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Research Article

Exponential Stability of Impulsive Stochastic Delay Differential Systems

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This paper investigates the stability of stochastic delay differential systems with two kinds of impulses, that is, destabilizing impulses and stabilizing impulses. Both the pth moment and almost sure exponential stability criteria are established by using the average impulsive interval. When the impulses are regarded as disturbances, a lower bound of average impulsive interval is obtained; it means that the impulses should not happen too frequently. On the other hand, when the impulses are used to stabilize the system, an upper bound of average impulsive interval is derived; namely, enough impulses are needed to stabilize the system. The effectiveness of the proposed results is illustrated by two examples.

1. Introduction

Impulsive dynamical systems have attracted considerable interest in science and engineering in recent years because they provide a natural framework for mathematical modeling of many real-world problems where the reactions undergo abrupt changes [1–3]. These systems have been found to have important applications in various fields, such as control systems with communication constraints [4], network system [5, 6], sampled-data systems [7, 8], and mechanical systems [9]. On the other hand, impulsive control based on impulsive systems can provide an efficient way to deal with plants that cannot endure continuous control inputs [3]. In recent years, the impulsive control theory has been generalized from deterministic systems to stochastic systems and has been shown to have extensive applications [10, 11].

Stability is one of the most important issues in the study of impulsive stochastic differential systems (see, e.g., [12–20]). When the continuous dynamical system is unstable, there is some literature that is concerned with the pth moment exponential stability with stabilizing impulses. For example, several criteria on the global exponential stability and instability are obtained in [17]. The pth moment exponential stability is discussed in [18] by using the vector Lyapunov functions. The authors in [19] investigated impulsive stabilization of stochastic delay differential systems, and both pth moment and almost sure exponential stability criteria are established by using the Lyapunov-Razumikhin method. Recently, both continuous dynamical stable system and continuous dynamical unstable system are studied in [20].

The average impulsive interval was proposed in [21], and it is useful to study the synchronization problem of dynamical networks with destabilizing impulses (see, e.g., [21–23]). The average impulsive interval can be used to control frequency of the impulsive occurrence. When the continuous dynamical system is stable and the impulsive effects are destabilizing, in order to maintain the stability of the system, the impulses should not happen too frequently. Therefore, there should exist a lower bound; if the average impulsive interval is not less than the bound, the stability can be maintained. On the other hand, when the continuous dynamical system is unstable, and the impulses are used to stabilize the unstable system, there should exist enough impulses to stabilize the system, that is, the frequency of impulsive occurrence should exceed a lower bound. Thus there exists an upper bound of the average impulsive interval; if the average interval is less than the upper bound, the system is stabilized by the impulses.

In this paper, by using the average impulsive interval, we investigate the pth moment and almost sure exponential stability for stochastic delay differential systems with two kinds of impulses, that is, destabilizing impulses and stabilizing impulses. When the continuous dynamical system is stable, the lower bound of the average impulsive interval is obtained, by which we can estimate how intensive impulsive disturbance the stable system can endure. On the other hand, when the continuous dynamical system is unstable, the upper bound of the average impulsive interval is derived. From this bound, we can estimate the minimum impulsive frequency needed to stabilize the system. The effectiveness of the proposed results is illustrated by two examples.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and definitions. We establish several stability criteria for impulsive stochastic delay differential systems in Section 3. In Section 4, two examples are given to illustrate the effectiveness of our results.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, P)$ be a complete probability space with some filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e., the filtration is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $B = (B(t), t \geq 0)$ be an m-dimensional $\mathcal{F}_t$-adapted Brownian motion.

For $x \in \mathbb{R}^d$, $|x|$ denotes the Euclidean norm of x. For $-\infty < a < b < \infty$, we say that a function from $[a,b]$ to $\mathbb{R}^d$ is piecewise continuous if the function has at most a finite number of jumps discontinuous on $(a,b]$ and continuous from the right for all points in $[a,b)$. Given $\tau > 0$, $PC([-\tau,0];\mathbb{R}^d)$ denotes the family of piecewise continuous functions from $[-\tau,0]$ to $\mathbb{R}^d$ with norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0}\varphi(\theta)$. For $p > 0$ and $t \geq t_0$, let $L^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^d)$ be the family of $\mathcal{F}_t$-adapted and $PC([-\tau,0];\mathbb{R}^d)$-valued random variables φ such that $E\|\varphi\|^p < \infty$. Let $\mathbb{N} = 1,2,\ldots$ and $\mathbb{R}^+ = [0,+\infty)$.

In this paper, we consider the following impulsive stochastic delay differential systems:

$$\begin{aligned} dx(t) &= f(t,x_t)dt + g(t,x_t)dB(t), \quad t \neq t_k,\ t \geq t_0, \\ \Delta x(t_k) &= x(t_k) - x(t_k^-) = I\left(t_k, x\left(t_k^-\right)\right), \quad k \in \mathbb{N}, \\ x_{t_0} &= \xi(t_0+\theta), \quad -\tau \leq \theta \leq 0, \end{aligned} \tag{2.1}$$

where $\{t_k, k \in \mathbb{N}\}$ is a strictly increasing sequence such that $t_k \to \infty$ as $(k \to \infty)$ and $x(t^-) = \lim_{s \uparrow t} x(s)$. x_t is defined by $x_t(\theta) = x(t+\theta)$, $-\tau \leq \theta \leq 0$. The mappings $I : \mathbb{R}^+ \times PC([-\tau,0];\mathbb{R}^d) \to \mathbb{R}^d$, $f : \mathbb{R}^+ \times PC([-\tau,0];\mathbb{R}^d) \to \mathbb{R}^d$, and $g : \mathbb{R}^+ \times PC([-\tau,0];\mathbb{R}^d) \to \mathbb{R}^{d\times m}$ are all Borel-measurable functions.

As a standing hypothesis, f, g, and I are assumed to satisfy necessary assumptions so that, for any $\xi \in L^p_{\mathcal{F}_{t_0}}([-\tau,0];\mathbb{R}^d)$, system (2.1) has a unique global solution $x(t) \in L^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^d)$. In addition, we suppose that $f(t,0) \equiv 0$, $g(t,0) \equiv 0$, and $I(t,0) \equiv 0$ for all $t \geq t_0$. Then system (2.1) admits a trivial solution $x(t) \equiv 0$.

Let $C^{1,2}(\mathbb{R}^+;\mathbb{R}^d \times [t_0-\tau,\infty))$ denote the family of all nonnegative functions $V(t,x)$ on $[t_0-\tau,\infty) \times \mathbb{R}^d$ that are continuously twice differentiable in x and once in t. For each $V \in C^{1,2}(\mathbb{R}^+;\mathbb{R}^d \times [t_0-\tau,\infty))$, define an operator $\mathcal{L}V : \mathbb{R}^+ \times PC([-\tau,0];\mathbb{R}^d) \to \mathbb{R}^d$ for system (2.1) by

$$\mathcal{L}V(t,x_t) = V_t(t,x) + V_x(t,x)f(t,x_t) + \frac{1}{2}\operatorname{trace}\left[g^T(t,x_t)V_{xx}(t,x)g(t,x_t)\right], \tag{2.2}$$

where

$$\begin{gathered} V_t(t,x) = \frac{\partial V(t,x)}{\partial t}, \qquad V_x(t,x) = \left(\frac{\partial V(t,x)}{\partial x_1},\ldots,\frac{\partial V(t,x)}{\partial x_d}\right), \\ V_{xx}(t,x) = \left(\frac{\partial^2 V(t,x)}{\partial x_i \partial x_j}\right)_{d\times d}. \end{gathered} \tag{2.3}$$

The purpose of this paper is to discuss the stability of system (2.1). Let us begin with the following definitions.

Definition 2.1. The trivial solution of system (2.1) is said to be

(1) pth moment exponentially stable if for any initial data $\xi \in L^p_{\mathcal{F}_{t_0}}([-\tau,0];\mathbb{R}^d)$, the solution $x(t)$ satisfies

$$E|x(t)|^p \leq CE\|\xi\|^p e^{-\lambda(t-t_0)}, \tag{2.4}$$

where λ and C are positive constants independent of t_0,

(2) almost exponentially stable if the solution $x(t)$ satisfies

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| < -\lambda, \tag{2.5}$$

for any initial data $\xi \in L^p_{\mathcal{F}_{t_0}}([-\tau,0];\mathbb{R}^d)$ and $\lambda > 0$.

Definition 2.2. The average impulsive interval of the impulsive sequence $\{t_k\}_{k\in\mathbb{N}}$ is equal to a positive number T_a if there exists a positive integer N_0 such that

$$\frac{t-t_0}{T_a} - N_0 \leq N(t,t_0) \leq \frac{t-t_0}{T_a} + N_0, \quad t \geq t_0, \tag{2.6}$$

where $N(t,t_0)$ denotes the number of impulsive times of the impulsive sequence $\{t_k\}_{k\in\mathbb{N}}$ on the interval (t_0,t).

3. Main Results

In this section, we will establish some stability criteria of stochastic delay differential system with destabilizing impulses or stabilizing impulses. The first theorem addresses the case where the continuous dynamics in the system (2.1) is stable. It is shown that under some conditions the impulsive disturbance do not destroy the stability of system (2.1).

Theorem 3.1. *Assume that there exist positive constants c_1, c_2, p, γ_1 such that*

(H_1) $c_1|x|^p \leq V(t,x) \leq c_2|x|^p$,

(H_2) *for* $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}$,

$$E\mathcal{L}V(t,\varphi(\theta)) \leq -\gamma_1 EV(t,\varphi(0)) \tag{3.1}$$

provided $\varphi \in L^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^d)$ *satisfying that* $EV(t+\theta,\varphi(\theta)) \leq qEV(t,\varphi(0))$, $\theta \in [-\tau,0]$,

(H_3) *there exists a positive constant* $\mu > 1$ *such that*

$$EV(t_k, x + I(t_k,x)) \leq \mu EV(t_k^-, x), \tag{3.2}$$

(H_4) $e^{\gamma_1 \tau} \leq q$, $T_a > \ln\mu/\gamma_1$.

Then the trivial solution of system (2.1) *is pth moment exponentially stable.*

Proof. According to (H_1), we see that

$$EV(t,x(t)) \leq Me^{-\gamma_1(t-t_0)}, \quad t \in [t_0-\tau, t_0], \tag{3.3}$$

where $M = c_2E\|\xi\|^p$. We will show that

$$EV(t,x(t)) \leq Me^{-\gamma_1(t-t_0)}, \quad t \in (t_0, t_1). \tag{3.4}$$

Suppose (3.4) is not true. Then there exist some $t \in (t_0,t_1)$ such that $EV(t,x(t)) > Me^{-\gamma_1(t-t_0)}$. Set $t^* = \inf\{t \in [t_0,t_1) : EV(t,x(t)) > Me^{-\gamma_1(t-t_0)}\}$. It follows that $t^* \in [t_0,t_1)$ and $EV(t^*,x(t^*)) = Me^{-\gamma_1(t^*-t_0)}$. Moreover, there is a sequence $\{s_m\}_{m\geq 1}$ and $s_m \downarrow t^*$ such that

$$EV(s_m,x(s_m)) > Me^{-\gamma_1(s_m-t_0)}, \quad s_m \in (t^*,t_1). \tag{3.5}$$

Consequently,

$$EV\left(t^*+\theta,\varphi(\theta)\right)\le e^{-\gamma_1\theta}EV(t^*,x(t^*))\le qEV(t^*,x(t^*)), \tag{3.6}$$

which implies that

$$E\mathcal{L}V\left(t^*,\varphi(\theta)\right)<-\gamma_1 EV\left(t^*,\varphi(0)\right). \tag{3.7}$$

Noticing that the solution $x(t)$ and functionals V, $\mathcal{L}V$ are continuous on $[t^*,t_1)$, thus we obtain

$$E\mathcal{L}V\left(t,\varphi(\theta)\right)\le-\gamma_1 EV\left(t,\varphi(0)\right),\quad t\in[t^*,t^*+h], \tag{3.8}$$

for sufficiently small $h>0$. Using Itô's formula, we derive

$$\begin{aligned}EV(t^*+h)&=EV(t^*)+\int_{t^*}^{t^*+h}E\mathcal{L}V\left(s,\varphi(\theta)\right)ds\\&\le EV(t^*)+\int_{t^*}^{t^*+h}E\left[-\gamma_1 V\left(s,\varphi(0)\right)\right]ds,\end{aligned} \tag{3.9}$$

which yields that

$$EV(t^*+h)\le Me^{-\gamma_1(t^*-t_0)}e^{-\gamma_1 h}. \tag{3.10}$$

This contradicts (3.5). Therefore (3.4) holds. Now, we assume that

$$EV(t,x(t))\le M\mu^{k-1}e^{-\gamma_1(t-t_0)},\quad t\in[t_{k-1},t_k). \tag{3.11}$$

We will prove that

$$EV(t,x(t))\le M\mu^{k}e^{-\gamma_1(t-t_0)},\quad t\in[t_{k},t_{k+1}). \tag{3.12}$$

In view of (3.11) and (H_3), we get (3.11) holds for $t=t_k$. Suppose (3.12) is not true; then, there exist some $t\in(t_k,t_{k+1})$ such that $EV(t,x(t))>M\mu^k e^{-\gamma_1(t-t_0)}$. Seting $t^*=\inf\{t\in[t_k,t_{k+1}):EV(t,x(t))>M\mu^k e^{-\gamma_1(t-t_0)}\}$, we have $t^*\in[t_k,t_{k+1})$ and $EV(t^*,x(t^*))=M\mu^k e^{-\gamma_1(t^*-t_0)}$. Moreover, there is a sequence $\{s_m\}_{m\ge1}$ and $s_m\downarrow t^*$ such that

$$EV(s_m,x(s_m))>M\mu^k e^{-\gamma_1(s_m-t_0)},\quad s_m\in(t^*,t_{k+1}). \tag{3.13}$$

For $-\tau\le\theta\le0$, there exists an integer $j\in[0,k]$ such that $t^*+\theta\in[t_j,t_{j+1})$. Hence

$$EV(t^*+\theta)\le M\mu^j e^{-\gamma_1(t^*+\theta-t_0)}\le e^{-\gamma_1\theta}M\mu^k e^{-\gamma_1(t^*-t_0)}\le qEV(t^*+\theta). \tag{3.14}$$

Thus, from (H_2), we have

$$E\mathcal{L}V(t^*,\varphi(\theta)) < -\gamma_1 EV(t^*,\varphi(0)). \tag{3.15}$$

Similarly, this can lead to a contradiction, which implies that (3.12) holds. From Definition 2.2, we see that

$$\frac{t-t_0}{T_a} - N_0 \leq N(t,t_0) \leq \frac{t-t_0}{T_a} + N_0, \quad t \geq t_0. \tag{3.16}$$

Consequently,

$$EV(t,x(t)) \leq M\mu^{N(t,t_0)} e^{-\gamma_1(t-t_0)} \leq M\mu^{N_0} e^{((t-t_0)\ln\mu)/T_a} e^{-\gamma_1(t-t_0)} = M\mu^{N_0} e^{-\lambda(t-t_0)}, \tag{3.17}$$

where $\lambda = \gamma_1 - \ln\mu/T_a > 0$. This completes the proof. □

Remark 3.2. Theorem 3.1 gives the conditions under which the impulsive disturbances do not destroy the stability of system (2.1). When the impulsive effects are destabilizing, the impulses should not happen too frequently. Therefore, in order to maintain the stability of continuous dynamical system, the average impulsive interval is used to control the impulsive frequency.

Remark 3.3. In Theorem 3.1, the impulses are regarded as disturbance; therefore, the condition $\mu > 1$ is reasonable. It is worth pointing out that in Theorem 3.1, for arbitrary small ε and any $T_a > 0$, the impulsive interval can be less than ε and simultaneously the average impulsive intervals are not less than T_a. That is, high-density impulses are allowed to happen in a certain interval, but we need low-density impulses to follow as a compensation.

In the following theorem, when the continuous dynamics in the system (2.1) is unstable, it is shown that the system (2.1) can be stabilized by impulses.

Theorem 3.4. *Let $\beta = \inf_{k\in\mathbb{N}}\{t_k - t_{k-1}\}$ and there is a positive integer l such that $(l-1)\beta < \tau \leq l\beta$. Assume that there exist positive constants c_1, c_2, p, γ_2 such that*

(H_1) $c_1|x|^p \leq V(t,x) \leq c_2|x|^p$,

(H_2) *for* $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}$,

$$E\mathcal{L}V(t,\varphi(\theta)) < \gamma_2 EV(t,\varphi(0)) \tag{3.18}$$

provided $\varphi \in L^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^d)$ *satisfying that* $EV(t,\varphi(\theta)) \leq qEV(t,\varphi(0))$, $\theta \in [-\tau,0]$,

(H_3) *there exists a positive constant* $\mu < 1$ *such that*

$$EV(t_k, x + I(t_k,x)) \leq \mu EV(t_k^-, x), \tag{3.19}$$

(H_4) $q\mu^l \geq 1$, $T_a < -\ln\mu/\gamma_2$.

Then the trivial solution of system (2.1) *is pth moment exponentially stable.*

Proof. In view of (H_1), we obtain

$$EV(t, x(t)) \le Me^{\gamma_2(t-t_0)}, \quad t \in [t_0 - \tau, t_0], \tag{3.20}$$

where $M = c_2 E\|\xi\|^p e^{\gamma_2 \tau}$. We will show that

$$EV(t, x(t)) \le Me^{\gamma_2(t-t_0)}, \quad t \in [t_0, t_1). \tag{3.21}$$

Suppose (3.21) is not true. Then there exist some $t \in (t_0, t_1)$ such that $EV(t, x(t)) > Me^{\gamma_2(t-t_0)}$. Set $t^* = \inf\{t \in (t_0, t_1) : EV(t, x(t)) > Me^{\gamma_2(t-t_0)}\}$, which yields that $EV(t, x(t)) \le Me^{\gamma_2(t-t_0)}$ for $t \in [t_0, t^*)$ and $EV(t^*, x(t^*)) = Me^{\gamma_2(t^*-t_0)}$. Moreover, there is a sequence $\{s_m\}_{m\ge1}$ and $s_m \downarrow t^*$ such that

$$EV(s_m, x(s_m)) > Me^{\gamma_2(s_m-t_0)}, \quad s_m \in (t^*, t_1). \tag{3.22}$$

Noticing that

$$EV(t^* + \theta) \le Me^{\gamma_2(t^*+\theta-t_0)} \le e^{\gamma_2\theta} EV(t^*, x(t^*)) \le qEV(t^*, x(t^*)), \tag{3.23}$$

we derive

$$E\mathcal{L}V(t^*, \varphi(0)) < \gamma_2 EV(t^*, \varphi(0)). \tag{3.24}$$

Since the solution $x(t)$ and functionals V, $\mathcal{L}V$ are continuous on $[t^*, t_1)$, we see that

$$E\mathcal{L}V(t, \varphi(\theta)) \le \gamma_2 EV(t, \varphi(0)), \quad t \in [t^*, t^* + h] \tag{3.25}$$

for sufficiently small $h > 0$. Using Itô's formula, we obtain

$$\begin{aligned} EV(t^* + h) &= EV(t^*) + \int_{t^*}^{t^*+h} E\mathcal{L}V(s, \varphi(\theta))ds \\ &\le EV(t^*) + \int_{t^*}^{t^*+h} E[\gamma_2 V(s, \varphi(0))]ds, \end{aligned} \tag{3.26}$$

which implies

$$EV(t^* + h) \le Me^{\gamma_2(t^*-t_0)} e^{\gamma_2 h}. \tag{3.27}$$

This contradicts (3.22). Thus (3.21) holds. Now, we assume that

$$EV(t, x(t)) \le M\mu^{k-1} e^{\gamma_2(t-t_0)}, \quad t \in [t_{k-1}, t_k). \tag{3.28}$$

We will prove that

$$EV(t, x(t)) \leq M\mu^k e^{\gamma_2(t-t_0)}, \quad t \in [t_k, t_{k+1}). \tag{3.29}$$

Using (H_2) and (3.28) implies that (3.29) holds for $t = t_k$. Suppose (3.29) is not true. Then, there exist some $t \in [t_k, t_{k+1})$ such that $EV(t, x(t)) > M\mu^k e^{\gamma_2(t-t_0)}$. Seting $t^* = \inf\{t \in [t_k, t_{k+1}) : EV(t, x(t)) > M\mu^k e^{\gamma_2(t-t_0)}\}$, we have $t^* \in [t_k, t_{k+1})$ and $EV(t^*, x(t^*)) = M\mu^k e^{\gamma_2(t^*-t_0)}$. Moreover, there is a sequence $\{s_m\}_{m\geq 1}$ and $s_m \downarrow t^*$ such that

$$EV(s_m, x(s_m)) > M\mu^k e^{\gamma_2(s_m-t_0)}, \quad s_m \in (t^*, t_{k+1}). \tag{3.30}$$

For $-\tau \leq \theta \leq 0$, there exists an integer j such that $t^* + \theta \in [t_j, t_{j+1})$, $k - l \leq j \leq k$; then,

$$EV(t^* + \theta) \leq M\mu^{k-l} e^{\gamma_2(t^*+\theta-t_0)} \leq \mu^{-l} M\mu^k e^{\gamma_2(t^*-t_0)} \leq qEV(t^*, x(t^*)). \tag{3.31}$$

It follows that

$$E\mathcal{L}V(t^*, \varphi(\theta)) < \gamma_2 EV(t^*, \varphi(0)). \tag{3.32}$$

Similarly, this can lead to a contradiction, which implies that (3.29) holds.

According to (3.16), we obtain

$$EV(t, x(t)) \leq M\mu^{N(t,t_0)} e^{\gamma_2(t-t_0)} \leq M\mu^{-N_0} e^{((t-t_0)\ln\mu)/T_a} e^{\gamma_2(t-t_0)} = M\mu^{-N_0} e^{-\lambda(t-t_0)}, \tag{3.33}$$

where $\lambda = -(\ln\mu/T_a + \gamma_2) > 0$. This completes the proof. □

Remark 3.5. Theorem 3.4 shows that an unstable stochastic delay differential system can be successfully stabilized by impulses. The average impulsive interval is used to estimate the impulsive frequency; namely, the impulsive frequency should exceed a lower bound so that there exist enough impulses to stabilize the unstable continuous dynamical system.

In Theorem 3.4, we need to assume that $q\mu^l \geq 1$ and $\mu < 1$, which means the impulsive interval cannot be small enough. However, if system (2.1) is an impulsive stochastic differential system without delay, then the system can still be exponential stability when $\inf_{k\in\mathbb{N}}\{t_k - t_{k-1}\}$ is extremely small.

Corollary 3.6. *Let $\theta \equiv 0$ and $\tau = 0$ in system* (2.1). *Assume that there exist positive constants c_1, c_2, p, γ_2 such that*

(H_1) $c_1|x|^p \leq V(t, x) \leq c_2|x|^p$,

(H_2) *for $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}$,*

$$E\mathcal{L}V(t, x(t)) < \gamma_2 EV(t, x(t)), \tag{3.34}$$

(H$_3$) *there exists a positive constant* $\mu < 1$ *such that*

$$EV(t_k, x + I(t_k, x)) \le \mu EV(t_k^-, x), \tag{3.35}$$

(H$_4$) $\ln \mu / T_a + \gamma_2 < 0$.

Then the trivial solution of system (2.1) *is pth moment exponentially stable.*

Proof. The proof is similar to the proof given in Theorem 3.4, so we omit the detailed proof. □

The following theorem shows that the trivial solution of system (2.1) is almost surely exponentially stable, under some additional conditions.

Theorem 3.7. *Assume that* $p \ge 1$, $\beta = \inf_{k\in\mathbb{N}}\{t_k - t_{k-1}\}$ *and there exists a positive integer* l *such that* $(l-1)\beta < \tau \le l\beta$. *Suppose that the conditions in Theorem 3.1 or Theorem 3.4 hold. Moreover, there exists a constant* $L > 0$, *such that*

$$E\left(\left|f(t,\varphi)\right|^p + \left|g(t,\varphi)\right|^p + \left|I(t,\varphi)\right|^p\right) < L \sup_{-\tau\le\theta\le 0} E\left|\varphi(\theta)\right|^p. \tag{3.36}$$

Then the trivial solution of system (2.1) *is almost surely exponentially stable.*

Proof. By Theorem 3.1 or Theorem 3.4, we derive that the trivial solution of system (2.1) is pth moment exponentially stable. Therefore, there exists a positive constant M_1 such that

$$E|x(t)|^p \le M_1 e^{-\lambda(t-t_0)}. \tag{3.37}$$

It is obvious that

$$\begin{aligned} E\left(\sup_{0\le s\le\tau}|x(t+s)|^p\right) \le 4^{p-1}\Bigg(&\mathrm{E}|x(t)|^p + E\left(\int_t^{t+\tau}\left|f(s,x_s)\right|ds\right)^p + E\left|\sup_{0\le s\le\tau}\int_t^{t+s} g(u,x_u)dB(u)\right|^p \\ &+E\left|\sum_{t\le t_k\le t+\tau} I(t_k^-, x(t_k^-))\right|^p\Bigg). \end{aligned} \tag{3.38}$$

Combining the Hölder inequality with (3.36) and (3.37) implies that

$$\begin{aligned} E\int_t^{t+\tau}\left|f(s,x_s)\right|^p ds &\le L\tau^{p-1}\int_t^{t+\tau}\sup_{-\tau\le\theta\le 0} E|x(s+\theta)|^p ds \\ &\le M_1 L\tau^p e^{-\lambda(t-\tau-t_0)}. \end{aligned} \tag{3.39}$$

By virtue of the Burkholder-Davis-Gundy inequality, (3.36), and (3.37), we have

$$\begin{aligned} E\left(\sup_{0\le s\le\tau}\int_t^{t+s}\left|g(u,x_u)\right|dB(u)\right)^p &\le L\tau^{p/2-1}C(p)\int_t^{t+\tau}\sup_{-\tau\le\theta\le 0}E|x(s+\theta)|^p ds \\ &\le M_1C(p)L\tau^{p/2}e^{-\lambda(t-\tau-t_0)}, \end{aligned} \tag{3.40}$$

where $C(p)$ is a positive constant depending on p only. Thanks to (3.36) and (3.37), we see that

$$\begin{aligned} E\left(\sum_{t\le t_k\le t+\tau}\left|I(t_k^-,x(t_k^-))\right|\right)^p &\le l^pE\sup_{t\le t_k\le t+\tau}\left|I(t_k^-,x(t_k^-))\right|^p \\ &\le l^pLM_1e^{-\lambda(t-\tau-t_0)}. \end{aligned} \tag{3.41}$$

Substituting (3.39)–(3.41) into (3.38) gives that

$$E\left(\sup_{0\le s\le\tau}|x(t+s)|^p\right)\le M_2e^{-\lambda t}, \tag{3.42}$$

where M_2 is a positive constant. Then for an arbitrary $\varepsilon\in(0,\lambda)$ and $n\in\mathbb{N}$, we derive

$$P\left(\omega:\sup_{0\le s\le\tau}|x(n\tau+s)|^p>e^{-(\lambda-\varepsilon)n\tau}\right)\le M_2e^{-\varepsilon n\tau}. \tag{3.43}$$

Using the Borel-Cantelly lemma, we see that there exists an $n_0(\omega)$ such that for almost all $\omega\in\Omega$, $n\ge n_0(\omega)$,

$$\sup_{0\le s\le\tau}|x(t+s)|^p\le e^{-(\lambda-\varepsilon)n\tau}, \tag{3.44}$$

where $n\tau\le t\le(n+1)\tau$. It follows that

$$\limsup_{n\to\infty}\frac{\log\sup_{n\tau\le t\le(n+1)\tau}|x(t)|}{(n+1)\tau}\le\frac{-(\lambda-\varepsilon)}{p},\ \text{a.s.} \tag{3.45}$$

Consequently,

$$\limsup_{t\to\infty}\frac{\log|x(t)|}{t}\le\frac{-(\lambda-\varepsilon)}{p},\ \text{a.s.} \tag{3.46}$$

Let $\varepsilon\to 0$. Then the result follows. □

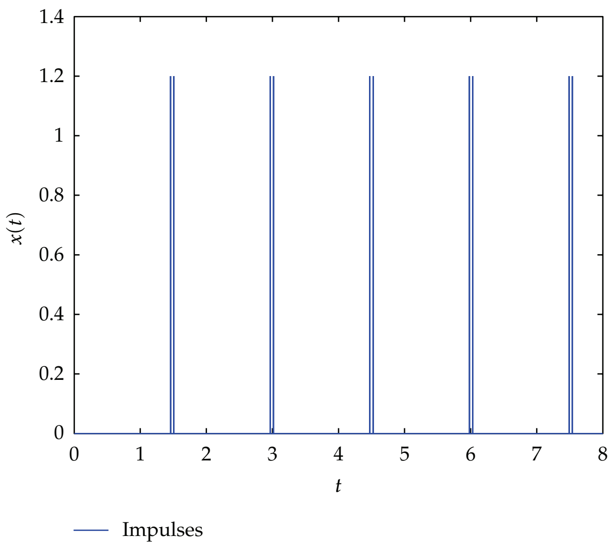

Figure 1: Impulses of Example 4.1.

4. Numerical Examples

In this section, two numerical examples are given to show the effectiveness of the main results derived in the preceding section.

Example 4.1. Consider an impulsive stochastic delay differential system as follows:

$$\begin{aligned} dx(t) &= [-x(t) + 0.125x(t-0.5)]dt + 0.5x(t-0.5)dB(t), \quad t \neq t_k, \ t \geq t_0, \\ \Delta x(t_k) &= 0.2x(t_k^-), \quad k \in \mathbb{N}, \\ x(t_0) &= 1.2. \end{aligned} \tag{4.1}$$

Choosing $p = 2$, $V(t,x) = x^2$, $c_1 = c_2 = 1$, and $q = 4/3$ in Theorem 3.1, then we have

$$\begin{aligned} E\mathcal{L}V(t,x) &= -2E|x(t)|^2 + 0.25Ex(t)x(t-0.5) + 0.25E|x(t-0.5)|^2 \\ &\leq -1.5E|x(t)|^2 + \frac{3q}{4}E|x(t)|^2 = -0.5E|x(t)|^2. \end{aligned} \tag{4.2}$$

Seting $\gamma_1 = 0.49$, then $E\mathcal{L}V(t,x) < -\gamma_1 EV(t,x)$. It is clear that $\mu = 1.44$, $e^{\gamma_1\tau} = 1.278 < q = 4/3$, $T_a \geq \ln\mu/\gamma_1 = 0.744$. For all $\varepsilon > 0$, we let $t_{2k-1} - t_{2(k-1)} = 1.488$, $t_{2k} - t_{2k-1} = \varepsilon$, $k \in \mathbb{N}$. Thus, by Theorem 3.1 the trivial solution of system (4.1) is pth moment exponential stability. Set $\varepsilon = 0.05$, which yields $l = 10$ in Theorem 3.7. Obviously, for system (4.1), condition (3.36) holds. Then by Theorem 3.7, the trivial solution of system (4.1) is also almost surely exponential stability.

Figure 1 describes the destabilizing impulsive sequence in the system (4.1) when $\varepsilon = 0.05$. It can be seen from Figure 2 that the destabilizing impulses do not destroy the stability of system (4.1).

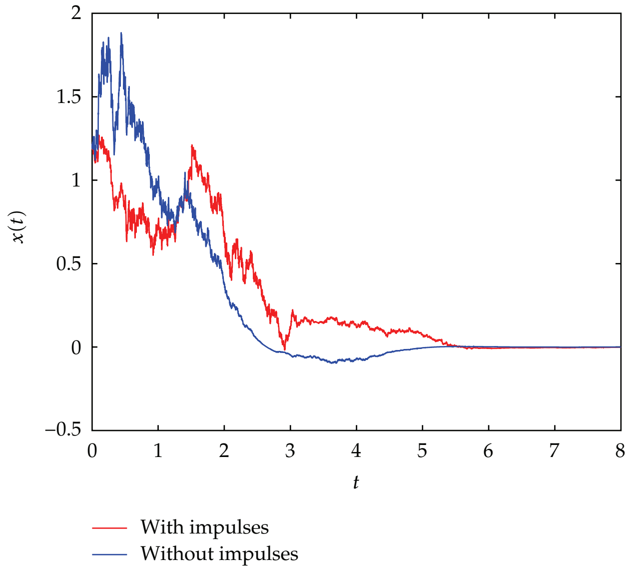

Figure 2: Impulsive disturbance of Example 4.1.

Example 4.2. Consider an impulsive stochastic delay differential system as follows:

$$
\begin{aligned}
dx(t) &= 0.25x(t-0.2)dt + 0.5x(t-0.2)dB(t), \quad t \neq t_k,\ t \geq t_0, \\
\Delta x(t_k) &= -0.5x(t_k^-), \quad k \in \mathbb{N}, \\
x(t_0) &= 1.2.
\end{aligned}
\tag{4.3}
$$

Clearly, for system (4.3), condition (3.36) holds. Let $p = 2$, $V(t,x) = x^2$, $c_1 = c_2 = 1$, and $q = 4$ in Theorem 3.4. Then we have

$$
\begin{aligned}
E\mathcal{L}V(t,x) &= 0.5Ex(t)x(t-0.2) + 0.25E|x(t-0.2)|^2 \\
&\leq E|x(t)|^2 + 0.75qE|x(t-0.2)|^2 \\
&\leq 4E|x(t)|^2.
\end{aligned}
\tag{4.4}
$$

Seting $\gamma_2 = 4.1$, then $E\mathcal{L}V(t,x) < \gamma_2 EV(t,x)$. It follows that $\mu = 0.25$, $T_a < -\ln\mu/\gamma_2 = 0.338$. Thus, we can choose $t_{2k-1} - t_{2(k-1)} = 0.2$, $t_{2k} - t_{2k-1} = 0.45$, $k \in \mathbb{N}$, which follows $T_a = 0.325$, $l = 1$, and $q\mu^l = 1 \geq 1$. Then by Theorems 3.4 and 3.7, the trivial solution of system (4.3) is pth moment and almost sure exponential stability.

The stabilizing impulsive sequence in the system (4.3) is described in Figure 3. It can be seen from Figure 4 that unstable continuous dynamics in the system (4.3) can be successfully stabilized by the impulses.

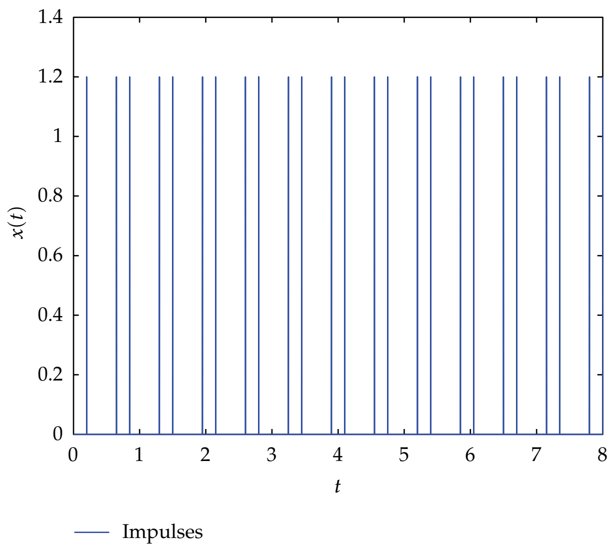

Figure 3: Impulses of Example 4.2.

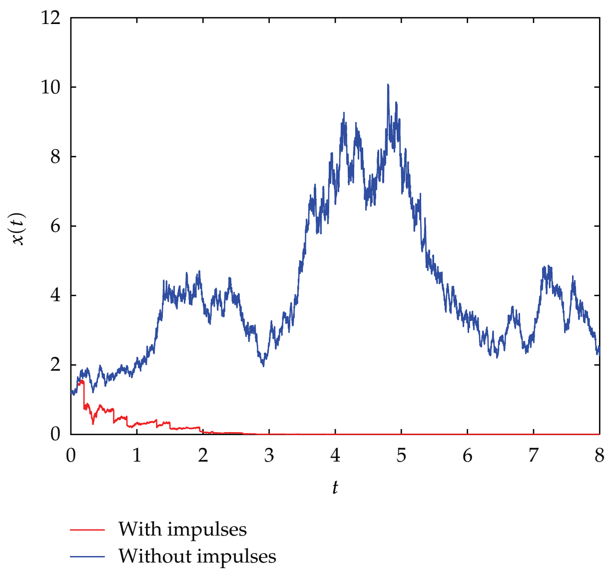

Figure 4: Impulsive control of Example 4.2.

5. Conclusion

The pth moment and almost sure exponential stability are investigated in this paper. By using the average impulsive interval, several sufficient conditions are established for stability of stochastic delay differential systems with destabilizing impulses or stabilizing impulses. Finally, two numerical simulation examples are offered to verify the effectiveness of the main results.

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