Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2011, Article ID 917892, 22 pages
doi:10.1155/2011/917892

Research Article

Mean-Square Convergence of Drift-Implicit One-Step Methods for Neutral Stochastic Delay Differential Equations with Jump Diffusion

Lin Hu[1,2] and Siqing Gan[1]

[1] *School of Mathematical Sciences and Computing Technology, Central South University, Changsha, Hunan 410075, China*
[2] *College of Science, Northeast Forestry University, Harbin, Heilongjiang 150040, China*

Correspondence should be addressed to Siqing Gan, siqinggan@yahoo.com.cn

Received 1 August 2011; Accepted 11 October 2011

Academic Editor: Xiaohua Ding

A class of drift-implicit one-step schemes are proposed for the neutral stochastic delay differential equations (NSDDEs) driven by Poisson processes. A general framework for mean-square convergence of the methods is provided. It is shown that under certain conditions global error estimates for a method can be inferred from estimates on its local error. The applicability of the mean-square convergence theory is illustrated by the stochastic θ-methods and the balanced implicit methods. It is derived from Theorem 3.1 that the order of the mean-square convergence of both of them for NSDDEs with jumps is 1/2. Numerical experiments illustrate the theoretical results. It is worth noting that the results of mean-square convergence of the stochastic θ-methods and the balanced implicit methods are also new.

1. Introduction

In stochastic numerical analysis, the order of convergence plays a crucial role in the design of numerical algorithms. Unlike in the deterministic modelling situation, there exist in the stochastic environment different types of convergence. Both in the literature and in practice, most attention is focused on two major types of convergence, that is, strong convergence and weak convergence. There is a rich literature on this subject; we here only mention [1–4] and the references therein.

For stochastic differential equations (SDEs), Milstein [1] presented a fundamental convergence theorem which established the order of mean-square convergence of explicit one-step methods. The conditions of this theorem use both properties of mean and mean-square deviation of one-step approximation. The theorem showed that under certain conditions global error estimates for a method can be inferred from estimates on its local error. Buckwar [4] extended the convergence theory in [1] to stochastic functional differential equations. Recently, Zhang and Gan [5] extend the theory to neutral stochastic differential delay equations (NSDDEs). Therefore, the convergence theory in [1] and its generalization have received some attention in the case of nonjump SDEs. However, in the jump-SDE context, which is becoming increasingly important in mathematical finance [6–8], to our best knowledge, no corresponding convergence theory of numerical methods for NSDDEs with jumps has been presented in the literature. Motivated by the work of Zhang and Gan [5], our aim is to establish a relationship between the consistent order and the convergence order of the methods for the NSDDEs with jumps.

In this paper, a class of drift-implicit one-step schemes are proposed for NSDDEs driven by Poisson processes. A general framework for mean-square convergence of the methods is provided. It is shown that under certain conditions global error estimates for a method can be inferred from estimates on its local error. The applicability of the theory about mean-square convergence is illustrated by the stochastic θ-methods and the balanced implicit methods. It is derived from Theorem 3.5 that the order of the mean square convergence of both of them for NSDDEs with jumps is 12. It is worth noting that the results of mean-square convergence of the stochastic θ-methods and the balanced implicit methods are also new.

2. Neutral Stochastic Delay Differential Equations with Jumps

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets). Let $W(t) := (W_1(t), \ldots, W_b(t))^T$ be a b-dimensional Wiener process, and $N(t)$ is a scalar Poisson process with intensity λ ($\lambda > 0$), both defined on the space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$. And $|\cdot|$ is used to denote both the norm in $\mathbb{R}^d$ and the trace norm (F-norm) in $\mathbb{R}^{d\times b}$. Also, $C([t_1, t_2]; \mathbb{R}^d)$ is used to represent the family of continuous mappings ψ from $[t_1, t_2]$ to $\mathbb{R}^d$. Finally, $L^p_{\mathcal{F}_t}([t_1, t_2]; \mathbb{R}^d)$ is used to denote a family of $\mathcal{F}_t$-measurable, $C([t_1, t_2]; \mathbb{R}^d)$-valued random variables $\psi = \{\psi(u) : t_1 \leq u \leq t_2\}$ such that $\|\psi\|^p_{\mathbb{E}} := \sup_{t_1\leq u\leq t_2} E|\psi(u)|^p < \infty$. $\mathbb{E}$ denote mathematical expectation with respect to $\mathbb{P}$.

Consider the neutral stochastic delay differential equations (NSDDEs) with Poisson-driven jumps

$$\begin{aligned} \mathrm{d}[x(t) - G(x(t-\tau))] &= f(x(t^-), x(t^- - \tau))\mathrm{d}t + g(x(t^-), x(t^- - \tau))\mathrm{d}W(t) \\ &\quad + u(x(t^-), x(t^- - \tau))\mathrm{d}N(t), \quad t \in [0, T], \end{aligned} \tag{2.1}$$

with initial data

$$x(t) = \psi(t), \quad t \in [-\tau, 0], \tag{2.2}$$

where $\psi(t) \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^d)$. Here, $\tau > 0$ is a constant, $x(t^-)$ denotes $\lim_{s\to t^-} x(s)$, $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times b}$, $u : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, and $G : \mathbb{R}^d \to \mathbb{R}^d$.

By the definition of Itô-interpreted stochastic differential equations, the integral version of (2.1) is expressed as follows:

$$\begin{aligned} x(t) = {} & G(x(t-\tau)) + x(0) - G(x(-\tau)) + \int_0^t f\left(x\left(s^-\right), x\left(s^- - \tau\right)\right) ds \\ & + \int_0^t g\left(x\left(s^-\right), x\left(s^- - \tau\right)\right) dW(s) + \int_0^t u\left(x\left(s^-\right), x\left(s^- - \tau\right)\right) dN(s), \quad t > 0. \end{aligned} \tag{2.3}$$

Definition 2.1 (see [9]). An $\mathbb{R}^d$-valued stochastic process $x(t)$ on $-\tau \le t \le T$ is called a solution to (2.1) with initial data (2.2) if it has the following properties:

(i) $\{x(t)\}_{0\le t\le T}$ is continuous and $\mathcal{F}_t$-adapted;

(ii) $\{f(x(t), x(t-\tau))\} \in \mathcal{L}^1([0,T];\mathbb{R}^d)$, $\{g(x(t), x(t-\tau))\} \in \mathcal{L}^2([0,T];\mathbb{R}^{d\times b})$, $\{u(x(t), x(t-\tau))\} \in \mathcal{L}^1([0,T];\mathbb{R}^d)$;

(iii) $x(t) = \varphi(t)$, $-\tau \le t \le 0$, and (2.3) holds for every $t \in [0,T]$ with probability 1, where $\mathcal{L}^p([0,T];\mathbb{R}^d)$ denotes the family of Borel measurable functions $\varpi : [0,T] \to \mathbb{R}^d$ such that $\int_0^T |\varpi(t)|^p dt < \infty$ a.s.

A solution $\{x(t)\}$ is said to be unique if any other solution $\{\overline{x}(t)\}$ is indistinguishable from $\{x(t)\}$, that is,

$$\mathbb{P}\{x(t) = \overline{x}(t)\ \forall t \in [0\,T]\} = 1. \tag{2.4}$$

In order to guarantee the existence and uniqueness of the solution, we impose the following hypothesis.

Assumption 1 (global Lipschitz condition). There exists a positive constant K such that for all $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$,

$$\left|v(x_1, y_1) - v(x_2, y_2)\right|^2 \leqslant K\left(|x_1 - x_2|^2 + \left|y_1 - y_2\right|^2\right), \quad v = f, g \text{ or } u. \tag{2.5}$$

Assumption 2 (linear growth condition). There exists a positive constant L such that for all $x, y \in \mathbb{R}^d$,

$$\left|v(x, y)\right|^2 \leqslant L\left(1 + |x|^2 + \left|y\right|^2\right), \quad v = f, g \text{ or } u. \tag{2.6}$$

Assumption 3. There is a constant $\eta \in (0,1)$ such that for all $x_1, x_2 \in \mathbb{R}^d$

$$|G(x_1) - G(x_2)| \le \eta |x_1 - x_2|. \tag{2.7}$$

Remark 2.2. In this paper, we always assume that $\eta \neq 0$. Otherwise, the system (2.1) reduces to the stochastic delay differential equations with jumps.

We use $J_1, J_2, \ldots$ to denote the constants which are independent of the stepsize h.

Lemma 2.3. *If Assumptions 1–3 hold, then* (2.1) *has a unique strong solution* $x(t)$ *on* $t \geq -\tau$*, and the solution of* (2.1) *satisfies*

$$\mathbb{E}\left(\sup_{-\tau\leq t\leq T}|x(t)|^2\right) \leq J_1, \tag{2.8}$$

where J_1 *is a positive constant which depends on constants* T*,* η*,* L*, and initial function* $\psi(t)$*.*

It is not hard to prove Lemma 2.3 in a similar way as the proof of Theorems 6.2.2 and 6.4.5 in [9].

Lemma 2.4. *Let Assumptions 2 and 3 hold. Assume that the initial function* $\psi(t)$ *is uniformly Lipschitz* L^2*-continuous, that is, there is a positive constant* $\overline{H}$ *such that*

$$\mathbb{E}\left|\psi(u_1) - \psi(u_2)\right|^2 \leq \overline{H}(u_1 - u_2), \quad \text{if } -\tau \leq u_2 < u_1 \leq 0, \tag{2.9}$$

then

$$\mathbb{E}|x(t_1) - x(t_2)|^2 \leq \widehat{H}(t_1 - t_2), \tag{2.10}$$

for all $0 \leq t_1 < t_2 \leq T$ *with* $t_1 - s\tau \in [-\tau, 0]$*,* $t_2 - s\tau \in [-\tau, 0]$*, where the constant* $\widehat{H}$ *depends on constants* $\overline{H}$*,* T*, initial function* $\psi(t)$*, and positive integer* s*.*

Lemma 2.4 is a modified version of [10, Lemma 2.1]. In a similar way, it is not hard to obtain the estimate (2.10).

In this paper, we will use the following inequality. For any $a, c > 0$ and $0 < \alpha < 1$, we have

$$(a + c)^2 \leq \frac{a^2}{\alpha} + \frac{c^2}{1 - \alpha}. \tag{2.11}$$

3. Implicit One-Step Schemes

Define a mesh with uniform step h which satisfies $\tau = mh$ for an integer number m (for convenience, we assume that $m \geqslant 2$), and suppose that q is a positive integer with $q = T/h$. Let $t_n = nh$, $n = 0, 1, \ldots, q$. The drift-implicit one-step methods for the simulation of the solution $x(t)$ of (2.1) are defined as follows:

$$\begin{aligned} Y_{n+1} = {} & G(Y_{n+1-m}) - G(Y_{n-m}) + Y_n \\ & + \Phi_f(h, Y_n, Y_{n+1}, Y_{n-m}, Y_{n+1-m}, \Delta W_n, \Delta N_n) \\ & + \Phi_g(h, Y_n, Y_{n-m}, \Delta W_n, \Delta N_n) + \Phi_u(h, Y_n, Y_{n-m}, \Delta W_n, \Delta N_n), \\ & \qquad n = 0, 1, \ldots, q-1, \end{aligned} \tag{3.1}$$

where Y_n is an approximation of the exact solution $x(t_n)$, and Φ_v ($v = f$, g, and u) are increment functions. $\Delta W_n = W(t_{n+1}) - W(t_n)$, $\Delta N_n = N(t_{n+1}) - N(t_n)$, and ΔW_n is independent of ΔN_n. $Y_{n-m} = \psi(t_n - \tau)$ when $n - m \leqslant 0$.

Remark 3.1. Now, we discuss the solvability of (3.1). If increment function Φ_f does not depend on Y_{n+1}, it is not difficult to find that the approximations Y_n can be computed iteratively. If Φ_f depends on Y_{n+1}, in order to guarantee the existence and uniqueness of a solution, the general approach is to assume Lipschitz continuity of Φ_f with respect to Y_{n+1} with the Lipschitz constant less than 1, and then to apply Banach's contraction mapping principle [4].

We denote by $\overline{Y}(t_{n+1})$ the value that is obtained when the exact solution values are inserted into the right-hand side of (3.1), that is,

$$\begin{aligned}\overline{Y}(t_{n+1}) = {} & G(x(t_{n+1} - \tau)) - G(x(t_n - \tau)) + x(t_n) \\ & + \Phi_f(h, x(t_n), x(t_{n+1}), x(t_n - \tau), x(t_{n+1} - \tau), \Delta W_n, \Delta N_n) \\ & + \Phi_g(h, x(t_n), x(t_n - \tau), \Delta W_n, \Delta N_n) + \Phi_u(h, x(t_n), x(t_n - \tau), \Delta W_n, \Delta N_n), \\ & n = 0, 1, \ldots, q-1.\end{aligned} \tag{3.2}$$

We introduce the following definitions, which are presented in the literature, see [1, 3], for example.

Definition 3.2. The *local error* of method (3.1) is the sequence of random variables

$$\delta_{n+1} := x(t_{n+1}) - \overline{Y}(t_{n+1}), \quad n = 0, 1, \ldots, q-1. \tag{3.3}$$

The *global error* of method (3.1) is the sequence of random variables

$$\varepsilon_n := x(t_n) - Y_n, \quad n = 0, 1, \ldots, q. \tag{3.4}$$

Definition 3.3. The numerical method (3.1) is said to be *consistent with order p_1 in the mean and with order p_2 in the mean square sense* if the following estimates hold with $p_2 \geqslant 1/2$ and $p_1 \geqslant p_2 + 1/2$:

$$\begin{aligned}\max_{0 \le n \le q-1} \|\mathbb{E}(\delta_{n+1} \mid \mathcal{F}_{t_n})\|_{L_2} \le H_0 h^{p_1}, \quad \text{as } h \longrightarrow 0, \\ \max_{0 \le n \le q-1} \|\delta_{n+1}\|_{L_2} \le H_1 h^{p_2}, \quad \text{as } h \longrightarrow 0,\end{aligned} \tag{3.5}$$

where the constants H_0 and H_1 do not depend on h but may depend on T and the initial data. Here, $\|z\|_{L_2} := (\mathbb{E}|z|^2)^{1/2}$.

Definition 3.4. The numerical method (3.1) is said to be *convergent with order p in the mean square sense*, on the mesh points, if the following estimate holds:

$$\max_{1\le n\le q}\|\varepsilon_n\|_{L_2} \le H_2 h^p, \quad \text{as } h \longrightarrow 0, \tag{3.6}$$

where the constant H_2 does not depend on h but may depend on T and the initial data.

In order to obtain the main result, the following properties of the increment functions are required. There exist positive constants L_f such that for $x_i, y_i \in \mathbb{R}^d$ $(i = 1, 2, 3, 4)$,

$$\left|\Phi_f(h, x_1, x_2, x_3, x_4, \Delta W_n, \Delta N_n) - \Phi_f(h, y_1, y_2, y_3, y_4, \Delta W_n, \Delta N_n)\right| \le hL_f \sum_{i=1}^{4} |x_i - y_i|, \tag{3.7}$$

and there exist the positive constants $L_g, L_u, L_{\overline{u}}$ such that for all $\mathcal{F}_{t_n}$-measurable random variables $x_1, x_3, y_1, y_3 \in \mathbb{R}^d$,

$$\begin{aligned}&\mathbb{E}\left|\Phi_g(h, x_1, x_3, \Delta W_n, \Delta N_n) - \Phi_g(h, y_1, y_3, \Delta W_n, \Delta N_n)\right|^2 \\ &\quad\le hL_g\left[\mathbb{E}|x_1 - y_1|^2 + \mathbb{E}|x_3 - y_3|^2\right],\end{aligned} \tag{3.8}$$

$$\begin{aligned}&\mathbb{E}\left|\Phi_u(h, x_1, x_3, \Delta W_n, \Delta N_n) - \Phi_u(h, y_1, y_3, \Delta W_n, \Delta N_n)\right|^2 \\ &\quad\le hL_u\left[\mathbb{E}|x_1 - y_1|^2 + \mathbb{E}|x_3 - y_3|^2\right],\end{aligned} \tag{3.9}$$

$$\mathbb{E}\left(\Phi_g(h, x_1, x_3, \Delta W_n, \Delta N_n) - \Phi_g(h, y_1, y_3, \Delta W_n, \Delta N_n) \mid \mathcal{F}_{t_n}\right) = 0, \tag{3.10}$$

$$\begin{aligned}&\left|\mathbb{E}\left[\Phi_u(h, x_1, x_3, \Delta W_n, \Delta N_n) - \Phi_u(h, y_1, y_3, \Delta W_n, \Delta N_n) \mid \mathcal{F}_{t_n}\right]\right| \\ &\quad\le hL_{\overline{u}}\left[|x_1 - y_1| + |x_3 - y_3|\right].\end{aligned} \tag{3.11}$$

Now, we state our result on the convergence of the one-step method (3.1).

Theorem 3.5. *Suppose that Assumption 3 and the conditions* (3.7)–(3.11) *hold. Assume that the one-step method* (3.1) *is consistent with order p_1 in the mean and order p_2 in the mean square sense, then the method* (3.1) *is convergent with order $p = p_2 - 1/2$ in the mean square sense.*

Proof. By (3.4), we have

$$\varepsilon_{n+1} = \widehat{\varepsilon}_{n+1} + \left[G(x(t_{n+1} - \tau)) - G(Y_{n+1-m})\right], \tag{3.12}$$

where $\widehat{\varepsilon}_{n+1}$ is defined as follows:

$$\widehat{\varepsilon}_{n+1} = x(t_{n+1}) - G(x(t_{n+1} - \tau)) - \left[Y_{n+1} - G(Y_{n+1-m})\right]. \tag{3.13}$$

Squaring and taking expectation on both sides of (3.12), using Assumption 3 and (2.11), we have

$$\begin{aligned}\mathbb{E}|\varepsilon_{n+1}|^2 &\leq \frac{1}{1-\eta}\mathbb{E}|\widehat{\varepsilon}_{n+1}|^2 + \frac{1}{\eta}\mathbb{E}|G(x(t_{n+1}-\tau)) - G(Y_{n+1-m})|^2 \\ &\leq \frac{1}{1-\eta}\mathbb{E}|\widehat{\varepsilon}_{n+1}|^2 + \eta\mathbb{E}|x(t_{n+1}-\tau) - Y_{n+1-m}|^2 \\ &= \frac{1}{1-\eta}\mathbb{E}|\widehat{\varepsilon}_{n+1}|^2 + \eta\mathbb{E}|\varepsilon_{n+1-m}|^2.\end{aligned} \tag{3.14}$$

It follows from (3.1), (3.2), and (3.13) that

$$\begin{aligned}\widehat{\varepsilon}_{n+1} &= x(t_{n+1}) - G(x(t_{n+1}-\tau)) - \overline{Y}(t_{n+1}) + \overline{Y}(t_{n+1}) - [Y_{n+1} - G(Y_{n+1-m})] \\ &= x(t_{n+1}) - \overline{Y}(t_{n+1}) + x(t_n) - G(x(t_n-\tau)) \\ &\quad + \Phi_f(h, x(t_n), x(t_{n+1}), x(t_n-\tau), x(t_{n+1}-\tau), \Delta W_n, \Delta N_n) \\ &\quad + \Phi_g(h, x(t_n), x(t_n-\tau), \Delta W_n, \Delta N_n) + \Phi_u(h, x(t_n), x(t_n-\tau), \Delta W_n, \Delta N_n) \\ &\quad - Y_n + G(Y_{n-m}) - \Phi_f(h, Y_n, Y_{n+1}, Y_{n-m}, Y_{n+1-m}, \Delta W_n, \Delta N_n) \\ &\quad - \Phi_g(h, Y_n, Y_{n-m}, \Delta W_n, \Delta N_n) - \Phi_u(h, Y_n, Y_{n-m}, \Delta W_n, \Delta N_n) \\ &= \delta_{n+1} + \widehat{\varepsilon}_n + R_n,\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}R_n &= \Phi_f(h, x(t_n), x(t_{n+1}), x(t_n-\tau), x(t_{n+1}-\tau), \Delta W_n, \Delta N_n) \\ &\quad - \Phi_f(h, Y_n, Y_{n+1}, Y_{n-m}, Y_{n+1-m}, \Delta W_n, \Delta N_n) \\ &\quad + \Phi_g(h, x(t_n), x(t_n-\tau), \Delta W_n, \Delta N_n) - \Phi_g(h, Y_n, Y_{n-m}, \Delta W_n, \Delta N_n) \\ &\quad + \Phi_u(h, x(t_n), x(t_n-\tau), \Delta W_n, \Delta N_n) - \Phi_u(h, Y_n, Y_{n-m}, \Delta W_n, \Delta N_n).\end{aligned} \tag{3.16}$$

Squaring and taking expectation on both sides of (3.15) yields

$$\mathbb{E}|\widehat{\varepsilon}_{n+1}|^2 \leq \mathbb{E}|\widehat{\varepsilon}_n|^2 + 2\mathbb{E}|\delta_{n+1}|^2 + 2\mathbb{E}|R_n|^2 + 2\mathbb{E}\langle\delta_{n+1}, \widehat{\varepsilon}_n\rangle + 2\mathbb{E}\langle\widehat{\varepsilon}_n, R_n\rangle. \tag{3.17}$$

We will now estimate the separate terms in (3.17) individually. Without loss of generality, we can assume that $0 < h < 1$. We notice that the method (3.1) is consistent with order p_2 in the mean square sense; thus, there exists a constant J_2 such that

$$\mathbb{E}|\delta_{n+1}|^2 \leq J_2 h^{2p_2}. \tag{3.18}$$

By (3.7)–(3.9), we obtain

$$
\begin{aligned}
\mathbb{E}|R_n|^2 \leq\ & 3\mathbb{E}\Big|\Phi_f(h, x(t_n), x(t_{n+1}), x(t_n-\tau), x(t_{n+1}-\tau), \Delta W_n, \Delta N_n) \\
& -\Phi_f(h, Y_n, Y_{n+1}, Y_{n-m}, Y_{n+1-m}, \Delta W_n, \Delta N_n)\Big|^2 \\
& + 3\mathbb{E}\Big|\Phi_g(h, x(t_n), x(t_n-\tau), \Delta W_n, \Delta N_n) - \Phi_g(h, Y_n, Y_{n-m}, \Delta W_n, \Delta N_n)\Big|^2 \\
& + 3\mathbb{E}|\Phi_u(h, x(t_n), x(t_n-\tau), \Delta W_n, \Delta N_n) - \Phi_u(h, Y_n, Y_{n-m}, \Delta W_n, \Delta N_n)|^2 \\
\leq\ & 3h^2L_f^2\mathbb{E}(|x(t_n)-Y_n| + |x(t_{n+1})-Y_{n+1}| + |x(t_n-\tau)-Y_{n-m}| \\
& +|x(t_{n+1}-\tau)-Y_{n+1-m}|)^2 + 3hL_g\Big(\mathbb{E}|x(t_n)-Y_n|^2 + \mathbb{E}|x(t_n-\tau)-Y_{n-m}|^2\Big) \\
& + 3hL_u\Big(\mathbb{E}|x(t_n)-Y_n|^2 + \mathbb{E}|x(t_n-\tau)-Y_{n-m}|^2\Big) \\
\leq\ & \Big(12h^2L_f^2 + 3hL_g + 3hL_u\Big)\Big(\mathbb{E}|x(t_n)-Y_n|^2 + \mathbb{E}|x(t_n-\tau)-Y_{n-m}|^2\Big) \\
& + 12h^2L_f^2\Big(\mathbb{E}|x(t_{n+1})-Y_{n+1}|^2 + \mathbb{E}|x(t_{n+1}-\tau)-Y_{n+1-m}|^2\Big) \\
=\ & \Big(12h^2L_f^2 + 3hL_g + 3hL_u\Big)\Big(\mathbb{E}|\varepsilon_n|^2 + \mathbb{E}|\varepsilon_{n-m}|^2\Big) + 12h^2L_f^2\Big(\mathbb{E}|\varepsilon_{n+1}|^2 + \mathbb{E}|\varepsilon_{n+1-m}|^2\Big),
\end{aligned}
\tag{3.19}
$$

which, by the inequality (3.14), yields

$$
\begin{aligned}
\mathbb{E}|R_n|^2 \leq\ & \Big(12h^2L_f^2 + 3hL_g + 3hL_u\Big)\left(\frac{1}{1-\eta}\mathbb{E}|\widehat{\varepsilon}_n|^2 + \eta\mathbb{E}|\varepsilon_{n-m}|^2\right) \\
& + \Big(12h^2L_f^2 + 3hL_g + 3hL_u\Big)\mathbb{E}|\varepsilon_{n-m}|^2 \\
& + 12h^2L_f^2\left(\frac{1}{1-\eta}\mathbb{E}|\widehat{\varepsilon}_{n+1}|^2 + \eta\mathbb{E}|\varepsilon_{n+1-m}|^2\right) + 12h^2L_f^2\mathbb{E}|\varepsilon_{n+1-m}|^2 \\
\leq\ & \Big(12h^2L_f^2 + 3hL_g + 3hL_u\Big)\Big(\mathbb{E}|\widehat{\varepsilon}_n|^2 + \mathbb{E}|\varepsilon_{n-m}|^2\Big)\max\left\{\frac{1}{1-\eta}, 1+\eta\right\} \\
& + 12h^2L_f^2\Big(\mathbb{E}|\widehat{\varepsilon}_{n+1}|^2 + \mathbb{E}|\varepsilon_{n+1-m}|^2\Big)\max\left\{\frac{1}{1-\eta}, 1+\eta\right\} \\
\leq\ & \Big(12TL_f^2 + 3L_g + 3L_u\Big)h\Big(\mathbb{E}|\widehat{\varepsilon}_n|^2 + \mathbb{E}|\varepsilon_{n-m}|^2 + \mathbb{E}|\widehat{\varepsilon}_{n+1}|^2 \\
& +\mathbb{E}|\varepsilon_{n+1-m}|^2\Big)\max\left\{\frac{1}{1-\eta}, 1+\eta\right\} \\
=\ & J_3h\Big(\mathbb{E}|\widehat{\varepsilon}_n|^2 + \mathbb{E}|\varepsilon_{n-m}|^2 + \mathbb{E}|\widehat{\varepsilon}_{n+1}|^2 + \mathbb{E}|\varepsilon_{\mathrm{n}+1-m}|^2\Big),
\end{aligned}
\tag{3.20}
$$

where $J_3 = (12TL_f^2 + 3L_g + 3L_u)\max\{1/1-\eta, 1+\eta\}$.

Since method (3.1) is consistent with order p_1 in the mean square, there exists a constant J_4 such that

$$\left(\mathbb{E}\left|\mathbb{E}(\delta_{n+1} \mid \mathcal{F}_{t_n})\right|^2\right)^{1/2} \leq J_4 h^{p_1}. \tag{3.21}$$

Noticing that $\widehat{\varepsilon}_n$ is $\mathcal{F}_{t_n}$-measurable and by (3.21), we arrive at

$$\begin{aligned}
2\mathbb{E}(\langle \widehat{\varepsilon}_n, \delta_{n+1}\rangle) &= 2\mathbb{E}[\mathbb{E}(\langle \widehat{\varepsilon}_n, \delta_{n+1}\rangle \mid \mathcal{F}_{t_n})] \\
&\leq 2\mathbb{E}|\mathbb{E}(\langle \widehat{\varepsilon}_n, \delta_{n+1}\rangle \mid \mathcal{F}_{t_n})| = 2\mathbb{E}|\langle \widehat{\varepsilon}_n, \mathbb{E}(\delta_{n+1} \mid \mathcal{F}_{t_n})\rangle| \\
&\leq 2\left(h\mathbb{E}|\widehat{\varepsilon}_n|^2\right)^{1/2}\left(h^{-1}\mathbb{E}|\mathbb{E}(\delta_{n+1} \mid \mathcal{F}_{t_n})|^2\right)^{1/2} \\
&\leq h\mathbb{E}|\widehat{\varepsilon}_n|^2 + h^{-1}\mathbb{E}|\mathbb{E}(\delta_{n+1} \mid \mathcal{F}_{t_n})|^2 \\
&\leq h\mathbb{E}|\widehat{\varepsilon}_n|^2 + (J_4)^2 h^{2p_1-1} \\
&= h\mathbb{E}|\widehat{\varepsilon}_n|^2 + J_5 h^{2p_1-1},
\end{aligned} \tag{3.22}$$

where $J_5 = (J_4)^2$. Applying the inequality $|\mathbb{E}(x \mid \mathcal{F})|^2 \leq \mathbb{E}(|x|^2 \mid \mathcal{F})$, (3.7), (3.10), and (3.11) yields

$$\begin{aligned}
|\mathbb{E}(R_n \mid \mathcal{F}_{t_n})|^2 = \Big|\mathbb{E}\Big[&\Phi_f(h, x(t_n), x(t_{n+1}), x(t_n-\tau), x(t_{n+1}-\tau), \Delta W_n, \Delta N_n) \\
&- \Phi_f(h, Y_n, Y_{n+1}, Y_{n-m}, Y_{n+1-m}, \Delta W_n, \Delta N_n) \\
&+ \Phi_g(h, x(t_n), x(t_n-\tau), \Delta W_n, \Delta N_n) - \Phi_g(h, Y_n, Y_{n-m}, \Delta W_n, \Delta N_n) \\
&+ \Phi_u(h, x(t_n), x(t_n-\tau), \Delta W_n, \Delta N_n) - \Phi_u(h, Y_n, Y_{n-m}, \Delta W_n, \Delta N_n) \mid \mathcal{F}_{t_n}\Big]\Big|^2 \\
= \Big|\mathbb{E}\big[&\Phi_f(h, x(t_n), x(t_{n+1}), x(t_n-\tau), x(t_{n+1}-\tau), \Delta W_n, \Delta N_n) \\
&- \Phi_f(h, Y_n, Y_{n+1}, Y_{n-m}, Y_{n+1-m}, \Delta W_n, \Delta N_n) \mid \mathcal{F}_{t_n}\big] \\
+ \mathbb{E}\big[&\Phi_u(h, x(t_n), x(t_n-\tau), \Delta W_n, \Delta N_n) - \Phi_u(h, Y_n, Y_{n-m}, \Delta W_n, \Delta N_n) \mid \mathcal{F}_{t_n}\big]\Big|^2 \\
\leq 2\mathbb{E}\Big[\Big|&\Phi_f(h, x(t_n), x(t_{n+1}), x(t_n-\tau), x(t_{n+1}-\tau), \Delta W_n, \Delta N_n) \\
&- \Phi_f(h, Y_n, Y_{n+1}, Y_{n-m}, Y_{n+1-m}, \Delta W_n, \Delta N_n)\Big|^2 \mid \mathcal{F}_{t_n}\Big] \\
+ 2\Big|\mathbb{E}\Big[&\Phi_u(h, x(t_n), x(t_n-\tau), \Delta W_n, \Delta N_n) - \Phi_u(h, Y_n, Y_{n-m}, \Delta W_n, \Delta N_n) \mid \mathcal{F}_{t_n}\Big]\Big|^2 \\
\leq 2h^2(L_f)^2\mathbb{E}\Big[&(|x(t_n) - Y_n| + |x(t_{n+1}) - Y_{n+1}| + |x(t_n-\tau) - Y_{n-m}| \\
&+ |x(t_{n+1}-\tau) - Y_{n+1-m}|)^2 \mid \mathcal{F}_{t_n}\Big] \\
+ 2h^2(L_{\overline{u}})^2&[|x(t_n) - Y_n| + |x(t_n-\tau) - Y_{n-m}|]^2 \\
\leq 8h^2(L_f)^2&\Big[|\varepsilon_n|^2 + \mathbb{E}\Big(|\varepsilon_{n+1}|^2 \mid \mathcal{F}_{t_n}\Big) + |\varepsilon_{n-m}|^2 + |\varepsilon_{n+1-m}|^2\Big] \\
+ 4h^2(L_{\overline{u}})^2&\Big[|\varepsilon_n|^2 + |\varepsilon_{n-m}|^2\Big]
\end{aligned}$$

$$= \Big[8h^2(L_f)^2 + 4h^2(L_{\overline{u}})^2\Big]|\varepsilon_n|^2 + 8h^2(L_f)^2\mathbb{E}\Big(|\varepsilon_{n+1}|^2 \mid \mathcal{F}_{t_n}\Big)$$
$$+ \Big[8h^2(L_f)^2 + 4h^2(L_{\overline{u}})^2\Big]|\varepsilon_{n-m}|^2 + 8h^2(L_f)^2|\varepsilon_{n+1-m}|^2. \tag{3.23}$$

Here, the fact used has been that $x(t_n)$, $x(t_n - \tau)$, $x(t_{n+1} - \tau)$, Y_n, Y_{n-m}, Y_{n+1-m} are $\mathcal{F}_{t_n}$-measurable. Using (3.14) and (3.23), we arrive at

$$2\mathbb{E}(\langle\widehat{\varepsilon}_n, R_n\rangle) = 2\mathbb{E}[\mathbb{E}(\langle\widehat{\varepsilon}_n, R_n\rangle \mid \mathcal{F}_{t_n})]$$
$$= 2[\mathbb{E}\langle\widehat{\varepsilon}_n, \mathbb{E}(R_n \mid \mathcal{F}_{t_n})\rangle] \le 2\mathbb{E}[|\widehat{\varepsilon}_n| \cdot |\mathbb{E}(R_n \mid \mathcal{F}_{t_n})|]$$
$$\le 2\Big(h\mathbb{E}|\widehat{\varepsilon}_n|^2\Big)^{1/2}\Big(h^{-1}\mathbb{E}|\mathbb{E}(R_n \mid \mathcal{F}_{t_n})|^2\Big)^{1/2}$$
$$\le h\mathbb{E}|\widehat{\varepsilon}_n|^2 + h^{-1}\mathbb{E}|\mathbb{E}(R_n \mid \mathcal{F}_{t_n})|^2$$
$$\le h\mathbb{E}|\widehat{\varepsilon}_n|^2 + \Big[8h(L_f)^2 + 4h(L_{\overline{u}})^2\Big]\mathbb{E}|\varepsilon_n|^2 + 8h(L_f)^2\mathbb{E}|\varepsilon_{n+1}|^2$$
$$+ \Big[8h(L_f)^2 + 4h(L_{\overline{u}})^2\Big]\mathbb{E}|\varepsilon_{n-m}|^2 + 8h(L_f)^2\mathbb{E}|\varepsilon_{n+1-m}|^2$$
$$\le h\mathbb{E}|\widehat{\varepsilon}_n|^2 + \Big[8h(L_f)^2 + 4h(L_{\overline{u}})^2\Big]\left[\frac{1}{1-\eta}\mathbb{E}|\widehat{\varepsilon}_n|^2 + \eta\mathbb{E}|\varepsilon_{n-m}|^2\right]$$
$$+ 8h(L_f)^2\left[\frac{1}{1-\eta}\mathbb{E}|\widehat{\varepsilon}_{n+1}|^2 + \eta\mathbb{E}|\varepsilon_{n+1-m}|^2\right]$$
$$+ \Big[8h(L_f)^2 + 4h(L_{\overline{u}})^2\Big]\mathbb{E}|\varepsilon_{n-m}|^2 + 8h(L_f)^2\mathbb{E}|\varepsilon_{n+1-m}|^2$$
$$= \left[1 + \Big(8(L_f)^2 + 4(L_{\overline{u}})^2\Big)\frac{1}{1-\eta}\right]h\mathbb{E}|\widehat{\varepsilon}_n|^2 + 8(L_f)^2\frac{1}{1-\eta}h\mathbb{E}|\widehat{\varepsilon}_{n+1}|^2$$
$$+ \Big(8(L_f)^2 + 4(L_{\overline{u}})^2\Big)(1+\eta)\ h\mathbb{E}|\varepsilon_{n-m}|^2 + 8(L_f)^2(1+\eta)h\mathbb{E}|\varepsilon_{n+1-m}|^2$$
$$\le \left[1 + \Big(8(L_f)^2 + 4(L_{\overline{u}})^2\Big)\max\left\{\frac{1}{1-\eta}, 1+\eta\right\}\right]h,$$
$$\Big[\mathbb{E}|\widehat{\varepsilon}_n|^2 + \mathbb{E}|\widehat{\varepsilon}_{n+1}|^2 + \mathbb{E}|\varepsilon_{n+1-m}|^2 + \mathbb{E}|\varepsilon_{n-m}|^2\Big]$$
$$= J_6 h\Big[\mathbb{E}|\widehat{\varepsilon}_n|^2 + \mathbb{E}|\widehat{\varepsilon}_{n+1}|^2 + \mathbb{E}|\varepsilon_{n+1-m}|^2 + \mathbb{E}|\varepsilon_{n-m}|^2\Big], \tag{3.24}$$

where $J_6 = 1+(8(L_f)^2+4(L_{\overline{u}})^2)\max\{1/(1-\eta), 1+\eta\}$. Inserting (3.18), (3.20), (3.22), and (3.24) into (3.17) yields

$$\begin{aligned}
\mathbb{E}|\widehat{\varepsilon}_{n+1}|^2 \leq\ & \mathbb{E}|\widehat{\varepsilon}_n|^2 + 2J_2h^{2p_2} + 2J_3h\Big(\mathbb{E}|\widehat{\varepsilon}_n|^2 + \mathbb{E}|\varepsilon_{n-m}|^2 + \mathbb{E}|\widehat{\varepsilon}_{n+1}|^2 + \mathbb{E}|\varepsilon_{n+1-m}|^2\Big) \\
& + h\mathbb{E}|\widehat{\varepsilon}_n|^2 + J_5h^{2p_1-1} + J_6h\Big(\mathbb{E}|\widehat{\varepsilon}_n|^2 + \mathbb{E}|\widehat{\varepsilon}_{n+1}|^2 + \mathbb{E}|\varepsilon_{n+1-m}|^2 + \mathbb{E}|\varepsilon_{n-m}|^2\Big) \\
=\ & (2J_3 + J_6)h\mathbb{E}|\widehat{\varepsilon}_{n+1}|^2 + (1 + (1 + 2J_3 + J_6)h)\mathbb{E}|\widehat{\varepsilon}_n|^2 \\
& + (2J_3 + J_6)h\Big(\mathbb{E}|\varepsilon_{n+1-m}|^2 + \mathbb{E}|\varepsilon_{n-m}|^2\Big) + (2J_2 + J_5)h^{2p_2}.
\end{aligned} \tag{3.25}$$

The following proof is analogous to that of Theorem 3.1 in [5]; thus, it is not hard to derive the convergence result. The proof is completed. □

Remark 3.6. Notice that if $u(\cdot) = 0$ in (2.1), then (2.1) reduces to the NSDDEs without jumps, our Theorem 3.5 coincides with Theorem 3.1 in [5], that is to say, Theorem 3.5 generalizes Theorem 3.1 in [5] to the case of NSDDEs with jumps.

4. The Examples

Theorem 3.5 presents the convergence result about the general implicit one-step methods for NSDDEs with jumps. In this section, we discuss the applicability of the theory presented in the previous section. We will give the convergence orders of the stochastic θ-methods and the balanced implicit methods.

Example 4.1. Consider the stochastic θ-methods for system (2.1),

$$\begin{aligned}
Y_{n+1} =\ & G(Y_{n+1-m}) + Y_n - G(Y_{n-m}) \\
& + \big[(1-\theta)f(Y_n, Y_{n-m}) + \theta f(Y_{n+1}, Y_{n+1-m})\big]h \\
& + g(Y_n, Y_{n-m})\Delta W_n + u(Y_n, Y_{n-m})\Delta N_n, \quad n = 0, 1, \ldots, q-1,
\end{aligned} \tag{4.1}$$

where $m = \tau/h$, $0 \leq \theta \leq 1$.

Lemma 4.2. *Let Assumption 1 hold, then the stochastic θ-methods* (4.1) *are consistent with order $p_1 = 3/2$ in the mean and order $p_2 = 1$ in the mean square sense.*

Proof. Combining (2.1), (3.2) with (4.1) yields

$$\begin{aligned}
\delta_{n+1} &= x(t_{n+1}) - \overline{Y}(t_{n+1}) \\
&= \int_{t_n}^{t_{n+1}} \big[f(x(s), x(s-\tau)) - f(x(t_n), x(t_n-\tau))\big]ds \\
&\quad + \int_{t_n}^{t_{n+1}} \big[g(x(s), x(s-\tau)) - g(x(t_n), x(t_n-\tau))\big]dW(s)
\end{aligned}$$

$$+\int_{t_n}^{t_{n+1}}[u(x(s),x(s-\tau))-u(x(t_n),x(t_n-\tau))]dN(s)$$
$$-\theta h\big[f(x(t_{n+1}),x(t_{n+1}-\tau))-f(x(t_n),x(t_n-\tau))\big]. \tag{4.2}$$

Noticing the compensated Poisson process, $\widetilde{N}(t)=N(t)-\lambda t$, which satisfies

$$\mathbb{E}\left(\int_a^b u(s)d\widetilde{N}(s)\mid\mathcal{F}_a\right)=0. \tag{4.3}$$

Using Assumption 1, (4.3), Hölder inequality, Lemma 2.4, the properties of conditional expectation, and *Jenson's* inequality: $|\mathbb{E}(x\mid\mathcal{F})|^2\le\mathbb{E}(|x|^2\mid\mathcal{F})$, we compute that

$$\begin{aligned}
\mathbb{E}|\mathbb{E}(\delta_{n+1}\mid\mathcal{F}_{t_n})|^2 &\le 5\mathbb{E}\left|\mathbb{E}\left(\int_{t_n}^{t_{n+1}}\big[f(x(s),x(s-\tau))-f(x(t_n),x(t_n-\tau))\big]ds\mid\mathcal{F}_{t_n}\right)\right|^2\\
&\quad+5\mathbb{E}\left|\mathbb{E}\left(\int_{t_n}^{t_{n+1}}\big[g(x(s),x(s-\tau))-g(x(t_n),x(t_n-\tau))\big]dW(s)\mid\mathcal{F}_{t_n}\right)\right|^2\\
&\quad+5\mathbb{E}\left|\mathbb{E}\left(\int_{t_n}^{t_{n+1}}[u(x(s),x(s-\tau))-u(x(t_n),x(t_n-\tau))]d\widetilde{N}(s)\mid\mathcal{F}_{t_n}\right)\right|^2\\
&\quad+5\lambda^2\mathbb{E}\left|\mathbb{E}\left(\int_{t_n}^{t_{n+1}}[u(x(s),x(s-\tau))-u(x(t_n),x(t_n-\tau))]ds\mid\mathcal{F}_{t_n}\right)\right|^2\\
&\quad+5h^2\mathbb{E}\big|\mathbb{E}\big(f(x(t_{n+1}),x(t_{n+1}-\tau))-f(x(t_n),x(t_n-\tau))\mid\mathcal{F}_{t_n}\big)\big|^2\\
&\le 5\mathbb{E}\left[\mathbb{E}\left(\left|\int_{t_n}^{t_{n+1}}f(x(s),x(s-\tau))-f(x(t_n),x(t_n-\tau))ds\right|^2\mid\mathcal{F}_{t_n}\right)\right]\\
&\quad+5\lambda^2\mathbb{E}\left[\mathbb{E}\left(\left|\int_{t_n}^{t_{n+1}}u(x(s),x(s-\tau))-u(x(t_n),x(t_n-\tau))ds\right|^2\mid\mathcal{F}_{t_n}\right)\right]\\
&\quad+5h^2\mathbb{E}\Big[\mathbb{E}\Big(\big|f(x(t_{n+1}),x(t_{n+1}-\tau))-f(x(t_n),x(t_n-\tau))\big|^2\mid\mathcal{F}_{t_n}\Big)\Big]\\
&\le 5\Big(1+\lambda^2\Big)Kh\int_{t_n}^{t_{n+1}}\mathbb{E}\Big(|x(s)-x(t_n)|^2+|x(s-\tau)-x(t_n-\tau)|^2\Big)ds
\end{aligned}$$

$$+ 5h^2K\mathbb{E}\left(|x(t_{n+1}) - x(t_n)|^2 + |x(t_{n+1} - \tau) - x(t_n - \tau)|^2\right)$$

$$\leq \left[10\left(1 + \lambda^2\right)K\widehat{H} + 10K\widehat{H}\right]h^3, \tag{4.4}$$

$$\begin{aligned}
\mathbb{E}|\delta_{n+1}|^2 \leq{}& 4h\mathbb{E}\int_{t_n}^{t_{n+1}} \left|f(x(s), x(s-\tau)) - f(x(t_n), x(t_n - \tau))\right|^2 ds \\
&+ 4\mathbb{E}\int_{t_n}^{t_{n+1}} \left|g(x(s), x(s-\tau)) - g(x(t_n), x(t_n - \tau))\right|^2 ds \\
&+ 8\lambda\int_{t_n}^{t_{n+1}} \mathbb{E}|u(x(s), x(s-\tau)) - u(x(t_n), x(t_n - \tau))|^2 ds \\
&+ 8\lambda^2 h\int_{t_n}^{t_{n+1}} \mathbb{E}|u(x(s), x(s-\tau)) - u(x(t_n), x(t_n - \tau))|^2 ds \\
&+ 4h^2\mathbb{E}\left|f(x(t_{n+1}), x(t_{n+1} - \tau)) - f(x(t_n), x(t_n - \tau))\right|^2 \\
\leq{}& \left(4h + 4 + 8\lambda + 8\lambda^2 h\right)K\int_{t_n}^{t_{n+1}} \mathbb{E}\left(|x(s) - x(t_n)|^2 + |x(s-\tau) - x(t_n - \tau)|^2\right) ds \\
&+ 4Kh^2\mathbb{E}\left(|x(t_{n+1}) - x(t_n)|^2 + |x(t_{n+1} - \tau) - x(t_n - \tau)|^2\right) \\
\leq{}& 8K\widehat{H}\left(1 + 2T + 2\lambda + 2\lambda^2 T\right)h^2,
\end{aligned} \tag{4.5}$$

where the compensated Poisson process $\widetilde{N}(t) = N(t) - \lambda t$ satisfies

$$\mathbb{E}\left|\int_{t_1}^{t_2} u(x(s))d\widetilde{N}(s)\right|^2 = \lambda\int_{t_1}^{t_2} \mathbb{E}|u(x(s))|^2 ds. \tag{4.6}$$

Hence, the stochastic θ-methods (4.1) are consistent with order 3/2 in the mean and order 1 in the mean square sense. The proof is completed. □

Theorem 4.3. *Let Assumption 1 hold, then the stochastic θ-methods* (4.1) *are convergent with order $p = 1/2$ in the mean square sense.*

Proof. Lemma 4.2 shows that the stochastic θ-methods (4.1) are consistent with order $p_1 = 3/2$ in the mean and order $p_2 = 1$ in the mean square sense. In order to prove that the stochastic θ-methods (4.1) are convergent with order $p = 1/2$ in the mean square sense, by Theorem 3.5, we only need to verify the conditions (3.7)–(3.11).

From (4.1), we find that

$$\begin{aligned}\Phi_f(h, Y_n, Y_{n+1}, Y_{n-m}, Y_{n+1-m}, \Delta W_n, \Delta N_n) &= \left[(1-\theta) f(Y_n, Y_{n-m}) + \theta f(Y_{n+1}, Y_{n+1-m})\right] h, \\ \Phi_g(h, Y_n, Y_{n-m}, \Delta W_n, \Delta N_n) &= g(Y_n, Y_{n-m}) \Delta W_n, \\ \Phi_u(h, Y_n, Y_{n-m}, \Delta W_n, \Delta N_n) &= u(Y_n, Y_{n-m}) \Delta N_n.\end{aligned} \tag{4.7}$$

For the random variables $x_i, y_i \in \mathbb{R}^d$ $(i = 1, 2, 3, 4)$, by Assumption 1, we have

$$\begin{aligned}&\left|\Phi_f(h, x_1, x_2, x_3, x_4, \Delta W_n, \Delta N_n) - \Phi_f(h, y_1, y_2, y_3, y_4, \Delta W_n, \Delta N_n)\right| \\ &\quad = \left|\left[(1-\theta) f(x_1, x_3) + \theta f(x_2, x_4)\right] h - \left[(1-\theta) f(y_1, y_3) + \theta f(y_2, y_4)\right] h\right| \\ &\quad \leq \sqrt{K} h \sum_{i=1}^{4} \left|x_i - y_i\right|.\end{aligned} \tag{4.8}$$

Noticing that $\mathbb{E}|\Delta W_n|^2 = bh$, $\mathbb{E}|\Delta N_n|^2 = \lambda h(1+\lambda h)$, and the random variables x_1, x_3, y_1, y_3 are $\mathcal{F}_{t_n}$-measurable, we derive that

$$\begin{aligned}&\mathbb{E}\left|\Phi_g(h, x_1, x_3, \Delta W_n, \Delta N_n) - \Phi_g(h, y_1, y_3, \Delta W_n, \Delta N_n)\right|^2 \\ &\quad \leq \mathbb{E}\left|g(x_1, x_3) \Delta W_n - g(y_1, y_3) \Delta W_n\right|^2 \\ &\quad \leq b\mathrm{K}h\mathbb{E}\left|x_1 - y_1\right|^2 + \mathbb{E}\left|x_3 - y_3\right|^2,\end{aligned} \tag{4.9}$$

$$\begin{aligned}&\mathbb{E}\left|\Phi_u(h, x_1, x_3, \Delta W_n, \Delta N_n) - \Phi_u(h, y_1, y_3, \Delta W_n, \Delta N_n)\right|^2 \\ &\quad \leq \mathbb{E}\left|u(x_1, x_3) \Delta N_n - u(y_1, y_3) \Delta N_n\right|^2 \\ &\quad \leq \lambda(1 + \lambda T) K h \left[\mathbb{E}\left|x_1 - y_1\right|^2 + \mathbb{E}\left|x_3 - y_3\right|^2\right].\end{aligned} \tag{4.10}$$

From (4.8)–(4.10), we see that the increment functions of the stochastic θ-methods (4.1) satisfy the inequalities (3.7)–(3.9) with $L_f = \sqrt{K}$, $L_g = bK$, and $L_u = \lambda(1 + \lambda T)K$. Noting that $\Delta W_n, \Delta N_n$ are independent of $\mathcal{F}_{t_n}$ and x_1, x_3, y_1, and y_3 are $\mathcal{F}_{t_n}$-measurable, then using $|\mathbb{E}(x \mid \mathcal{F})| \leq \mathbb{E}(|x| \mid \mathcal{F})$, we find that

$$\begin{aligned}&\mathbb{E}\left[\Phi_g(h, x_1, x_3, \Delta W_n, \Delta N_n) - \Phi_g(h, y_1, y_3, \Delta W_n, \Delta N_n) \mid \mathcal{F}_{t_n}\right] \\ &\quad = \mathbb{E}\left[g(x_1, x_3) \Delta W_n - g(y_1, y_3) \Delta W_n \mid \mathcal{F}_{t_n}\right] \\ &\quad = \left[g(x_1, x_3) - g(y_1, y_3)\right] \mathbb{E}(\Delta W_n \mid \mathcal{F}_{t_n}) = 0, \\ &\left|\mathbb{E}\left[\Phi_u(h, x_1, x_3, \Delta W_n, \Delta N_n) - \Phi_u(h, y_1, y_3, \Delta W_n, \Delta N_n) \mid \mathcal{F}_{t_n}\right]\right| \\ &\quad = \left|\mathbb{E}\left[u(x_1, x_3) \Delta N_n - u(y_1, y_3) \Delta N_n \mid \mathcal{F}_{t_n}\right]\right| \\ &\quad = \left|\left[u(x_1, x_3) - u(y_1, y_3)\right]\right| \cdot \mathbb{E}(\Delta N_n \mid \mathcal{F}_{t_n}) \\ &\quad \leq \lambda \sqrt{K} h \left[\left|x_1 - y_1\right| + \left|x_3 - y_3\right|\right].\end{aligned} \tag{4.11}$$

Here, the fact used has been that $\mathbb{E}\Delta W_n = 0$ and $\mathbb{E}\Delta N_n = \lambda h$. From above, we find that the increment functions of the stochastic θ-methods (4.1) satisfy the estimations (3.10) and (3.11) with $L_{\overline{u}} = \lambda\sqrt{K}$. Therefore, the conditions (3.7)–(3.11) are satisfied. By Lemma 4.2 and Theorem 3.5, it is not difficult to find that the stochastic θ-methods (4.1) are convergent with order $p = 1/2$ in the mean square sense. The proof is completed. □

Remark 4.4. For the case of $G(\cdot) = 0$, (2.1) reduces to the stochastic delay differential equations with jumps

$$\begin{aligned}
\mathrm{d}x(t) &= f\left(x(t^-), x(t^- - \tau)\right)\mathrm{d}t + g\left(x(t^-), x(t^- - \tau)\right)\mathrm{d}W(t) \\
&\quad + u\left(x(t^-), x(t^- - \tau)\right)\mathrm{d}N(t), \quad 0 < t < T, \\
x(t) &= \varphi(t), \quad -\tau \le t \le 0,
\end{aligned} \tag{4.12}$$

where $\varphi \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^d)$.

Theorem 4.3 implies that the stochastic θ-methods for (4.12) are convergent with order $p = 1/2$, which coincides with Theorem 3.2 in [11].

Example 4.5. Consider the balanced implicit methods for system (2.1),

$$\begin{aligned}
Y_{n+1} &= G(Y_{n+1-m}) + Y_n - G(Y_{n-m}) + f(Y_n, Y_{n-m})h \\
&\quad + g(Y_n, Y_{n-m})\Delta W_n + u(Y_n, Y_{n-m})\Delta N_n \\
&\quad + C(Y_n, Y_{n-m})[Y_n - G(Y_{n-m}) - Y_{n+1} + G(Y_{n+1-m})].
\end{aligned} \tag{4.13}$$

Here, the $d \times d$ matrix $C(Y_n, Y_{n-m})$ is given by

$$C(Y_n, Y_{n-m}) = C_0(Y_n, Y_{n-m})h + C_1(Y_n, Y_{n-m})|\Delta W_n| = C_{0n}h + C_{1n}|\Delta W_n|, \tag{4.14}$$

where the $C_{0n} = C_0(Y_n, Y_{n-m})$, $C_{1n} = C_1(Y_n, Y_{n-m})$ are, in general matrix, called control functions which are often chosen as constants.

Furthermore, the control functions must satisfy some conditions.

Assumption 4. The $C_0(x, y)$ and $C_1(x, y)$ represent bounded $d \times d$-matrix-valued functions for $x, y \in \mathbb{R}^d$. For any real numbers $\alpha_0 \in [0, \overline{\alpha}]$, $\alpha_1 \ge 0$, where $\overline{\alpha} \ge h$ for all step sizes h considered and $x, y \in \mathbb{R}^d$, the matrix $M(x, y) = I + \alpha_0 C_0(x, y) + \alpha_1 C_1(x, y)$ has an inverse and satisfies the condition

$$\left|(M(x, y))^{-1}\right| \le H < \infty. \tag{4.15}$$

Here, I is the unit matrix, and H is a positive constant. Notice that if C_{0n}, C_{1n} satisfy Assumption 4, the methods (4.13) are well defined and can be rewritten as

$$Y_{n+1} = G(Y_{n+1-m}) + Y_n - G(Y_{n-m}) + [I + C(Y_n, Y_{n-m})]^{-1} \times \left(f(Y_n, Y_{n-m})h + g(Y_n, Y_{n-m})\Delta W_n + u(Y_n, Y_{n-m})\Delta N_n \right). \tag{4.16}$$

Lemma 4.6 (see [12]). *If $F(y,\omega)$ is independent of $\mathcal{F}$, $y \in \mathbb{R}^d$, $\omega \in \Omega$, $\mathbb{E}(F(y,\omega)) = \varphi(y)$, and ς is $\mathcal{F}$-measurable, then $\mathbb{E}(F(\varsigma,\omega) \mid \mathcal{F}) = \varphi(\varsigma)$.*

For simplicity, from now on, we suppose that C_{0n}, C_{1n} in (4.14) are constants, that is to say, $C_{0n} = C_0$, $C_{1n} = C_1$.

Lemma 4.7. *Let Assumptions 1, 2, and 4 hold, then the balanced implicit methods* (4.13) *are consistent with order $p_1 = 3/2$ in the mean and order $p_2 = 1$ in the mean square sense.*

Proof. Without loss of generality, we can assume that $0 < h < 1$. From Lemma 4.2, we know that the Euler-Maruyama method is consistent with order $3/2$ in the mean, thus using (4.4), we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\mathbb{E}\left[x(t_{n+1}) - \overline{Y}^B(t_{n+1}) \mid \mathcal{F}_{t_n}\right]\right|^2\right] \\
&= \mathbb{E}\left\{\left|\mathbb{E}\left[x(t_{n+1}) - \overline{Y}^E(t_{n+1}) \mid \mathcal{F}_{t_n}\right] + \mathbb{E}\left[\overline{Y}^E(t_{n+1}) - \overline{Y}^B(t_{n+1}) \mid \mathcal{F}_{t_n}\right]\right|^2\right\} \\
&\le 2\mathbb{E}\left[\left|\mathbb{E}\left[x(t_{n+1}) - \overline{Y}^E(t_{n+1}) \mid \mathcal{F}_{t_n}\right]\right|^2\right] + 2\mathbb{E}\left[\left|\mathbb{E}\left[\overline{Y}^E(t_{n+1}) - \overline{Y}^B(t_{n+1}) \mid \mathcal{F}_{t_n}\right]\right|^2\right] \\
&\le \left[20\left(1+\lambda^2\right)K\widehat{H} + 20K\widehat{H}\right]h^3 + 2\mathbb{E}\left[\left|\mathbb{E}\left[\overline{Y}^E(t_{n+1}) - \overline{Y}^B(t_{n+1}) \mid \mathcal{F}_{t_n}\right]\right|^2\right],
\end{aligned} \tag{4.17}$$

where $\overline{Y}^E(t_{n+1})$ and $\overline{Y}^B(t_{n+1})$ are defined as follows:

$$\begin{aligned}
\overline{Y}^E(t_{n+1}) &= G(x(t_{n+1} - \tau)) + x(t_n) - G(x(t_n - \tau)) + f(x(t_n), x(t_n - \tau))h \\
&\quad + g(x(t_n), x(t_n - \tau))\Delta W_n + u(x(t_n), x(t_n - \tau))\Delta N_n, \\
\overline{Y}^B(t_{n+1}) &= G(x(t_{n+1} - \tau)) + x(t_n) - G(x(t_n - \tau)) + [I + C_n]^{-1} \\
&\quad \times \left(f(x(t_n), x(t_n - \tau))h + g(x(t_n), x(t_n - \tau))\Delta W_n + u(x(t_n), x(t_n - \tau))\Delta N_n \right),
\end{aligned} \tag{4.18}$$

where $C_n = C_0 h + C_1|\Delta W_n|$. By (4.18), we have

$$\begin{aligned}
\overline{Y}^E(t_{n+1}) - \overline{Y}^B(t_{n+1}) &= \left(I - [I + C_n]^{-1}\right) \\
&\quad \cdot \left(f(x(t_n), x(t_n - \tau))h + g(x(t_n), x(t_n - \tau))\Delta W_n + u(x(t_n), x(t_n - \tau))\Delta N_n\right) \\
&= [I + C_n]^{-1} \cdot ((I + C_n) - I) \\
&\quad \cdot \left(f(x(t_n), x(t_n - \tau))h + g(x(t_n), x(t_n - \tau))\Delta W_n + u(x(t_n), x(t_n - \tau))\Delta N_n\right) \\
&= [I + C_n]^{-1} \cdot C_n \\
&\quad \cdot \left(f(x(t_n), x(t_n - \tau))h + g(x(t_n), x(t_n - \tau))\Delta W_n + u(x(t_n), x(t_n - \tau))\Delta N_n\right).
\end{aligned} \tag{4.19}$$

Noticing that $x(t_n), x(t_n - \tau)$ are $\mathcal{F}_{t_n}$-measurable, ΔW_n is independent of $\mathcal{F}_{t_n}$, and using Lemma 4.6, we find that

$$\mathbb{E}\left[(I + C_n)^{-1} \cdot C_n \cdot g(x(t_n), x(t_n - \tau))\Delta W_n \mid \mathcal{F}_{t_n}\right] = 0. \tag{4.20}$$

We notice that C_0 and C_1 are constants; thus, there exists a positive constant B, such that $|C_i| \leq B$ $(i = 0, 1)$. Since $\Delta W_n, \Delta N_n$ are independent of $\mathcal{F}_{t_n}$, by Assumption 4, (4.19), (4.20), $|\mathbb{E}(x \mid \mathcal{F})| \leq \mathbb{E}(|x| \mid \mathcal{F})$, $\mathbb{E}|\Delta W_n| \leq \sqrt{bh}$, and $\mathbb{E}\Delta N_n = \lambda h$, we obtain

$$\begin{aligned}
&\left|\mathbb{E}\left[\overline{Y}^E(t_{n+1}) - \overline{Y}^B(t_{n+1}) \mid \mathcal{F}_{t_n}\right]\right| \\
&\quad \leq \left|f(x(t_n), x(t_n - \tau))\right| h \cdot \mathbb{E}\left(\left|(I + C_n)^{-1}\right| \cdot |C_0 h + C_1|\Delta W_n|| \mid \mathcal{F}_{t_n}\right) \\
&\qquad + |u(x(t_n), x(t_n - \tau))| \cdot \mathbb{E}\left(\left|(I + C_n)^{-1}\right| \cdot |C_0 h + C_1|\Delta W_n|| \cdot \Delta N_n \mid \mathcal{F}_{t_n}\right) \\
&\quad \leq \left|f(x(t_n), x(t_n - \tau))\right| Hh \cdot \mathbb{E}(|C_0 h + C_1|\Delta W_n|| \mid \mathcal{F}_{t_n}) \\
&\qquad + |u(x(t_n), x(t_n - \tau))| H \cdot \mathbb{E}(|C_0 h + C_1|\Delta W_n|| \cdot \Delta N_n \mid \mathcal{F}_{t_n}) \\
&\quad \leq \left(\left|f(x(t_n), x(t_n - \tau))\right| + |u(x(t_n), x(t_n - \tau))|\lambda\right) H h^{3/2} B\left(\sqrt{h} + \sqrt{b}\right).
\end{aligned} \tag{4.21}$$

It follows from Assumption 2, Lemma 2.3, and (4.21) that

$$\begin{aligned}
&\mathbb{E}\left[\left|\mathbb{E}\left[\overline{Y}^E(t_{n+1}) - \overline{Y}^B(t_{n+1}) \mid \mathcal{F}_{t_n}\right]\right|^2\right] \\
&\quad \leq 2H^2 B^2 \left(\sqrt{T} + \sqrt{b}\right)^2 h^3 \left[\mathbb{E}\left|f(x(t_n), x(t_n - \tau))\right|^2 + \mathbb{E}|u(x(t_n), x(t_n - \tau))|^2 \lambda^2\right] \\
&\quad \leq 2LH^2 B^2 \left(1 + \lambda^2\right)\left(\sqrt{T} + \sqrt{b}\right)^2 (1 + 2J_1) h^3.
\end{aligned} \tag{4.22}$$

Inserting (4.22) into (4.17) yields

$$\mathbb{E}\left[\left|\mathbb{E}\left[x(t_{n+1}) - \overline{Y}^B(t_{n+1}) \mid \mathcal{F}_{t_n}\right]\right|^2\right] \leq \left(20\left(1+\lambda^2\right)K\widehat{H} + 20K\widehat{H} + 4LH^2B^2\left(1+\lambda^2\right)\left(\sqrt{T}+\sqrt{b}\right)^2(1+2J_1)\right)h^3, \tag{4.23}$$

which implies that the balanced implicit methods (4.13) are consistent with order 3/2 in the mean. In the following, we will show that the balanced implicit methods (4.13) are consistent with order 1 in the mean square sense. Using Assumptions 2 and 4, Lemma 2.3, $|C_i| \leq B$ $(i = 0, 1)$, $\mathbb{E}|\Delta W_n|^2 = bh$, $\mathbb{E}|\Delta W_n|^4 \leq 3b^2h^2$, $\mathbb{E}\Delta N_n^2 = \lambda h(1+\lambda h)$, and (4.19), we compute that

$$\begin{aligned}
&\mathbb{E}\left[\left|\overline{Y}^E(t_{n+1}) - \overline{Y}^B(t_{n+1})\right|^2\right] \\
&\leq 3h^2H^2\mathbb{E}\left(|C_0h + C_1|\Delta W_n||^2 \cdot \left|f(x(t_n), x(t_n-\tau))\right|^2\right) \\
&\quad + 3H^2\mathbb{E}\left(|C_0h + C_1|\Delta W_n||^2 \cdot \left|g(x(t_n), x(t_n-\tau))\right|^2|\Delta W_n|^2\right) \\
&\quad + 3H^2\mathbb{E}\left(|C_0h + C_1|\Delta W_n||^2 \cdot |u(x(t_n), x(t_n-\tau))|^2\Delta N_n^2\right) \\
&\leq 3h^2H^2\mathbb{E}\left(\left(2|C_0|^2h^2 + 2|C_1|^2|\Delta W_n|^2\right) \cdot \left|f(x(t_n), x(t_n-\tau))\right|^2\right) \\
&\quad + 3H^2\mathbb{E}\left(\left(2|C_0|^2h^2|\Delta W_n|^2 + 2|C_1|^2|\Delta W_n|^4\right) \cdot \left|g(x(t_n), x(t_n-\tau))\right|^2\right) \\
&\quad + 3H^2\mathbb{E}\left(\left(2|C_0|^2h^2 + 2|C_1|^2|\Delta W_n|^2\right) \cdot |u(x(t_n), x(t_n-\tau))|^2\Delta N_n^2\right) \\
&\leq 3h^2H^2\left(2B^2h^2 + 2B^2bh\right)\mathbb{E}\left[\left|f(x(t_n), x(t_n-\tau))\right|^2\right] \\
&\quad + 3H^2\left(2B^2h^2bh + 2B^2 \cdot 3b^2h^2\right)\mathbb{E}\left[\left|g(x(t_n), x(t_n-\tau))\right|^2\right] \\
&\quad + 3H^2\lambda h(1+\lambda h)\left(2B^2h^2 + 2B^2bh\right)\mathbb{E}\left[|u(x(t_n), x(t_n-\tau))|^2\right] \\
&\leq 6H^2B^2L\left[T^2 + 2bT + 3b^2 + \lambda(1+\lambda T)(T+b)\right](1+2J_1)h^2.
\end{aligned} \tag{4.24}$$

Theorem 4.3 implies that the Euler-Maruyama method is convergent with order $p = 1/2$. Thus, by (4.5) and (4.24), we have

$$\begin{aligned}
\mathbb{E}\left[\left|x(t_{n+1}) - \overline{Y}^B(t_{n+1})\right|^2\right] &\leq 2\mathbb{E}\left[\left|x(t_{n+1}) - \overline{Y}^E(t_{n+1})\right|^2\right] + 2\mathbb{E}\left[\left|\overline{Y}^E(t_{n+1}) - \overline{Y}^B(t_{n+1})\right|^2\right] \\
&\leq \left[16K\widehat{H}\left(1 + 2T + 2\lambda + 2\lambda^2T\right) + 12H^2B^2L\right. \\
&\qquad \left.\times\left[T^2 + 2bT + 3b^2 + \lambda(1+\lambda T)(T+b)\right](1+2J_1)\right]h^2.
\end{aligned} \tag{4.25}$$

The proof is completed. □

Theorem 4.8. *Let Assumptions 1–4 hold, then the balanced implicit methods* (4.16) *are convergent with order $p = 1/2$ in the mean square sense.*

Proof. By (4.16), the increment functions Φ_f, Φ_g, and Φ_u of the balanced implicit methods (4.16) are given as follows:

$$\Phi_f(h, Y_n, Y_{n+1}, Y_{n-m}, Y_{n+1-m}, \Delta W_n, \Delta N_n) = [I + C(Y_n, Y_{n-m})]^{-1} f(Y_n, Y_{n-m})h, \tag{4.26}$$

$$\begin{aligned}
\Phi_g(h, Y_n, Y_{n-m}, \Delta W_n, \Delta N_n) &= [I + C(Y_n, Y_{n-m})]^{-1} g(Y_n, Y_{n-m})\Delta W_n, \\
\Phi_u(h, Y_n, Y_{n-m}, \Delta W_n, \Delta N_n) &= [I + C(Y_n, Y_{n-m})]^{-1} u(Y_n, Y_{n-m})\Delta N_n.
\end{aligned} \tag{4.27}$$

For $x_i, y_i \in \mathbb{R}^d$ $(i = 1, 2, 3, 4)$, by Assumptions 1–4, (4.26), we arrive at

$$\begin{aligned}
&\left|\Phi_f(h, x_1, x_2, x_3, x_4, \Delta W_n, \Delta N_n) - \Phi_f(h, y_1, y_2, y_3, y_4, \Delta W_n, \Delta N_n)\right| \\
&\quad = \left|[I + C_0h + C_1|\Delta W_n|]^{-1} f(x_1, x_3)h - [I + C_0h + C_1|\Delta W_n|]^{-1} f(y_1, y_3)h\right| \\
&\quad \le H\sqrt{K}h\left(\left|x_1 - y_1\right| + \left|x_3 - y_3\right|\right).
\end{aligned} \tag{4.28}$$

Noticing that $\mathbb{E}|\Delta W_n|^2 = bh$, $\mathbb{E}\Delta N_n^2 = \lambda h(1 + \lambda h)$, the random variables x_1, x_3, y_1, and y_3 are $\mathcal{F}_{t_n}$-measurable and using Assumptions 1–4, (4.27), we obtain

$$\begin{aligned}
&\mathbb{E}\left|\Phi_g(h, x_1, x_3, \Delta W_n, \Delta N_n) - \Phi_g(h, y_1, y_3, \Delta W_n, \Delta N_n)\right|^2 \\
&\quad = \mathbb{E}\left|[I + C_n)]^{-1} g(x_1, x_3)\Delta W_n - [I + C_n]^{-1} g(y_1, y_3)\Delta W_n\right|^2 \\
&\quad \le H^2 bKh\left[\mathbb{E}\left|x_1 - y_1\right|^2 + \mathbb{E}\left|x_3 - y_3\right|^2\right],
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
&\mathbb{E}\left|\Phi_u(h, x_1, x_3, \Delta W_n, \Delta N_n) - \Phi_u(h, y_1, y_3, \Delta W_n, \Delta N_n)\right|^2 \\
&\quad = \mathbb{E}\left|[I + C_n]^{-1} u(x_1, x_3)\Delta N_n - [I + C_n]^{-1} u(y_1, y_3)\Delta N_n\right|^2 \\
&\quad \le H^2 K\lambda(1 + \lambda T)h\left[\mathbb{E}\left|x_1 - y_1\right|^2 + \mathbb{E}\left|x_3 - y_3\right|^2\right].
\end{aligned} \tag{4.30}$$

Since $\Delta W_n, \Delta N_n$ are independent of $\mathcal{F}_{t_n}$ and the random variables x_1, x_3, y_1, and y_3 are $\mathcal{F}_{t_n}$-measurable; thus, by the inequality $|\mathbb{E}[x \mid \mathcal{F}]| \le \mathbb{E}[|x| \mid \mathcal{F}]$, $\mathbb{E}\Delta N_n = \lambda h$, and Lemma 4.6, we compute that

$$\begin{aligned}
&\mathbb{E}\left[\Phi_g(h, x_1, x_3, \Delta W_n, \Delta N_n) - \Phi_g(h, y_1, y_3, \Delta W_n, \Delta N_n) \mid \mathcal{F}_{t_n}\right] \\
&\quad = \mathbb{E}\left[(I + C_n)^{-1} g(x_1, x_3)\Delta W_n \mid \mathcal{F}_{t_n}\right] - \mathbb{E}\left[(I + C_n)^{-1} g(y_1, y_3)\Delta W_n \mid \mathcal{F}_{t_n}\right] \\
&\quad = 0,
\end{aligned} \tag{4.31}$$

$$
\begin{aligned}
&\left|\mathbb{E}\left[\Phi_u(h,x_1,x_3,\Delta W_n,\Delta N_n)-\Phi_u(h,y_1,y_3,\Delta W_n,\Delta N_n)\mid \mathcal{F}_{t_n}\right]\right| \\
&\quad\le \mathbb{E}\left|[I+C_n]^{-1}u(x_1,x_3)\Delta N_n-[I+C_n]^{-1}u(y_1,y_3)\Delta N_n\mid \mathcal{F}_{t_n}\right| \\
&\quad\le \left|u(x_1,x_3)-u(y_1,y_3)\right|\cdot \mathbb{E}\left[\left|(I+C_n)^{-1}\right|\cdot \Delta N_n\mid \mathcal{F}_{t_n}\right] \\
&\quad\le H\sqrt{K}\lambda h\left[\left|x_1-y_1\right|+\left|x_3-y_3\right|\right].
\end{aligned}
\tag{4.32}
$$

From (4.28)–(4.32), we see that the increment functions of the balanced implicit methods (4.13) satisfy the conditions (3.7)–(3.11) with $L_f = H\sqrt{K}$, $L_g = H^2Kb$, $L_u = H^2K\lambda(1+\lambda T)$, and $L_{\overline{u}} = H\sqrt{K}\lambda$. A combination of Lemma 4.7 and Theorem 3.5 leads to the conclusion that the balanced implicit methods (4.13) are convergent with order $p = 1/2$ in the mean square sense. The proof is completed. □

Remark 4.9. For the case of $G(\cdot) = 0$ and $\tau = 0$, (2.1) reduces to the stochastic differential equations with jumps

$$
\begin{gathered}
\mathrm{d}x(t) = f\left(t,x(t^-)\right)\mathrm{d}t + g\left(t,x(t^-)\right)\mathrm{d}W(t) + u\left(t,x(t^-)\right)\mathrm{d}N(t), \quad 0<t<T, \\
x(0^-) = x_0.
\end{gathered}
\tag{4.33}
$$

From Theorem 4.8, it is not difficult to find that the balanced implicit methods for (4.33) are convergent with order $p = 1/2$ in the mean square sense, which coincides with Theorem 2.1 in [13].

5. Numerical Experiments

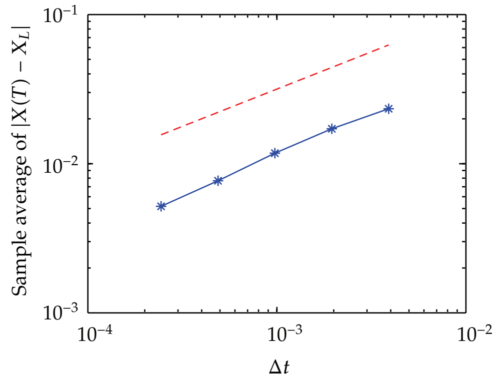
In this section, several numerical examples are given to illustrate our theoretical results in the previous sections. Consider the nonlinear equation

$$
\begin{aligned}
\mathrm{d}[(x(t)-x(t-1))] = {}& \left[-x(t)+\frac{x(t-1)}{1+x^2(t-1)}\right]\mathrm{d}t + [\sin(x(t))+\cos(x(t-1))]\mathrm{d}W(t) \\
&+ \frac{1}{2}x(t)\mathrm{d}N(t), \quad t\in[0,T],
\end{aligned}
\tag{5.1}
$$

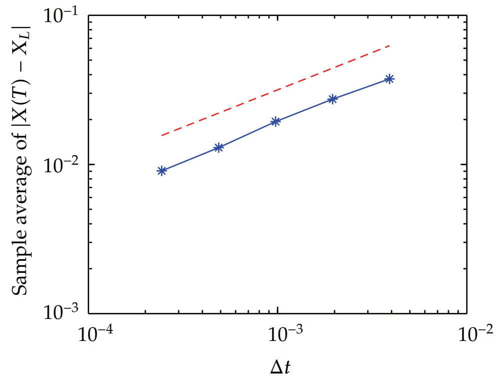
with initial data

$$
x(t)\equiv 1, \quad t\in[-1,0]. \tag{5.2}
$$

To show the convergence of the θ-methods (4.1) and the balanced implicit methods (4.16), we choose $\theta = 0.5$, $C_0 = 1$, and $C_1 = 0.5$. In all the numerical experiments, we identify the numerical solution using very small stepsize $h = \Delta t$ as the exact solution and compare this with the numerical approximations using $h = 4\Delta t, 8\Delta t, 16\Delta t, 32\Delta t, 64\Delta t$ for $\Delta t = 2^{-14}$ over 2000 different discretized Brownian paths. The mean-square errors $\varepsilon^r = \varepsilon_{2^{1+r}\Delta t}, r = 1,2,3,4,5$, all measured at time $T = 1$, are estimated by trajectory averaging, that is,



(a)



(b)

Figure 1: The convergence of the numerical methods. (a) Stochastic θ-method (4.1); (b) the balanced implicit methods (4.16).

$\varepsilon^r \cong 1/2000\sum_{j=1}^{2000}|x_{r,T}(\omega_j) - Y_{r,q}(\omega_j)|^2$. We plot our approximation to $\sqrt{\varepsilon^r}$ against h on a log-log scale in Figure 1. For reference, a dashed line of slope one-half is added in two graphs. In Figure 1., we show the convergence of the θ-method (4.1) in the left picture and the balanced implicit methods (4.16) in the right picture, respectively.

We see that the slopes of the two curves appear to match well in two pictures in Figure 1, which is consistent with the strong order of one-half implied in Theorem 4.3 and Theorem 4.8.

6. Conclusion

In this paper, we consider a family of implicit one-step methods for the NSDDEs with jumps. A relationship between the consistent order and the convergence order is established. A general framework for mean-square convergence of the methods is provided. The applicability of the mean-square convergence theory is illustrated by the stochastic θ-methods and the balanced implicit methods. We have generalized the existing results. The examples presented in Section 4 show that the main result in this paper can be applied not only to semi-implicit methods but also to full implicit methods.

Acknowledgments

This research is supported with funds provided by the National Natural Science Foundation of China (no. 10871207 and 11171352) and the project sponsored by the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.

References

[1] G. N. Milstein, *Numerical Integration of Stochastic Differential Equations*, vol. 313 of *Mathematics and Its Applications*, Kluwer Academic Publishers Group, London, UK, 1995.

[2] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, vol. 23 of *Applications of Mathematics (New York)*, Springer, Berlin, Germany, 1992.

[3] E. Buckwar, "Introduction to the numerical analysis of stochastic delay differential equations," *Journal of Computational and Applied Mathematics*, vol. 125, no. 1-2, pp. 297–307, 2000.

[4] E. Buckwar, "One-step approximations for stochastic functional differential equations," *Applied Numerical Mathematics*, vol. 56, no. 5, pp. 667–681, 2006.
[5] H. Zhang and S. Gan, "Mean square convergence of one-step methods for neutral stochastic differential delay equations," *Applied Mathematics and Computation*, vol. 204, no. 2, pp. 884–890, 2008.
[6] R. Cont and P. Tankov, *Financial Modelling with Jump Processes*, Chapman & Hall/CRC Financial Mathematics Series, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2004.
[7] P. Glasserman and N. Merener, "Convergence of a discretization scheme for jump-diffusion processes with state-dependent intensities," *Proceedings of The Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences*, vol. 460, no. 2041, pp. 111–127, 2004.
[8] P. Glasserman, *Monte Carlo Methods in Financial Engineering*, vol. 53 of *Applications of Mathematics (New York)*, Springer, New York, NY, USA, 2004.
[9] X. Mao, *Stochastic Differential Equations and Their Applications*, Horwood Publishing Series in Mathematics & Applications, Horwood Publishing Limited, Chichester, UK, 1997.
[10] S. Gan, H. Schurz, and H. Zhang, "Mean square convergence of stochastic θ-methods for nonlinear neutral stochastic differential delay equations," *International Journal of Numerical Analysis and Modeling*, vol. 8, no. 2, pp. 201–213, 2011.
[11] L.-s. Wang, C. Mei, and H. Xue, "The semi-implicit Euler method for stochastic differential delay equations with jumps," *Applied Mathematics and Computation*, vol. 192, no. 2, pp. 567–578, 2007.
[12] I. I. Gihman and A. V. Skorohod, *Stokhasticheskie Differentsialnye Uravneniya*, Naukova Dumka, Kyiv, Ukrania, 1968.
[13] L. Hu and S. Gan, "Convergence and stability of the balanced methods for stochastic differential equations with jumps," *International Journal of Computer Mathematics*, vol. 88, no. 10, pp. 2089–2108, 2011.