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Research Article

Uniform Convergence and Transitive Subsets

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Let (X, d) be a metric space and a sequence of continuous maps $f_n : X \to X$ that converges uniformly to a map f. We investigate the transitive subsets of f_n whether they can be inherited by f or not. We give sufficient conditions such that the limit map f has a transitive subset. In particular, we show the transitive subsets of f_n that can be inherited by f if f_n converges uniformly strongly to f.

1. Introduction

A topological dynamical system is a pair (X, f), where X is a compact metric space with metric d and $f : X \to X$ is a continuous map. When X is finite, it is a discrete space and there is no any nontrivial convergence. Hence, we assume that X contains infinitely many points. Define $\mathbb{N}$ by the set of all positive integers.

In [1], Blanchard and Huang introduced the concepts of weakly mixing subset and partial weak mixing, derived from a result given by Xiong and Yang [2] and showed "partial weak mixing implies Li-Yorke chaos" and "Li-Yorke chaos does not imply partial weak mixing". A closed set A with at least two elements is said to be *weakly mixing* if for any $k \in \mathbb{N}$, any choice of nonempty open subsets $V_1, V_2, \dots, V_k$ of A and nonempty open subsets $U_1, U_2, \dots, U_k$ of X with $A \cap U_i \neq \emptyset$, $i = 1, 2, \dots, k$, there exists a $m \in \mathbb{N}$ such that $f^m(V_i) \cap U_i \neq \emptyset$ for $1 \leq i \leq k$. A topological dynamical system (X, f) is called *partial weak mixing* if X contains a weakly mixing subset. Motivated by the idea of Blanchard and Huang's notion of "weakly mixing subset", Oprocha and Zhang [3] extended the notion of weakly mixing subset and gave the concept of "transitive subset" and discussed its basic properties.

It is a well-known fact that if a sequence of continuous maps converges uniformly, then the uniform limit map is continuous. Abu-Saris and Al-Hami [4] studied uniform convergence and chaotic behavior. Later Abu-Saris et al. [5] pointed out some wrong claims in [4] and corrected them. Román-Flores [6] gave sufficient conditions for the topological transitivity of uniform limit map $f : X \to X$ of a sequence of continuous maps $f_n : X \to X$, where X is a compact metric space. Fedeli and Le Donne [7] studied the dynamical behavior of the uniform limit of a sequence of continuous self-maps on a compact metric space satisfying topological transitivity or other related properties and gave some conditions for the transitivity of a limit. Bhaumik and Choudhury [8] investigated the chaotic behavior of the uniform limit map $f : I \to I$ of a sequence of continuous topologically transitive maps $f_n : I \to I$, where I is a compact interval. Recently, Yan, Zeng, and Zhang et al. [9] studied transitivity and sensitive dependence on initial conditions for uniform limits.

In this paper, motivated by the idea of Román-Flores [6], we give sufficient conditions such that the limit map f has a transitive subset. In particular, we prove that A is a transitive subset of (X, f) if A is a transitive subset of (X, f_n) for every $n \in \mathbb{N}$ when a sequence of continuous maps f_n converges strongly uniformly to a map f, where (X, d) is a compact metric space. Moreover, we give an example to show that if A is a transitive subset of (X, f), then A cannot be a transitive subset of (X, f_n) for some $n \in \mathbb{N}$.

2. Preliminaries

Topological transitivity (see [10–12]) are global characteristic of topological dynamical systems. Let (X, f) be a topological dynamical system. (X, f) is *topologically transitive* if for any nonempty open subsets U and V of X there exists a $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. For a topological dynamical system (X, f), the orbit of x is the set $\text{orb}(x, f) = \{f^n(x) : n \in \mathbb{N}\}$ for every $x \in X$. (X, f) is *point transitive* if there exists a point $x_0 \in X$ with dense orbit, that is, $\overline{\text{orb}(x_0, f)} = X$. Such a point x_0 is called a transitive point of (X, f). By [13], if X is a compact metric space without isolated points, then the topologically transitive and point transitive are equivalent.

Definition 2.1 (see[3]). A closed subset A is called a transitive subset of (X, f) if for any choice of nonempty open subset V^A of A and nonempty open subset U of X with $A \cap U \neq \emptyset$, there exists a $n \in \mathbb{N}$ such that $f^n(V^A) \cap U \neq \emptyset$.

Remark 2.2. (1) By Definition 2.1, (X, f) is transitive if and only if X is a transitive subset of (X, f).

(2) If $a \in X$ is a transitive point of (X, f), then $\{a\}$ is a transitive subset of (X, f).

Definition 2.3 (see[14]). Let (X, τ) be a topological space. A and B are two nonempty subsets of X. B is dense in A if $A \subseteq \overline{A \cap B}$.

In fact, we easily prove that B is dense in A if and only if $V^A \cap B \neq \emptyset$ for any nonempty open set V^A of A.

Proposition 2.4. *Let (X, f) be a topological dynamical system and A be a nonempty closed set of X. Then the following conditions are equivalent.*

(1) *A is a transitive subset of (X, f).*

(2) *Let V^A be a nonempty open subset of A and U a nonempty open subset of X with $A \cap U \neq \emptyset$. Then there exists $n \in \mathbb{N}$ such that $V^A \cap f^{-n}(U) \neq \emptyset$.*

(3) *Let U be a nonempty open set of X with $U \cap A \neq \emptyset$. Then $\bigcup_{n\in\mathbb{N}} f^{-n}(U)$ is dense in A.*

Proof. (1) $\Rightarrow$ (2) Let A be a transitive subset of (X, f). Then for any choice of nonempty open set V^A of A and nonempty open set U of X with $A \cap U \neq \emptyset$, there exists $n \in \mathbb{N}$ such that $f^n(V^A) \cap U \neq \emptyset$. Since $f^n(V^A \cap f^{-n}(U)) = f^n(V^A) \cap U$, it follows that $V^A \cap f^{-n}(U) \neq \emptyset$.

(2) $\Rightarrow$ (3) Let V^A be a nonempty open set of A and U be a nonempty open set of X with $A \cap U \neq \emptyset$. By the assumption of (2), there exists $n \in \mathbb{N}$ such that $V^A \cap f^{-n}(U) \neq \emptyset$. Furthermore,

$$V^A \cap \bigcup_{n \in \mathbb{N}} f^{-n}(U) = \bigcup_{n \in \mathbb{N}} \left(V^A \cap f^{-n}(U) \right) \neq \emptyset. \tag{2.1}$$

Hence, $\bigcup_{n \in \mathbb{N}} f^{-n}(U)$ is dense in A.

(3) $\Rightarrow$ (1) Let V^A be a nonempty open set of A and U a nonempty open set of X with $A \cap U \neq \emptyset$. Since $\bigcup_{n \in \mathbb{N}} f^{-n}(U)$ is dense in A, it follows that $V^A \cap \bigcup_{n \in \mathbb{N}} f^{-n}(U) \neq \emptyset$. Hence, there exists $n \in \mathbb{N}$ such that $V^A \cap f^{-n}(U) \neq \emptyset$. Moreover, $f^n(V^A \cap (f^{-n}(U)) = f^n(V^A) \cap U$, which implies $f^n(V^A) \cap U \neq \emptyset$. Therefore, A is a transitive subset of (X, f). □

Definition 2.5. Let (X, d) be a metric space and a sequence of continuous maps $f_n : X \to X$, for each $n \in \mathbb{N}$. $\{f_n : n \in \mathbb{N}\}$ is said to converge strongly uniformly to f if for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for any $x \in X$, $l \in \mathbb{N}$ and $n \geq n_0$ satisfying

$$d\left((f_n)^l(x), f^l(x) \right) < \varepsilon. \tag{2.2}$$

If $\{f_n : n \in \mathbb{N}\}$ converges strongly uniformly to f, $\{f_n : n \in \mathbb{N}\}$ is called a strong uniform convergent sequence.

The following example is from [9, 15]; we show that the example is a strong uniformly convergence example.

Example 2.6. Let $I = [0, 1]$. Denote $I_i^j = [j - 1/3^{i-1}, j/3^{i-1}]$ for any $i \in \mathbb{N}$ and $j = 1, 2, \ldots, 3^{i-1}$. Let $f_i^j : I_i^j \to I_i^j$ satisfy

$$f_i^j(x) = \frac{j-1}{3^{i-1}} + f_i^1\left(x - \frac{j-1}{3^{i-1}} \right) \text{ for any } x \in I_i^j, \text{ where} \tag{2.3}$$

$$f_i^1(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq \dfrac{1}{3^i}, \\ 3x - \dfrac{1}{3^{i-1}}, & \text{if } \dfrac{1}{3^i} < x < \dfrac{2}{3^i}, \\ \dfrac{1}{3^{i-1}}, & \text{if } \dfrac{2}{3^i} \leq x \leq \dfrac{1}{3^{i-1}}. \end{cases} \tag{2.4}$$

For any $n \in \mathbb{N}$, we define $f_n : I \to I$ satisfying

$$f_n(x) = f_n^j(x) \text{ for any } x \in I_n^j \text{ and } j = 1, 2, \ldots, 3^{n-1}. \tag{2.5}$$

Then it is easy to see that $f_n : I \to I$ is a continuous map for each $n \in \mathbb{N}$ and f_n converges strongly uniformly to id_I, the identity on I.

3. Main Results

Let $C(X,X)$ denote the set of continuous maps $f : X \to X$. In the sequel, as in usual, $d_\infty(f,g)$ denotes the uniform metric on $C(X,X)$, that is, $d_\infty(f,g) = \sup_{x\in X} d(f(x),g(x))$. A topological space X is perfect if X is closed and has no isolated points. Clearly, if X is a perfect space, then any nonempty open set U of X has no isolated points.

From the idea of Román-Flores [6], we obtain the following theorem.

Theorem 3.1. *Let (X,d) be a compact metric space and a sequence of continuous maps $f_n : X \to X$ that converges uniformly to a map f. Assume that A is a perfect set of X and A is a transitive subset of (X,f_n) for all $n \in \mathbb{N}$. Additionally, suppose that*

(1) $d_\infty((f_n)^n, f^n) \to 0$ as $n \to \infty$,

(2) $\{(f_n)^n(x) : n \in \mathbb{N}\}$ is dense in A, for some $x \in X$.

Then A is a transitive subset of (X,f).

Proof. Let V^A be a nonempty open set of A and U a nonempty open set of X with $A\cap U \neq \emptyset$. Since condition (2), there exists $x_0 \in X$ such that $\{(f_n)^n(x_0) : n \in \mathbb{N}\}$ is dense in A. Furthermore, by condition (1) and A is perfect, we obtain that the sequence $\{f^n(x_0) : n \in \mathbb{N}\}$ is also dense in A. Moreover, V^A is a nonempty open set of A; there exists $k \in \mathbb{N}$ such that $z = f^k(x_0) \in V^A$. Let $G = (U \cap A) \setminus \{f(x_0), f^2(x_0), \ldots, f^k(x_0)\}$. Then G is a nonempty open set of A. Since A is a perfect metric space and $\{f^n(x_0) : n \in \mathbb{N}\}$ is dense in A, there exists $l > k$ such that $f^l(x_0) \in G \subseteq (U \cap A)$. Hence, we have

$$f^l(x_0) = f^{l-k}\left(f^k(x_0)\right) = f^{l-k}(z) \in f^{l-k}\left(V^A\right) \cap (U \cap A). \tag{3.1}$$

Consequently, $f^{l-k}(V^A) \cap U \neq \emptyset$. Therefore, A is a transitive subset of (X,f). □

Theorem 3.2. *Let (X,d) be a compact metric space. Assume a sequence of continuous maps $f_n : X \to X$ that converges strongly uniformly to a map f and A is a transitive subset of dynamical systems (X,f_n) for each $n \in \mathbb{N}$. Then A is a transitive subset of (X,f).*

Proof. Let V^A be a nonempty open set of A and U a nonempty open set of X with $A\cap U \neq \emptyset$. Since X is a compact metric space and $A \cap U \neq \emptyset$, there exists a nonempty open set W of X such that $\overline{W} \subseteq U$ and $W \cap A \neq \emptyset$.

Let $W_n = \bigcup_{k=1}^{\infty} (f_n)^{-k}(W)$ for each $n \in \mathbb{N}$. Since A is a transitive subset of (X,f_n) for each $n \in \mathbb{N}$, by Proposition 2.4, then W_n is an open set of X and W_n is dense in A. We denote $W^\infty = \bigcap_{n=1}^{\infty} W_n$. By Baire theorem, W^∞ is dense in A. Furthermore, we have $V^A \cap W^\infty \neq \emptyset$. Take a point $y_0 \in V^A \cap W^\infty$. There exists $k_n \in \mathbb{N}$ such that $y_0 \in (f_n)^{-k_n}(W)$ for each $n \in \mathbb{N}$. Denote $x_n = (f_n)^{k_n}(y_0)$ for each $n \in \mathbb{N}$. Without loss of generality, we may assume $\lim_{n\to\infty} x_n = x \in \overline{W}$ because X is a compact metric space. Choose a $\delta > 0$ such that $B(x,\delta) = \{y \in X : d(x,y) < \delta\} \subseteq U$. Since maps sequence $\{f_n : n \in \mathbb{N}\}$ converges strongly uniformly to f and $\lim_{n\to\infty} x_n = x$, there exists $n_0 \in \mathbb{N}$ such that

$$d\left((f_{n_0})^{k_{n_0}}(y_0), f^{k_{n_0}}(y_0)\right) < \frac{\delta}{2} \text{ and } d(x_{n_0}, x) = d\left((f_{n_0})^{k_{n_0}}(y_0), x\right) < \frac{\delta}{2}. \tag{3.2}$$

It follows that $d(x, f^{k_{n_0}}(y_0)) < \delta$, which implies $f^{k_{n_0}}(y_0) \in U$. Therefore, $f^{k_{n_0}}(V^A) \cap U \neq \emptyset$. This shows that A is a transitive subset of (X,f). □

The following example is from [4]. We give the example which shows if maps sequence $\{f_n : n \in \mathbb{N}\}$ converges uniformly to a map f and A is a transitive subset of (X, f_n) for each $n \in \mathbb{N}$, then A cannot be a transitive subset of (X, f).

Example 3.3 (see [4]). *Let S^1 be the unit circle and $T_\lambda : S^1 \to S^1$ a translation map such that*

$$T_\lambda(\theta) = \theta + 2\lambda\pi, \quad \lambda \in \mathbb{R}. \tag{3.3}$$

Let λ be an irrational number, $\lambda_n = \lambda/n$, and $T_n = T_{\lambda_n} : S^1 \to S^1$ such that $T_n(\theta) = \theta + (2\lambda/n)\pi$. Let maps sequence $\{T_n : n \in \mathbb{N}\}$ converge uniformly to a map T_0. Then T_0 is not topologically transitive on S^1; that is, S^1 is not a transitive subset of dynamical system (S^1, T_0).

It is well known that if $\lambda = q/p$ is a rational number, then all points are periodic of period q, and so the set of periodic points is, obviously, dense in S^1. Moreover, by Jacobi's Theorem [16], if λ is an irrational number, then T_λ is topologically transitive on S^1. Therefore, S^1 is a transitive subset of (S^1, T_λ). Since $\lambda_n = \lambda/n$ is an irrational number for each $n \in \mathbb{N}$, then $T_n = T_{\lambda_n} : S^1 \to S^1$ is topologically transitive for each $n \in \mathbb{N}$, which implies S^1 is a transitive subset of (S^1, T_n) for each $n \in \mathbb{N}$. Moreover, maps sequence $\{T_{\lambda_n} : n \in \mathbb{N}\}$ converges uniformly to a map $T_0 = id$, where id is identity map. Therefore, T_0 is not topologically transitive on S^1, which implies S^1 is not a transitive subset of (S^1, T_0).

Let $f_n : X \to X$ be a continuous map for each $n \in \mathbb{N}$, and maps sequence $\{f_n : n \in \mathbb{N}\}$ converges uniformly to a map f. The following example shows that A is a transitive subset of (X, f), but there exists $k \in \mathbb{N}$ such that A is not a transitive subset of (X, f_k).

Example 3.4. Let

$$f_n(x) = \begin{cases} \dfrac{2n}{n-2}x, & \text{if } 0 \le x \le \dfrac{n-2}{2n}, \\ 1, & \text{if } \dfrac{n-2}{2n} \le x \le \dfrac{n+2}{2n}, \quad n = 3, 4, \ldots \\ \dfrac{2n}{n-2}(1-x), & \text{if } \dfrac{n+2}{2n} \le x \le 1. \end{cases} \tag{3.4}$$

Observe that the given sequence converges uniformly to tent map

$$f(x) = \begin{cases} 2x, & \text{if } 0 \le x \le \dfrac{1}{2}, \\ 2(1-x), & \text{if } \dfrac{1}{2} \le x \le 1, \end{cases} \tag{3.5}$$

Figures 1 and 2, which is known to be topologically transitive on $I = [0, 1]$ (see [16]). We will prove that $[1/4, 3/4]$ is a transitive subset of (X, f).

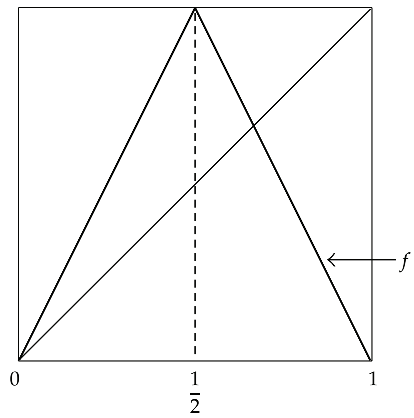

Figure 1

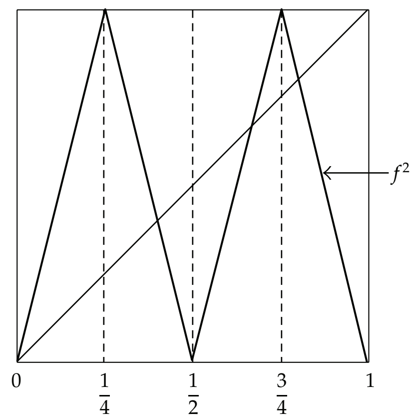

Figure 2

Let $S(f^k)$ denote the set of extreme value points of f^k for every $k \in \mathbb{N}$; then $S(f^k) = \{1/2^k, 2/2^k, \dots, (2^k - 1)/2^k\}$. Since $S(f) = \{1/2\}$, $f(1/2) = 1$, $f(0) = 0$, and $f(1) = 0$, we have

$$f^k(x) = \begin{cases} 1, & \text{if } x = \dfrac{1}{2^k}, \dfrac{3}{2^k}, \dots, \dfrac{2^k - 1}{2^k}, \\ \\ 0, & \text{if } x = 0, \dfrac{2}{2^k}, \dfrac{4}{2^k}, \dots, \dfrac{2^k - 2}{2^k}, 1. \end{cases} \tag{3.6}$$

Let $I_k^j = [j/2^k, (j+1)/2^k]$ for $0 \le j \le 2^k - 1$. Then $f^k(I_k^j) = [0,1]$. For any nonempty open set U of $[1/4, 3/4]$. Without loss of generality, we take $U = (x_0 - \varepsilon, x_0 + \varepsilon)$ for a given $\varepsilon > 0$ and $x_0 \in \operatorname{int}[1/4, 3/4]$, where $\operatorname{int}[1/4, 3/4]$ denotes the interior of $[1/4, 3/4]$. When $l \in \mathbb{N}$ and $l > \log_2(1/\varepsilon)$, then there exists $j \in \mathbb{N}$ and $0 \le j \le 2^l - 1$ such that $I_l^j \subseteq U$. Furthermore, we have $f^l(U) = [0,1]$. Thus, for any nonempty open set U of $[1/4, 3/4]$ and nonempty open set V of $[0,1]$ with $V \cap [1/4, 3/4] \ne \emptyset$, there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V \ne \emptyset$. This shows that

$[1/4, 3/4]$ is a transitive subset of (I, f). Moreover, $f_4(x) = 1$ and $(f_4)^n(x) = 0 (n \geq 2)$ for all $x \in [1/4, 3/4]$, which implies that $[1/4, 3/4]$ is not a transitive subset of (I, f_4).

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