\qquad + \left[\frac{2\left(1+a_2\right)a_1^2 v_n^2}{4 - a_2 a_1^2}\right. \\ &\qquad\qquad \left. + \frac{2a_1\left(u_n - 2 + a_2 a_1^2 + a_2 u_n + 2a_2\right)v_n}{4 - a_2 a_1^2} - u_n\right]\phi_n \\ &\qquad + \frac{a_1^2 + a_2 a_1^2}{4 - a_2 a_1^2} v_n^3 + \frac{a_1\left(a_2 a_1^2 + 2a_2 u_n + 4a_2 + 2u_n\right)}{4 - a_2 a_1^2} v_n^2 \\ &\qquad + \frac{\left(1+a_2\right)u_n^2 v_n}{4 - a_2 a_1^2} + \frac{a_2\left(4 + a_1^2\right)u_n v_n}{4 - a_2 a_1^2}, \\ &f_2\left(\phi_n, u_n, v_n\right) = v_n^3 + 2\left(\phi_n + \frac{u_n + 4}{a_1}\right)v_n^2 \\ &\qquad\qquad + \frac{\left(a_1\phi_n + u_n\right)\left(a_1\phi_n + u_n + 4\right)v_n}{a_1^2}. \end{aligned} \tag{23}$$

Then, we can consider $v_n = g(\phi_n, u_n) = \varepsilon_1 \phi_n^2 + \varepsilon_2 \phi_n u_n + \varepsilon_3 u_n^2 + o((|\phi_n| + |u_n|)^3)$, which must satisfy

$$\begin{aligned} &g\left(-\phi_n + f_1\left(\phi_n, g\left(\phi_n, u_n\right), v_n\right), u_{n+1}\right) \\ &\quad = \frac{4}{a_1^2} g\left(\phi_n, u_n\right) + g^3\left(\phi_n, u_n\right) \\ &\qquad + 2\left(\phi_n + \frac{u_n + 4}{a_1}\right) g^2\left(\phi_n, u_n\right) \\ &\qquad + \frac{\left(a_1\phi_n + u_n\right)\left(a_1\phi_n + u_n + 4\right) g\left(\phi_n, u_n\right)}{a_1^2}. \end{aligned} \tag{24}$$

Thus, we have $\varepsilon_k = 0$, $k = 1, 2, 3$. We obtain the center manifold as follows:

$$\begin{aligned} v_n = g\left(\phi_n, u_n\right) = &\varepsilon_4 \phi_n^2 u_n + \varepsilon_5 \phi_n u_n^2 \\ &+ \varepsilon_6 u_n^3 + o\left(\left(\left|\phi_n\right| + \left|u_n\right|\right)^4\right). \end{aligned} \tag{25}$$

And (19) is restricted to the center manifold, which is given by

$$f: \phi_{n+1} = -\phi_n + f_1\left(\phi_n, g\left(\phi_n, u_n\right), v_n\right). \tag{26}$$

Direct calculations show that

$$\left(\frac{\partial f}{\partial \sigma}\frac{\partial^2 f}{\partial p^2}+2\frac{\partial^2 f}{\partial p\partial \sigma}\right)\Bigg|_{(0,0)}=-2,$$
$$\left(\frac{1}{2}\left(\frac{\partial^2 f}{\partial p^2}\right)^2+\frac{1}{3}\left(\frac{\partial^3 f}{\partial p^3}\right)\right)\Bigg|_{(0,0)}=\frac{a_1^2}{2}. \tag{27}$$

Hence, system (5) undergoes a flip bifurcation at P_1. This completes the proof. □

The positive fixed point is so important to the biological system that people usually are very interested in it. We will next pay attention to the only positive fixed point P_2 of (5).

Theorem 3. *Equation* (5) *undergoes a transcritical bifurcation at* P_2 *when* $a_1 = a_0$, $a_1 \neq 2$, *and* $\Delta > 0$, *where*

$$\Delta=\frac{(1+a_2)^2a_0^2}{(1-a_2)^2}+8(a_1-a_0)+\frac{4a_1^2a_2^2-4a_2(a_2+1)}{(1-a_2)^2}. \tag{28}$$

Theorem 4. *If* $\Delta > 0$, $a_1 \neq a_0$, *and* $a_0 = (2-2a_2-a_1+3a_1a_2)/2a_2$, (5) *undergoes a flip bifurcation at* P_2.

Since the analysis is similar to the case at P_1, the previously mentioned proofs are omitted.

We next give the condition of existence of Hopf bifurcation by using the hopf bifurcation theorem in [17].

The characteristic equation of the Jacobian matrix $J(P_2)$ can be written as $\lambda^2 - t_2\lambda + d_2 = 0$, where

$$t_2 = 2 + a_0 - 2a_1 - \frac{2(a_0-a_1)}{1-a_2},$$
$$d_2 = t_2 - 1 + 2a_0 - 2a_1. \tag{29}$$

The eigenvalues of the Jacobian matrix of (5) at P_2 are $\lambda_{1,2} = (t_2 \pm \sqrt{t_2^2-4d_2})/2$. The eigenvalues $\lambda_{1,2}$ are complex conjugates for $\Delta < 0$. We translate the fixed point $P_2(1,(a_0-a_1)/(1-a_2))$ to the origin by $x_n = \varphi_n - 1$, $y_n = \delta_n - ((a_0-a_1)/(1-a_2))$, and the system (5) becomes

$$\varphi_{n+1} = (t_2-1)\varphi_n + (a_2-1)\delta_n - \left(\delta_n + \frac{a_0-a_1a_2}{1-a_2}\right)\varphi_n^2 - 2\varphi_n\delta_n,$$
$$\delta_{n+1} = \frac{2(a_0-a_1)}{1-a_2}\varphi_n + \delta_n + 2\varphi_n\delta_n + \left(\delta_n+\frac{a_0-a_1}{1-a_2}\right)\varphi_n^2. \tag{30}$$

The eigenvalues of the matrix associated with the linearized map (30) at fixed point $(0,0)$ are complex conjugates which are written as $\lambda,\overline{\lambda} = (t_2 \pm i\sqrt{4d_2-t_2^2})/2$, where

$$|\lambda| = \sqrt{1+3a_0-4a_1-\frac{2(a_0-a_1)}{1-a_2}}. \tag{31}$$

Now assume that $3a_2 < 1$, and let $a_{00} = (2a_1(1-2a_2))/(1-3a_2)$. Then we have

$$\frac{d(|\lambda|)}{da_0}\Bigg|_{a_0=a_{00}} = \frac{1-3a_2}{2(1-a_2)} > 0, \quad |\lambda(a_{10})| = 1,$$
$$\lambda(a_{00}) = 1 + \frac{a_1a_2-a_1}{1-3a_2} \pm i\frac{\sqrt{a_1(1-a_2)(2-a_1+a_1a_2-6a_2)}}{1-3a_2},$$
$$\lambda^j \neq 1, \quad j = 1,2,3,4. \tag{32}$$

Let

$$\mathrm{T} = \begin{pmatrix} a_2-1 & 0 \\ \frac{2-t_2}{2} & -\frac{\sqrt{4d_2-t_2^2}}{2}\end{pmatrix},$$
$$\begin{pmatrix}\varphi_n\\ \delta_n\end{pmatrix} = \mathrm{T}\begin{pmatrix}\phi_n\\ \sigma_n\end{pmatrix}. \tag{33}$$

The system (30) becomes

$$\begin{pmatrix}\phi_{n+1}\\ \sigma_{n+1}\end{pmatrix} = \begin{pmatrix}\frac{t_2}{2} & -\frac{\sqrt{4d_2-t_2^2}}{2}\\ \frac{\sqrt{4d_2-t_2^2}}{2} & \frac{t_2}{2}\end{pmatrix}\begin{pmatrix}\phi_n\\ \sigma_n\end{pmatrix} + \begin{pmatrix}f_1(\phi_n,\sigma_n)\\ f_2(\phi_n,\sigma_n)\end{pmatrix}, \tag{34}$$

where

$$f_1(\phi_n,\sigma_n) = \left(\frac{a_0+a_0a_2}{2}-a_1a_2\right)\phi_n^3 + \left[\frac{(a_2-1)\sqrt{4d_2-t_2^2}}{2}\sigma_n + \frac{a_2(2a_0-a_1-a_1a_2)}{a_2-1}\right]\phi_n^2 + \sqrt{4d_2-t_2^2}\sigma_n\phi_n,$$
$$f_2(\phi_n,\sigma_n) = \frac{(2-t_2)(2a_2-t_2)(a_2-1)}{2\sqrt{4d_2-t_2^2}}\phi_n^3 + (a_2-1)\left(a_2-\frac{t_2}{2}\right)\phi_n^2\sigma_n - \frac{(2-t_2)(2-t_2+a_1a_2-a_0)}{\sqrt{4d_2-t_2^2}}\phi_n^2 - \frac{2(a_2-1)(2-t_2-a_0+a_1)}{\sqrt{4d_2-t_2^2}}\phi_n^2 + (2a_2-t_2)\sigma_n\phi_n. \tag{35}$$

Notice that (34) is exactly on the center manifold in the form, in which the coefficient l [18] is given by $l = -\mathrm{Re}[((1-2\lambda)\overline{\lambda}^2/(1-\lambda))l_{11}l_{20}] - (1/2)|l_{11}|^2 - |l_{02}|^2 + \mathrm{Re}(\overline{\lambda}l_{21})$, where

$$
\begin{aligned}
l_{11} &= \frac{1}{4}\left[\left(f_{1\phi\phi} + f_{1\sigma\sigma}\right) + \left(f_{2\phi\phi} + f_{2\sigma\sigma}\right)i\right] \\
&= \frac{3a_0a_2 - 2a_1a_2^2 - a_0}{4a_2 - 4} - \frac{1}{4\sqrt{4d_2 - t_2^2}} \\
&\quad \times \left[(4-2t_2)(2 - t_2 + a_1a_2 - a_0)\right. \\
&\qquad \left. + (a_2 - 1)(4 - 2t_2 - 2a_0 + 2a_1)\right] i, \\
l_{20} &= \frac{1}{8}\left[\left(f_{1\phi\phi} - f_{1\sigma\sigma} + 2f_{2\phi\sigma}\right) + \left(f_{2\phi\phi} - f_{2\sigma\sigma} - 2f_{1\phi\sigma}\right)i\right] \\
&= \frac{1}{4}(a_2 - a_1a_2 - 2 + 2a_0) - \frac{1}{8\sqrt{4d_2 - t_2^2}} \\
&\quad \times \left[(4-2t_2)(2 - t_2 + a_1a_2 - a_0) + (a_2 - 1)\right. \\
&\qquad \left. \times (4 - 2t_2 - 2a_0 + 2a_1) - 8d_2 + 2t_2^2\right] i, \\
l_{02} &= \frac{1}{8}\left[\left(f_{1\phi\phi} - f_{1\sigma\sigma} - 2f_{2\phi\sigma}\right) + \left(f_{2\phi\phi} - f_{2\sigma\sigma} + 2f_{1\phi\sigma}\right)i\right] \\
&= \frac{1}{8}(2a_0 - 2a_0a_2 - 2 - 4a_2 + 3t_2) \\
&\quad + \frac{1}{8}\left(2a_0 - a_1a_2 - 2 + t_2 + 2\sqrt{4d_2 - t_2^2}\right) i, \\
l_{21} &= \frac{1}{16}\left[\left(f_{1\phi\phi\phi} + f_{1\phi\sigma\sigma} + f_{2\phi\phi\sigma} + f_{2\sigma\sigma\sigma}\right)\right. \\
&\qquad \left. + \left(f_{2\phi\phi\phi} + f_{2\phi\sigma\sigma} - f_{1\phi\phi\sigma} - f_{1\sigma\sigma\sigma}\right)i\right] \\
&= \frac{1}{8}(1 - a_2)(3 + a_2 - 2t_2) \\
&\quad + \frac{1}{8\sqrt{4d_2 - t_2^2}}\left(6a_2 - 3t_2 + 3a_2t_2 + 8d_2 + t_2^2\right) i.
\end{aligned}
\tag{36}
$$

From the previous analysis, we have Theorem 5.

Theorem 5. *System* (5) *undergoes a Hopf bifurcation at fixed point* P_2, *if* $l \neq 0$, $\Delta < 0$ *and*

$$a_1 \neq \frac{2(1-a_2) + 2a_2a_1}{1 + a_3}. \tag{37}$$

3. Numerical Simulations

With development of scientific computation, computer becomes a powerful tool to study nonlinear systems, especially for a system without explicit solution. It not only is able to explore new complex dynamical behaviors, for example, periodic orbits and chaos in different regions [16, 19], but also can do rigorous analysis by combining modern dynamical systems theory and reliable computation, for example, topological horseshoes [20–23]. In this section, we use the bifurcation diagrams, Lyapunov exponents, and phase portraits to illustrate the previous analytic results and find new dynamics of the model (5) as the parameters vary. The attractors of (5) are also given by using the method in [20]. The bifurcation parameters are considered in the following three cases:

(I) varying a_0 in range $1.6 \leq a_0 \leq 2.2$ and fixing $a_1 = 0.9$, $a_3 = 0.01$;

(II) varying a_1 in range $0.8 \leq a_1 \leq 1.2$ and fixing $a_0 = 2$, $a_3 = 0.01$;

(III) varying a_2 in range $0.3 \leq a_2 \leq 0.6$ and fixing $a_0 = 2$, $a_1 = 0.9$.

For case (I). The bifurcation diagram of (5) in a_0-x_n and a_0-y_n space for $a_1 = 0.9$, $a_3 = 0.01$ is given in Figures 1(a) and 1(b) to show the dynamical changes of susceptible and infective, respectively, as a_0 varies. There is obvious phenomenon of bifurcation when we select the stepsize to be 10^{-4}. The spectrum of Lyapunov exponents of the system (5) with respect to parameter a_1 is given in Figure 1(c).

Moreover, we can see that the orbit with initial values $(1, 1)$ approaches to the stable fixed point P_2 for $a_0 < 1.8$ approximately, and Hopf bifurcation occurs at $a_0 \approx 1.8$. When increased to $a_0 \approx 2.05$, (5) becomes stable. In Figures 1(b) and 1(c), we observe the period -4,8 windows within the chaotic regions and boundary crisis at $a_0 \approx 2.05$. For $a_0 \in (1.95, 2.05)$ the maximum Lyapunov exponents are positive which correspond to chaotic region. To well see the dynamics, the attractor in the system (5) and time series of x_n and y_n are given in Figures 1(d) and 1(e), respectively.

For case (II). Figure 2(a) is the bifurcation diagram of (5) for x_n and y_n and depicts that there are period -4,5 windows within the chaotic regions and boundary crisis at $a_1 \approx 1$. Figure 2(b) shows the spectrum of Lyapunov exponents of the system (5) with respect to parameter a_1. For $a_1 \in (0.85, 0.95)$ the maximum Lyapunov exponents are positive which correspond to chaotic region.

For case (III). Figures 3(a) and 3(b) are the bifurcation diagrams of (5) for x_n and y_n, respectively, and depict that there are period -2,4 windows within the chaotic regions and boundary crisis at $a_2 = 0.5$. Figure 3(c) shows the spectrum of Lyapunov exponents of (5) with respect to parameter a_2. Figure 3(d) is the attractor of (5) with $a_2 = 0.46$.

4. Conclusion

In this paper, we investigate the behaviors of a discrete-time SIS epidemic model with nonlinear incidence rate, and find many complex and new interesting dynamical phenomena. Without the recovery rate of infectious hosts, (1) becomes the SI model (see [16]). Our theoretical analysis and numerical simulations have demonstrated that the model exhibits the variety of dynamical behaviors, which includes that the discrete epidemic model undergoes transcritical bifurcation,

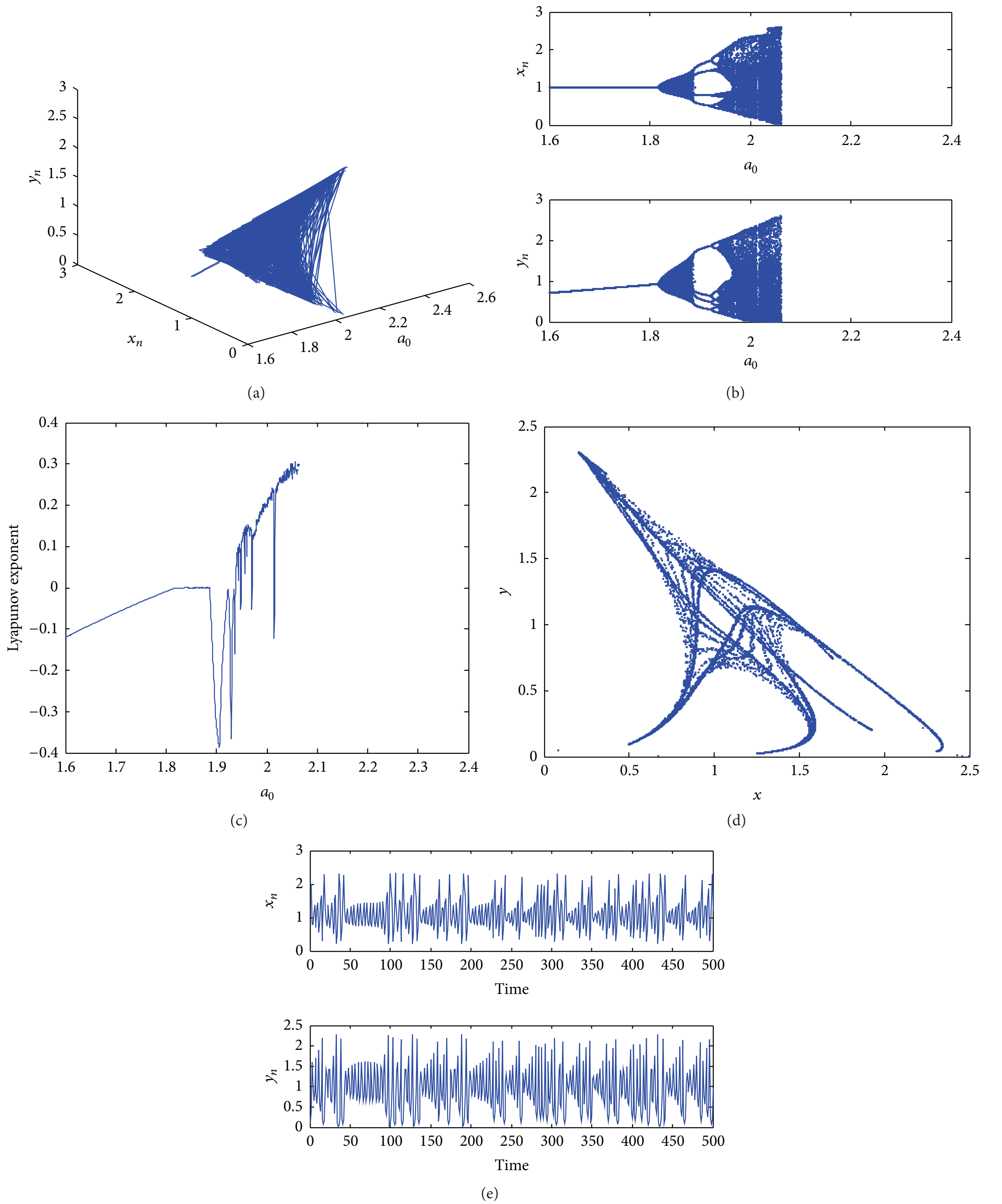
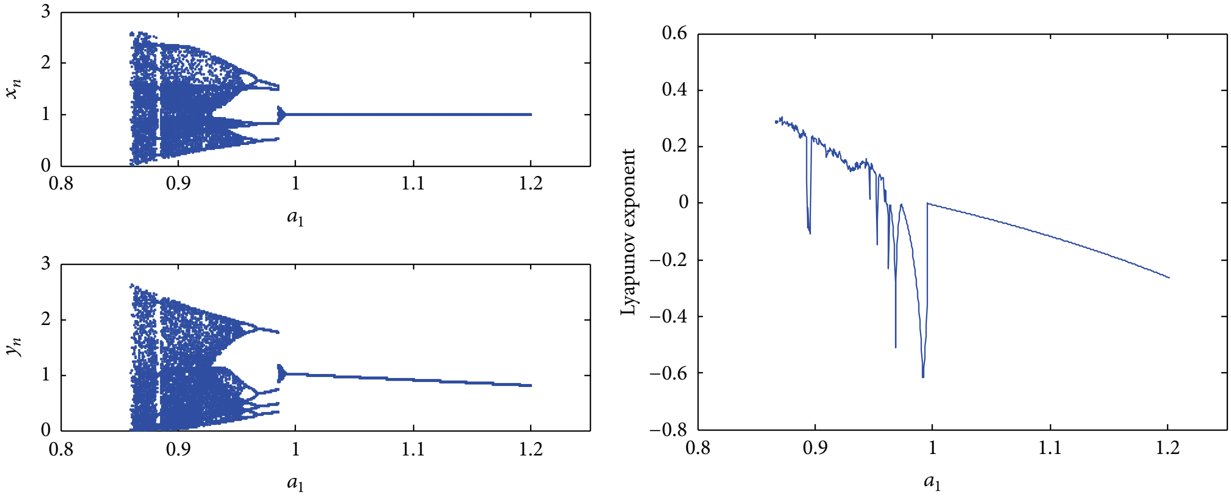
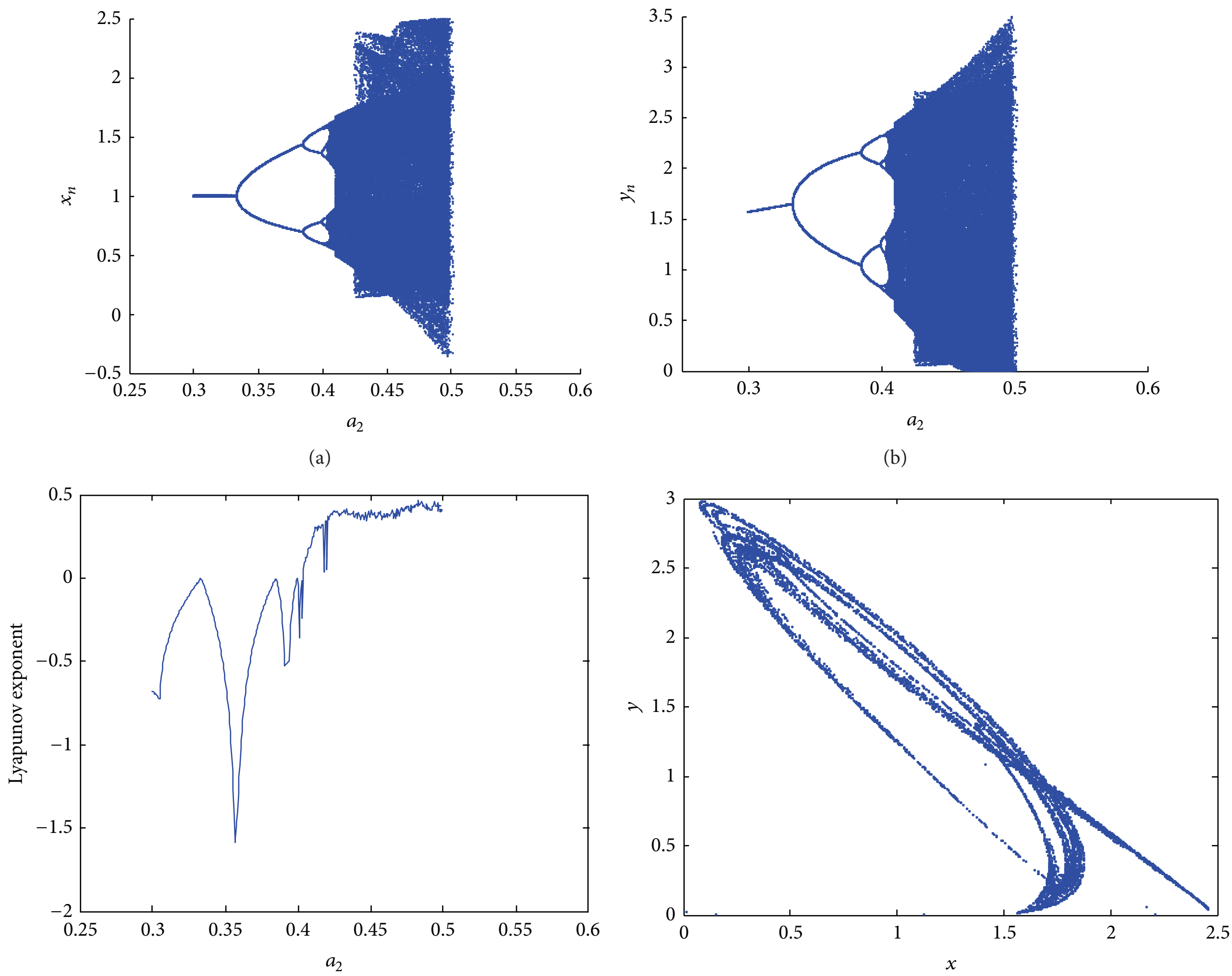


FIGURE 1: (a) Bifurcation diagram in (a_0, x_n, y_n) space. (b) The bifurcation diagram in a_0-x_n plane and in a_0-y_n plane. (c) Spectrum of Lyapunov exponents corresponding to (b). (d) The attractor of (5) with $a_0 = 2$ (e) Time series of x_n and y_n of the same parameters.



(a) (b)

Figure 2



(c) (d)

Figure 3

flip bifurcation, Hopf bifurcation, and chaos. Furthermore, chaos can cause the population to run a higher risk of extinction due to the unpredictability [24, 25]. Thus, how to control chaos in the epidemic model is very important, which needs further consideration.

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References

[1] L. Cai, X. Li, M. Ghosh, and B. Guo, "Stability analysis of an HIV/AIDS epidemic model with treatment," *Journal of Computational and Applied Mathematics*, vol. 229, no. 1, pp. 313–323, 2009.

[2] Y.-H. Hsieh, C.-C. King, C. W. S. Chen, M.-S. Ho, S.-B. Hsu, and Y.-C. Wu, "Impact of quarantine on the 2003 SARS outbreak: a retrospective modeling study," *Journal of Theoretical Biology*, vol. 244, no. 4, pp. 729–736, 2007.

[3] S. Iwami, Y. Takeuchi, X. Liu, and S. Nakaoka, "A geographical spread of vaccine-resistance in avian influenza epidemics," *Journal of Theoretical Biology*, vol. 259, no. 2, pp. 219–228, 2009.

[4] D. Summers, J. G. Cranford, and B. P. Healey, "Chaos in periodically forced discrete-time ecosystem models," *Chaos, Solitons and Fractals*, vol. 11, no. 14, pp. 2331–2342, 2000.

[5] J. Vandermeer, L. Stone, and B. Blasius, "Categories of chaos and fractal basin boundaries in forced predator-prey models," *Chaos, Solitons and Fractals*, vol. 12, no. 2, pp. 265–276, 2001.

[6] A. Gray, D. Greenhalgh, X. Mao, and J. Pan, "The SIS epidemic model with Markovian switching," *Journal of Mathematical Analysis and Applications*, vol. 394, no. 2, pp. 496–516, 2012.

[7] A. Gray, D. Greenhalgh, L. Hu, X. Mao, and J. Pan, "A stochastic differential equation SIS epidemic model," *SIAM Journal on Applied Mathematics*, vol. 71, no. 3, pp. 876–902, 2011.

[8] J. Liu and T. Zhang, "Bifurcation analysis of an SIS epidemic model with nonlinear birth rate," *Chaos, Solitons & Fractals*, vol. 40, no. 3, pp. 1091–1099, 2009.

[9] D. Gao and S. Ruan, "An SIS patch model with variable transmission coefficients," *Mathematical Biosciences*, vol. 232, no. 2, pp. 110–115, 2011.

[10] Y. Li and J. Cui, "The effect of constant and pulse vaccination on SIS epidemic models incorporating media coverage," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 5, pp. 2353–2365, 2009.

[11] Q. Wu and X. Fu, "Modelling of discrete-time SIS models with awareness interactions on degree-uncorrelated networks," *Physica A*, vol. 390, no. 3, pp. 463–470, 2011.

[12] P. L. Salceanu and H. L. Smith, "Persistence in a discrete-time stage-structured fungal disease model," *Journal of Biological Dynamics*, vol. 3, no. 2-3, pp. 271–285, 2009.

[13] J. E. Franke and A.-A. Yakubu, "Disease-induced mortality in density-dependent discrete-time S-I-S epidemic models," *Journal of Mathematical Biology*, vol. 57, no. 6, pp. 755–790, 2008.

[14] J. Li, Z. Ma, and F. Brauer, "Global analysis of discrete-time SI and SIS epidemic models," *Mathematical Biosciences and Engineering*, vol. 4, no. 4, pp. 699–710, 2007.

[15] L. J. S. Allen, Y. Lou, and A. L. Nevai, "Spatial patterns in a discrete-time SIS patch model," *Journal of Mathematical Biology*, vol. 58, no. 3, pp. 339–375, 2009.

[16] L. Li, G.-Q. Sun, and Z. Jin, "Bifurcation and chaos in an epidemic model with nonlinear incidence rates," *Applied Mathematics and Computation*, vol. 216, no. 4, pp. 1226–1234, 2010.

[17] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, vol. 42, Springer, New York, NY, USA, 1983.

[18] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, vol. 2, Springer, New York, NY, USA, 1990.

[19] T. Chen, J. He, and Q. Yin, "Dynamics evolution of credit risk contagion in the CRT market," *Dynamics in Nature and Society*, vol. 2013, Article ID 206201, 9 pages, 2013.

[20] Q. Li, L. Zhang, and F. Yang, "An algorithm to automatically detect the Smale horseshoes," *Discrete Dynamics in Nature and Society*, Article ID 283179, 9 pages, 2012.

[21] Q. Li and X.-S. Yang, "A simple method for finding topological horseshoes," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 20, no. 2, pp. 467–478, 2010.

[22] X.-S. Yang, "Topological horseshoes and computer assisted verification of chaotic dynamics," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 19, no. 4, pp. 1127–1145, 2009.

[23] L. Qingdu and T. Song, "Algorithm for finding horseshoes in three-dimensinal hyperchaltic maps and its application," *Acta Physica Sinica*, vol. 62, no. 2, Article ID 020510, 2013.

[24] M. P. Hassell, H. N. Comins, and R. M. May, "Spatial structure and chaos in insect population dynamics," *Nature*, vol. 353, pp. 255–258, 1991.

[25] A. A. Berryman and J. A. Millstein, "Are ecological systems chaotic and if not, why not?" *Trends in Ecology & Evolution*, vol. 4, no. 1, pp. 26–28, 1989.