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Research Article

A Fixed Point Theorem for Contraction Type Maps in Partially Ordered Metric Spaces and Application to Ordinary Differential Equations

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We present a fixed point theorem for generalized contraction in partially ordered complete metric spaces. As an application, we give an existence and uniqueness for the solution of a periodic boundary value problem.

1. Introduction

The contraction mapping theorem and the abstract monotone iterative technique are well known and are applicable to a variety of situations. Recently, there has been a trend to weaken the requirement on the contraction by considering metric spaces endowed with partial order (see [1–7]). It is of interest to determine if it is still possible to establish the existence of a unique fixed point assuming that the operator considered is monotone in such a setting. Such a fixed point theorem is useful, for example, in establishing the existence of a unique solution to periodic boundary value problems, besides many others.

That approach was initiated by Ran and Reurings in [8], where some applications to matrix equations were studied. This fixed point theorem was refined and extended in [7, 9] and applied to the periodic boundary value problem in the monotone case. In this paper, we consider a special case of the following periodic boundary value problem

$$\begin{gathered} u'(t) = f(t, u(t)) \quad \text{if } t \in I = [0, T], \\ u(0) = u(T), \end{gathered} \tag{1.1}$$

where $T > 0$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous function.

Definition 1.1. A lower solution for (1.1) is a function $u \in C^1(I, \mathbb{R})$ such that

$$u'(t) \leq f(t, u(t)) \quad \text{for } t \in I = [0, T],$$
$$u(0) \leq u(T). \tag{1.2}$$

Very recently, Harjani and Sadarangani [4] proved the following existence theorem.

Theorem 1.2. *Consider problem* (1.1) *with* $f : I \times \mathbb{R} \to \mathbb{R}$ *is continuous and suppose that there exists* $\lambda > 0$ *such that for* $x, y \in \mathbb{R}$ *with* $y \geq x$

$$0 \leq f(t, y) + \lambda y - [f(t, x) + \lambda x] \leq \lambda \phi(y - x), \tag{1.3}$$

where $\phi : [0, \infty) \to [0, \infty)$ *can be written by* $\phi(x) = x - \psi(x)$ *with* $\psi : [0, \infty) \to [0, \infty)$ *continuous increasing positive in* $(0, \infty)$, $\psi(0) = 0$ *and* $\lim_{t \to \infty} \psi(t) = \infty$. *Then the existence of a lower solution of* (1.1) *provides the existence of a unique solution of* (1.1).

In Section 2, we prove a new fixed point theorem in partially ordered complete metric spaces. In Section 3, existence of a unique solution for problem (1.1) is obtained under suitable conditions.

2. Fixed Point Theorem

Let S denote the class of those functions $\alpha : [0, \infty) \to [0, 1)$ which satisfies the condition

$$\limsup_{s \to t^+} \alpha(s) < 1, \quad \forall t \in [0, \infty). \tag{2.1}$$

We prove the main theorem of the paper as follows.

Theorem 2.1. *Let* $(X, \preccurlyeq)$ *be a partially order metric space that there exists a metric* d *in* X *such that* (X, d) *is a complete metric space. Let* $f : X \to X$ *be an increasing mapping such that there exists* $x_0 \in X$ *with* $x_0 \preccurlyeq f(x_0)$. *Suppose that there exists* $\alpha \in S$ *such that*

$$d(f(x), f(y)) \leq \alpha(d(x, y)) d(x, y), \tag{2.2}$$

for all comparable $x, y \in X$. *If*

$$f \text{ is continuous} \tag{2.3}$$

or

$$\text{if an increasing sequence } \{x_n\} \to x \text{ in } X, \text{ then } x_n \preccurlyeq x, \quad \forall n \in \mathbb{N}. \tag{2.4}$$

Besides, if

$$\text{for each } x, y \in X, \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y, \tag{2.5}$$

then f have a unique fixed point.

Proof. We first show that f has a fixed point. Since $x_0 \preccurlyeq f(x_0)$ and f is increasing function, we obtain by induction that

$$x_0 \preccurlyeq f(x_0) \preccurlyeq f^2(x_0) \preccurlyeq f^3(x_0) \preccurlyeq \cdots \preccurlyeq f^n(x_0) \preccurlyeq f^{n+1}(x_0) \cdots. \tag{2.6}$$

Put $x_n = f^n(x_0)$, $n = 1, 2, \ldots$. Since $x_n \preccurlyeq x_{n+1}$ for each $n \in \mathbb{N}$ then by (2.2)

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d\Big(f^{n+1}(x_0), f^{n+2}(x_0)\Big) \\ &\le \alpha(d(x_n, x_{n+1}))d(x_n, x_{n+1}) \\ &\le d(x_n, x_{n+1}). \end{aligned} \tag{2.7}$$

And so the sequence $\{d(x_{n+1}, x_n)\}$ is nonincreasing and bounded below. Thus there exists $\tau \ge 0$ such that $\lim_{n\to\infty} d(x_{n+1}, x_n) = \tau$. Since $\limsup_{s\to\tau^+} \alpha(s) < 1$ and $\alpha(\tau) < 1$ then there exist $r \in [0, 1)$ and $\epsilon > 0$ such that $\alpha(s) < r$ for all $s \in [\tau, \tau + \epsilon]$. We can take $\nu \in \mathbb{N}$ such that $\tau \le d(x_{n+1}, x_n) \le \tau + \epsilon$ for all $n \in \mathbb{N}$ with $n \ge \nu$. Then since

$$d(x_{n+1}, x_{n+2}) \le \alpha(d(x_n, x_{n+1}))d(x_n, x_{n+1}) \le rd(x_n, x_{n+1}), \tag{2.8}$$

for all $n \in \mathbb{N}$ with $n \ge \nu$ we have

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \le \sum_{n=1}^{\nu} d(x_n, x_{n+1}) + \sum_{n=1}^{\infty} r^n d(x_\nu, x_{\nu+1}) < \infty, \tag{2.9}$$

and hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ converges to some point $z \in X$. To prove that z is a fixed point of f, if f is a continuous, then

$$z = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f^n(x_0) = \lim_{n\to\infty} f^{n+1}(x_0) = f\left(\lim_{n\to\infty} f^n(x_0)\right) = f(z); \tag{2.10}$$

hence $z = f(z)$. If case (2.4) holds then

$$\begin{aligned} d(f(z), z) &\le d(f(z), f(x_n)) + d(f(x_n), z) \\ &\le \alpha(d(x_n, z))d(x_n, z) + d(x_{n+1}, z) \\ &\le d(x_n, z) + d(x_{n+1}, z). \end{aligned} \tag{2.11}$$

Since $d(x_n, z) \to 0$ then we get $f(z) = z$. To prove the uniqueness of the fixed point, let y be another fixed point of f. From (2.5) there exists $x \in X$ which is comparable to y and z. Monotonicity implies that $f^n(x)$ is comparable to $f^n(y) = y$ and $f^n(z) = z$ for $n = 0, 1, \ldots$. Moreover,

$$\begin{aligned} d(z, f^n(x)) &= d(f^n(z), f^n(x)) \\ &\leq \alpha\Big(d\Big(f^{n-1}(z), f^{n-1}(x)\Big)\Big) d\Big(f^{n-1}(z), f^{n-1}(x)\Big) \\ &\leq d\Big(f^{n-1}(z), f^{n-1}(x)\Big) \\ &= d\Big(z, f^{n-1}(x)\Big). \end{aligned} \tag{2.12}$$

Consequently, the sequence $\zeta_n^z := d(z, f^n(x))$ is nonnegative and decreasing and so $\lim_{n\to\infty} d(z, f^n(x)) = \zeta_z \in \mathbb{R}$. Similarly we can show that the sequence $\zeta_n^y := d(y, f^n(x))$ is nonnegative and decreasing and so $\lim_{n\to\infty} d(y, f^n(x)) = \zeta_y \in \mathbb{R}$. Now similarly the above method we can choose r_1, r_2 in $[0, 1)$ and $\tau_1 \in \mathbb{N}$ such that

$$\begin{aligned} d(z, f^n(x)) &\leq \alpha\Big(d\Big(z, f^{n-1}(x)\Big)\Big) d\Big(z, f^{n-1}(x)\Big) \leq r_1 d\Big(z, f^{n-1}(x)\Big), \\ d(y, f^n(x)) &\leq \alpha\Big(d\Big(y, f^{n-1}(x)\Big)\Big) d\Big(y, f^{n-1}(x)\Big) \leq r_2 d\Big(y, f^{n-1}(x)\Big), \end{aligned} \tag{2.13}$$

for all $n \in \mathbb{N}$ with $n > \tau_1$. Finally

$$d(z, y) \leq d(z, f^n(x)) + d(f^n(x), y) \leq r_1^{n-\tau_1} d(z, f^{\tau_1} x_0) + r_2^{n-\tau_1} d(y, f^{\tau_1} x_0), \tag{2.14}$$

for all $n \in \mathbb{N}$ with $n > \tau_1$. Therefor if in (2.14) taking $n \to \infty$ yields $d(z, y) = 0$. □

3. Application to Ordinary Differential Equation

In this section we present an example where Theorem 2.1 can be applied. This example is inspired in [2, 4, 7].

Definition 3.1. Let $\mathfrak{B}$ denote the class of those functions $\phi : [0, \infty) \to [0, \infty)$ which satisfies the following condition:

(i) ϕ is increasing,

(ii) for each $x > 0$, $\phi(x) < x$,

(iii) $\beta(x) = \phi(x)/x \in S$.

For example, $\phi(x) = ax$, where $0 \leq a < 1$, $\phi(x) = x/(x+1)$, and $\phi(x) = \ln(1 + x)$ are in $\mathfrak{B}$.

Theorem 3.2. *Consider problem* (1.1) *with $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous and suppose that there exists $\lambda > 0$ such that for $x, y \in \mathbb{R}$ with $y \geq x$*

$$0 \leq f(t, y) + \lambda y - [f(t, x) + \lambda x] \leq \lambda \phi(y - x), \tag{3.1}$$

where $\phi \in \mathfrak{B}$*. Then the existence of a lower solution of* (1.1) *provides the existence of a unique solution of* (1.1).

Proof. Problem (1.1) is equivalent to the integral equation

$$u(t) = \int_0^T G(t,s)\left[f(s,u(s)) + \lambda u(s)\right] ds, \tag{3.2}$$

where

$$G(t,s) = \begin{cases} \dfrac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & 0 \le s < t \le T, \\ \dfrac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & 0 \le t < s \le T. \end{cases} \tag{3.3}$$

Define $F : C(I,\mathbb{R}) \to C(I,\mathbb{R})$ by

$$(Fu)(t) = \int_0^T G(t,s)\left[f(s,u(s)) + \lambda u(s)\right] ds. \tag{3.4}$$

Note that if $u \in C(I,\mathbb{R})$ is a fixed point of F then $u \in C^1(I,\mathbb{R})$ is a solution of (1.1). Now, we check that hypotheses in Theorem 2.1 are satisfied. Indeed, $X = C(I,\mathbb{R})$ is a partially ordered set if we define the following order relation in X:

$$x, y \in C(I,\mathbb{R}), \qquad x \le y \quad \text{iff } x(t) \le y(t), \qquad \forall t \in I. \tag{3.5}$$

The mapping F is increasing since, by hypotheses, for $u \ge v$

$$f(t,u) + \lambda u \ge f(t,v) + \lambda v \tag{3.6}$$

which implies for $t \in I$, using that $G(t,s) > 0$ for $(t,s) \in I \times I$, that

$$\begin{aligned} (Fu)(t) &= \int_0^T G(t,s)\left[f(s,u(s)) + \lambda u(s)\right] ds \\ &\ge \int_0^T G(t,s)\left[f(s,v(s)) + \lambda v(s)\right] ds = (Fv)(t). \end{aligned} \tag{3.7}$$

Beside, for $u \ge v$

$$\begin{aligned} d(Fu,Fv) &= \sup_{t\in I}\left|(Fu)(t) - (Fv)(t)\right| \\ &\le \sup_{t\in I}\int_0^T G(t,s)\left|f(s,u(s)) + \lambda u(s) - \left(f(s,v(s)) + \lambda v(s)\right)\right| ds \\ &\le \sup_{t\in I}\int_0^T G(t,s)\cdot\lambda\phi(u(s) - v(s))\, ds. \end{aligned} \tag{3.8}$$

As the function $\phi(x)$ is increasing and $u \geq v$ then $\phi(u(s) - v(s)) \leq \phi(d(u,v))$ we obtain

$$\begin{aligned} d(Fu, Fv) &\leq \sup_{t\in I} \int_0^T G(t,s)\cdot\lambda\phi(u(s)-v(s))ds \\ &\leq \lambda\phi(d(u,v))\sup_{t\in I}\int_0^T G(t,s)ds \\ &= \lambda\phi(d(u,v))\sup_{t\in I}\frac{1}{e^{\lambda T}-1}\left(\frac{1}{\lambda}e^{\lambda(T+s-t)}]_0^t + \frac{1}{\lambda}e^{\lambda(s-t)}]_t^T\right) \\ &= \lambda\phi(d(u,v))\cdot\frac{1}{\lambda(e^{\lambda T}-1)}\left(e^{\lambda T}-1\right) = \phi(d(u,v)). \end{aligned} \tag{3.9}$$

Then for $u \geq v$

$$d(Fu, Fv) \leq \alpha(d(u,v))d(u,v). \tag{3.10}$$

Finally, let $\beta(t)$ be a lower solution of (1.1), and we will show that $\beta \leq F(\beta)$.
Indeed,

$$\beta'(t) + \lambda\beta(t) \leq f(t,\beta(t)) + \lambda\beta(t), \quad \text{for } t \in I. \tag{3.11}$$

Multiplying by $e^{\lambda t}$ we get

$$\left(\beta(t)e^{\lambda t}\right)' \leq \left[f(t,\beta(t)) + \lambda\beta(t)\right]e^{\lambda t}, \quad \text{for } t \in I, \tag{3.12}$$

and this gives us

$$\beta(t)e^{\lambda t} \leq \beta(0) + \int_0^t\left[f(s,\beta(s)) + \lambda\beta(s)\right]e^{\lambda s}ds, \quad \text{for } t \in I \tag{3.13}$$

which implies that

$$\beta(0)e^{\lambda T} \leq \beta(T)e^{\lambda T} \leq \beta(0) + \int_0^T\left[f(s,\beta(s)) + \lambda\beta(s)\right]e^{\lambda s}ds, \tag{3.14}$$

and so

$$\beta(0) \leq \int_0^T \frac{e^{\lambda s}}{e^{\lambda T}-1}\left[f(s,\beta(s)) + \lambda\beta(s)\right]ds. \tag{3.15}$$

From this equality and (3.13) we obtain

$$\beta(t)e^{\lambda t} \leq \int_0^t \frac{e^{\lambda(T+s)}}{e^{\lambda T}-1}\left[f(s,\beta(s)) + \lambda\beta(s)\right]ds + \int_t^T \frac{e^{\lambda s}}{e^{\lambda T}-1}\left[f(s,\beta(s)) + \lambda\beta(s)\right]ds, \tag{3.16}$$

and, consequently,

$$\beta(t) \le \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}\left[f\left(s,\beta(s)\right)+\lambda\beta(s)\right]ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}\left[f\left(s,\beta(s)\right)+\lambda\beta(s)\right]ds. \tag{3.17}$$

Hence

$$\beta(t) \le \int_0^T G(t,s)\left[f\left(s,\beta(s)\right)+\lambda\beta(s)\right]ds = \left(F(\beta)\right)(t), \quad \text{for } t \in I. \tag{3.18}$$

Finally, Theorem 2.1 gives that F has a unique fixed point. □

Example 3.3. Let $\phi_0 : [0,\infty) \to [0,\infty)$ be a defined

$$\phi_0(t) = \begin{cases} 0, & 0 \le t \le 3, \\ 3t-9, & 3 < t \le 4, \\ \dfrac{3}{4}t, & 4 < t. \end{cases} \tag{3.19}$$

Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and suppose that there exists $\lambda > 0$ such that for $x, y \in \mathbb{R}$ with $y \ge x$

$$0 \le f(t,y) + \lambda y - \left[f(t,x)+\lambda x\right] \le \lambda \phi_0(y-x). \tag{3.20}$$

Then be Theorem 2.1, the existence of a lower solution for (1.1) provides the existence of a unique solution of (1.1).

The example discussed above cannot be the result of Harjani and Sadarangani noted Theorem 1.2, because $\psi(x) = x - \phi_0(x)$ is not increasing.

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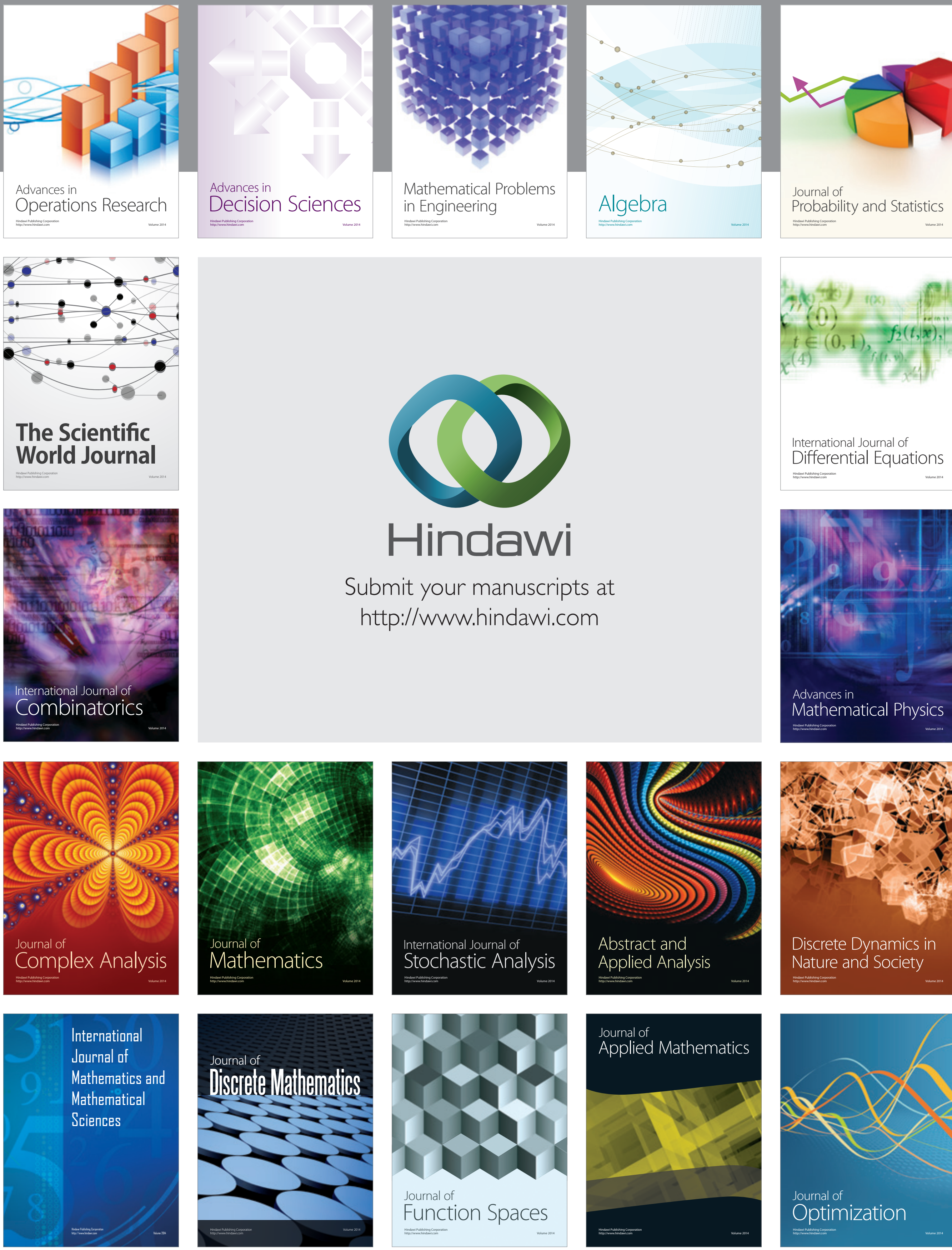
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