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Research Article

Pinning Synchronization of Delayed Neural Networks with Nonlinear Inner-Coupling

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Without assuming the symmetry and irreducibility of the outer-coupling weight configuration matrices, we investigate the pinning synchronization of delayed neural networks with nonlinear inner-coupling. Some delay-dependent controlled stability criteria in terms of linear matrix inequality (LMI) are obtained. An example is presented to show the application of the criteria ob-tained in this paper.

1. Introduction and Model Description

In the past few decades, the problem of control and synchronization in complex networks has attracted increasing attention. There are attempts to control the dynamics of a complex network and guide it to a desired state, such as an equilibrium point or a periodic orbit of the network. Since a complex network has a large number of nodes, it is difficult to control it by adding controllers to all nodes. To reduce the number of the controllers, Wang and Chen investigated pinning control for complex networks [1]. Pinning control applies local feedback injections to a small fraction of nodes on a large-size network, thereby achieving some intended global performances over the entire network.

In [1], Wang and Chen showed that, due to the extremely inhomogeneous connectivity distribution of scale-free networks, it is much effective to pin some most-highly connected nodes than to pin randomly selected nodes. In [2], Li et al. further investigated the control of complete random networks and scale-free networks via virtual control and showed that the control actions applied to the pinned nodes can be propagated to the rest of network nodes through the couplings in the network and eventually result in the synchronization of the whole network. In [3], Chen et al. proved that, if the coupling strength is large enough, then even one single pinning controller is able to control network. In the sequel, [4–9] also studied the global pinning controllability of complex networks and some sufficient pinning conditions were established. The common feature of the work in [1–9] is that there are no coupling delays in the network. However, due to the limited speeds of transmission and spreading as well as traffic congestion, signals traveling through a network are often associated with time delays, which are very common in biological and physical networks. Therefore, time delays should be modeled in order to simulate more realistic networks. In [10–13], the pinning synchronization of complex networks with homogeneous time delay is studied. In [14], Xiang et al. considered the pinning control of complex networks with heterogeneous delays via linearized method.

The previous researches on pinning control of complex dynamical networks have mainly focused on such networks with some specific coupling schemes. That is, there is a common outer-coupling strength for all connections and the inner-coupling is linear. Moreover, most of the existed studies assume the coupling configuration matrices are symmetric. In a real neural network, however, this is not always the case. Many real neural networks are direct graphs, such as the WWW, whose coupling configuration matrix is not symmetric. Additionally, as pointed in [15–17], synchronization is influenced not only by the topology, but also by the strength of the connections. So, we should consider nonuniform coupling strength while studying complex dynamical networks. Xiang et al. [14] considered the pinning control of complex networks with nonuniform coupling strength. In [18], Zhou et al. investigated the pinning adaptive synchronization of complex network with nonuniform coupling strength as well as time delay under the assumption of the symmetry of the nondelayed and delayed outer-coupling weight configuration matrices, they introduced some specific pinning control technique.

Motivated by the above discussion, we consider a general complex dynamical network described by [18]

$$\dot{x}_i(t) = f(x_i(t), t) + \sum_{j=1}^{N} a_{ij} g(x_j(t)) + \sum_{j=1}^{N} b_{ij} h(x_j(t-\tau(t))), \quad i = 1, 2, \ldots, N, \tag{1.1}$$

where $x_i(t)$ denotes the state vector of the ith node, $f : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ represents the activity of an individual subsystem, $g : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^n$ are the inner-coupling and delayed inner-coupling vector functions, and $\tau(t)$ is the coupling delay function. a_{ij} and b_{ij} are the nondelayed and delayed outer-coupling strength, respectively. If there is a link from node i to node j $(j \neq i)$, then $a_{ij} > 0$ and $b_{ij} > 0$; otherwise, $a_{ij} = b_{ij} = 0$, and $a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}$, $b_{ii} = -\sum_{j=1, j\neq i}^{N} b_{ij}$, $i = 1, 2, \ldots, N$. We will give some synchronization criteria by adding nonlinear and adaptive feedback controllers to a small fraction of nodes of network (1.1).

Let $C([-\tau, 0], \mathbb{R}^n)$ be the Banach space of continuous functions mapping $[-\tau, 0]$ into $\mathbb{R}^n$ with the norm $\|\phi\| = \sup_{-\tau\leq\theta\leq 0}\|\phi(\theta)\|$, where $\tau = \sup_{t\in\mathbb{R}^+}\{\tau(t)\}$. For the complex network (1.1), its initial conditions are given by $x_i(t) = \phi_i(t) \in C([-\tau, 0], \mathbb{R}^n)$. We always assume that (1.1) has a unique solution with respect to initial conditions.

2. Preliminaries

In this section, we present some lemmas and assumptions required throughout this paper.

Lemma 2.1 (see [19] (Schur Complement)). *The following linear matrix inequality (LMI)*

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{pmatrix} < 0, \tag{2.1}$$

where $S_{11} = S_{11}^T, S_{22} = S_{22}^T$, is equivalent to the following condition:

$$S_{22} < 0, \qquad S_{11} - S_{12}S_{22}^{-1}S_{12}^T < 0. \tag{2.2}$$

Lemma 2.2 (see [18]). *Assume that M is a diagonal matrix whose i_kth $(1 \le i_k \le N, 1 \le k \le l, 1 \le l \le N)$ diagonal elements are m and the others are 0, where $m > 0$ is a constant. Then, for a symmetric matrix G which has the same dimension with M, $G - M < 0$ is equivalent to $G_l < 0$ when m is large enough, where G_l denotes the minor matrix of the matrix G by removing all the i_kth row-column pairs of G.*

Assumption 1. Suppose that the delay function $\tau(t)$ is differentiable and satisfies $\dot{\tau}(t) \le \alpha$, where $0 \le \alpha < 1$ is a constant.

Assumption 2 (see [3]). Assume that there is a positive definite diagonal matrix $P = \operatorname{diag}(p_1, p_2, \ldots, p_n)$ and a diagonal matrix $\Delta = \operatorname{diag}(\delta_1, \delta_2, \ldots, \delta_n)$, such that f satisfies the following inequality:

$$(x-y)^T P(f(x) - f(y) - \Delta(x-y)) \le -\eta(x-y)^T(x-y) \tag{2.3}$$

for some $\eta > 0$, all $x, y \in \mathbb{R}^n$, and $t > 0$.

Assumption 3. Assume that there exist positive constants β_1 and β_2 such that

$$\beta_1\|x-y\|^2 \le (x-y)^T(g(x)-g(y)) \le \|x-y\|\|g(x)-g(y)\| \le \beta_2\|x-y\|^2 \tag{2.4}$$

for all $x, y \in \mathbb{R}^n$.

Assumption 4. Assume that there exists a positive constant $\mu > 0$ satisfying

$$\|h(x) - h(y)\| \le \mu\|x-y\| \tag{2.5}$$

for all $x, y \in \mathbb{R}^n$.

Remark 2.3. In [18], the delayed inner-coupling vector function h was required to satisfy

$$e_1\|x-y\|^2 \le (x-y)^T(h(x)-h(y)) \le \|x-y\|\|h(x)-h(y)\| \le e_2\|x-y\|^2 \tag{2.6}$$

for all $x, y \in \mathbb{R}^n$, where e_1 and e_2 are positive constants. Obviously, our assumption is weaker than that.

3. Synchronization Criteria of Directed Networks

The objective in this section is to stabilize network (1.1) onto a homogeneous trajectory satisfying $\dot{s}(t) = f(s(t), t)$, where $s(t) \in \mathbb{R}^n$ is a solution of an isolate node. To achieve this goal, we first add feedback pinning controllers to a small fraction of nodes in the network. The pinning controlled network is described as follows:

$$\dot{x}_i(t) = f(x_i(t), t) + \sum_{j=1}^{N} a_{ij} g(x_j(t)) + \sum_{j=1}^{N} b_{ij} h(x_j(t-\tau(t))) + u_i(t), \tag{3.1}$$

with nonlinear feedback controllers given by

$$u_i(t) = -d_i(g(x_i(t)) - g(s(t))), \quad i = 1, 2, \ldots, N, \tag{3.2}$$

where the feedback gain satisfies $d_i > 0$ for $i = 1, 2, \ldots, l$ and $d_i = 0$ for $i = l+1, \ldots, N$.

For the convenience of later use, we introduce some notations employed through this section. We let $\underline{p} = \min_{1\le j\le n}\{p_j\}$; $|x| = (|x_1|, |x_2|, \ldots, |x_n|)^T$ for all $x \in \mathbb{R}^n$; $D = \operatorname{diag}(d_1, d_2, \ldots, d_l, 0, \ldots, 0)$; $\tilde{\tilde{A}} = \operatorname{diag}(a_{11}, a_{22}, \ldots, a_{NN})$, $\tilde{A} = A - \tilde{\tilde{A}}$; $e_i(t) = x_i(t) - s(t) \in \mathbb{R}^n$, $g(e_i(t)) = g(e_i(t)+s(t)) - g(s(t))$, $h(e_i(t-\tau(t))) = h(e_i(t-\tau(t)) + s(t-\tau(t))) - h(s(t-\tau(t)))$, $i = 1, 2, \ldots, N$; $\tilde{e}_j(t) = (e_{1j}(t), e_{2j}(t), \ldots, e_{Nj}(t))^T \in \mathbb{R}^N$, $\tilde{g}_j(e(t)) = (g_j(e_1(t)), g_j(e_2(t)), \ldots, g_j(e_N(t)))^T \in \mathbb{R}^N$, $\tilde{h}_j(e(t-\tau(t))) = (h_j(e_1(t-\tau(t))), h_j(e_2(t-\tau(t))), \ldots, h_j(e_N(t-\tau(t))))^T \in \mathbb{R}^N$, $j = 1, 2, \ldots, n$; $\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$ for all $A \in \mathbb{R}^{N\times N}$.

Theorem 3.1. *Under the Assumptions 1–4, the pinning controlled directed network* (3.1) *is globally asymptotically stable at the homogenous trajectory $s(t)$ if there exist positive diagonal matrices R and Q, such that*

$$\begin{gathered} p_j\delta_j I_N + \beta_1 \underline{p}\tilde{\tilde{A}} - \beta_1 \underline{p} D + \frac{p_j^2}{4}\tilde{A}R^{-1}\tilde{A}^T < 0, \\ -\eta I_N + \mu^2 Q + \beta_2^2 R + \frac{(1-\alpha)p_j^2}{4} B Q^{-1} B^T < 0 \end{gathered} \tag{3.3}$$

for $j = 1, 2, \ldots, n$.

Proof. From Lemma 2.1, conditions (3.3) are equivalent to

$$\begin{gathered} \Omega_1 = \begin{pmatrix} p_j\delta_j I_N + \beta_1\underline{p}\tilde{\tilde{A}} - \beta_1\underline{p}D & \dfrac{p_j}{2}\tilde{A} \\ \dfrac{p_j}{2}\tilde{A}^T & -R \end{pmatrix} < 0, \\ \Omega_2 = \begin{pmatrix} -\eta I_N + \mu^2 Q + \beta_2^2 R & \dfrac{p_j}{2} B \\ \dfrac{p_j}{2} B^T & -(1-\alpha) Q \end{pmatrix} < 0. \end{gathered} \tag{3.4}$$

We consider a Lyapunov function as

$$V(t)=\frac{1}{2}\sum_{i=1}^{N}e_i^T(t)Pe_i(t)+\mu^2\sum_{j=1}^{n}\int_{t-\tau(t)}^{t}\widetilde{e}_j^T(s)Q\widetilde{e}_j(s)ds. \tag{3.5}$$

Differentiating the function $V(t)$ along the trajectories of (3.1), one obtains

$$\begin{aligned}\frac{dV(t)}{dt}=\sum_{i=1}^{N}e_i^T(t)P\Bigg\{&f(x_i(t))-f(s(t))-\Delta e_i(t)+\Delta e_i(t)+\sum_{j=1}^{N}a_{ij}g\big(e_j(t)\big)\\&+\sum_{j=1}^{N}b_{ij}h\big(e_j(t-\tau(t))\big)-d_ig(e_i(t))\Bigg\}+\mu^2\sum_{j=1}^{n}\widetilde{e}_j^T(t)Q\widetilde{e}_j(t)\\&-\mu^2\sum_{j=1}^{n}(1-\dot{\tau}(t))\widetilde{e}_j^T(t-\tau(t))Q\widetilde{e}_j(t-\tau(t)).\end{aligned} \tag{3.6}$$

From Assumption 3, one can obtain

$$\begin{aligned}\sum_{i=1}^{N}e_i^TPa_{ii}g(e_i(t))&=\sum_{i=1}^{N}a_{ii}\sum_{j=1}^{n}e_{ij}(t)p_jg_j(e_i(t))\le\underline{p}\sum_{i=1}^{N}a_{ii}\sum_{j=1}^{n}e_{ij}(t)g_j(e_i(t))\\&=\underline{p}\sum_{i=1}^{N}a_{ii}e_i^T(t)g(e_i(t))\le\beta_1\underline{p}\sum_{i=1}^{N}a_{ii}e_i^T(t)e_i(t).\end{aligned} \tag{3.7}$$

Similar to (3.7), we can get

$$\sum_{i=1}^{N}e_i^TPd_ig(e_i(t))\ge\beta_1\underline{p}\sum_{i=1}^{N}d_ie_i^T(t)e_i(t). \tag{3.8}$$

It follows from (3.7) and (3.8), and combining with Assumptions 1-2, we can derive

$$\begin{aligned}\frac{dV(t)}{dt}\le&-\eta\sum_{i=1}^{N}e_i^T(t)e_i(t)+\sum_{i=1}^{N}e_i^T(t)P\Delta e_i(t)+\beta_1\underline{p}\sum_{i=1}^{N}a_{ii}e_i^T(t)e_i(t)\\&+\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}a_{ij}e_i^T(t)Pg\big(e_j(t)\big)+\sum_{i=1}^{N}\sum_{j=1}^{N}b_{ij}e_i^T(t)Ph\big(e_j(t-\tau(t))\big)\\&-\beta_1\underline{p}\sum_{i=1}^{N}d_ie_i^T(t)e_i(t)+\mu^2\sum_{j=1}^{n}\widetilde{e}_j^T(t)Q\widetilde{e}_j(t)-\mu^2\sum_{j=1}^{n}(1-\alpha)\widetilde{e}_j^T(t-\tau(t))Q\widetilde{e}_j(t-\tau(t))\\=&-\eta\sum_{j=1}^{n}\widetilde{e}_j^T(t)\widetilde{e}_j(t)+\sum_{j=1}^{n}p_j\delta_j\widetilde{e}_j^T(t)\widetilde{e}_j(t)+\beta_1\underline{p}\sum_{j=1}^{n}\widetilde{e}_j^T(t)\widetilde{\widetilde{A}}\widetilde{e}_j(t)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\tilde{e}_j^T(t)\tilde{A}\tilde{g}_j(e(t))-\sum_{j=1}^{n}\tilde{g}_j^T(e(t))R\tilde{g}_j(e(t))+\sum_{j=1}^{n}\tilde{g}_j^T(e(t))R\tilde{g}_j(e(t))\\
&+\sum_{j=1}^{n}\tilde{e}_j^T(t)B\tilde{h}_j(e(t-\tau(t)))+\mu^2\sum_{j=1}^{n}\tilde{e}_j^T(t)Q\tilde{e}_j(t)\\
&-\beta_1\underline{p}\sum_{j=1}^{n}\tilde{e}_j^T(t)D\tilde{e}_j(t)-\mu^2\sum_{j=1}^{n}(1-\alpha)\tilde{e}_j^T(t-\tau(t))Q\tilde{e}_j(t-\tau(t)).
\end{aligned}
\tag{3.9}
$$

From Assumption 4, one can obtain

$$
\begin{aligned}
\sum_{j=1}^{n}\tilde{h}_j^T(e(t-\tau(t)))Q\tilde{h}_j(e(t-\tau(t)))&=\sum_{j=1}^{n}\sum_{i=1}^{N}q_ih_j^2(e_i(t-\tau(t)))\\
&=\sum_{i=1}^{N}q_i\|h(e_i(t-\tau(t)))\|^2\le\mu^2\sum_{i=1}^{N}q_ie_i^T(t-\tau(t))e_i(t-\tau(t))\\
&=\mu^2\sum_{j=1}^{n}\tilde{e}_j^T(t-\tau(t))Q\tilde{e}_j(t-\tau(t)).
\end{aligned}
\tag{3.10}
$$

Similar to (3.10), from Assumption 3, we can derive

$$
\begin{aligned}
\sum_{j=1}^{n}\tilde{g}_j^T(e(t))R\tilde{g}_j(e(t))&=\sum_{j=1}^{n}\sum_{i=1}^{N}r_ig_j^2(e_i(t))\\
&=\sum_{i=1}^{N}r_i\|g(e_i(t))\|^2\le\beta_2^2\sum_{i=1}^{N}r_ie_i^T(t)e_i(t)=\beta_2^2\sum_{j=1}^{n}\tilde{e}_j^T(t)R\tilde{e}_j(t).
\end{aligned}
\tag{3.11}
$$

From (3.7)–(3.11), we can get

$$
\begin{aligned}
\frac{dV(t)}{dt}&=\sum_{j=1}^{n}\left(\tilde{e}_j^T(t),\tilde{g}_j^T(e(t))\right)\Omega_1\begin{pmatrix}\tilde{e}_j(t)\\ \tilde{g}_j(e(t))\end{pmatrix}\\
&\quad+\sum_{j=1}^{n}\left(\tilde{e}_j^T(t),\tilde{h}_j^T(e(t-\tau(t)))\right)\Omega_2\begin{pmatrix}\tilde{e}_j(t)\\ \tilde{h}_j(e(t-\tau(t)))\end{pmatrix}<0.
\end{aligned}
\tag{3.12}
$$

From Lyapunov stability theory, the controlled system (3.1) is globally asymptotically stable at $s(t)$. This completes the proof. □

Remark 3.2. It is hard to compare our results with existing ones, because the issues are different. However, from the aspect of the network model, network (1.1) contains the models studied in [1–13], moreover, because the coupling configuration matrices A and B are assumed to be asymmetric, which are more consistent with the real-world network.

In the coupled system (3.1), if d_i, $i = 1,2,\ldots,l$ are defined as

$$\dot{d}_i(t) = \varepsilon_i e_i^T(t)e_i(t), \quad i = 1,2,\ldots,l, \tag{3.13}$$

where ε_i are positive constants, then the controller $u_i(t) = -d_i(t)(g(x_i(t)) - g(s(t)))$ is said to be adaptive pinning controller. In the following, we will prove that under some conditions the system (3.1) would get global synchronization with nonlinear adaptive pinning controllers.

Theorem 3.3. *Under the Assumptions 1–4, the controlled directed network* (3.1) *is globally asymptotically stable with adaptive pinning controllers* (3.13) *if there exist positive diagonal matrices R and Q, such that*

$$\begin{gathered}\left(p_j\delta_j I_N + \beta_1\underline{p}\tilde{\tilde{A}} + \frac{p_j^2}{4}\tilde{A}R^{-1}\tilde{A}^T\right)_l < 0,\\ -\eta I_N + \mu^2 Q + \beta_2^2 R + \frac{(1-\alpha)p_j^2}{4}BQ^{-1}B^T < 0,\end{gathered} \tag{3.14}$$

for $j = 1,2,\ldots,n$.

Proof. Construct the Lyapunov function as

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)Pe_i(t) + \mu^2\sum_{j=1}^{n}\int_{t-\tau(t)}^{t}\tilde{e}_j^T(s)Q\tilde{e}_j(s)ds + \beta_1\underline{p}\sum_{i=1}^{l}\frac{\left(d_i(t) - \hat{d}\right)^2}{2\varepsilon_i}, \tag{3.15}$$

where $\hat{d}$ is a sufficiently large positive constant to be determined. From Lemma 2.2, when $\hat{d}$ is large enough,

$$\left(p_j\delta_j I_N + \beta_1\underline{p}\tilde{\tilde{A}} + \frac{p_j^2}{4}\tilde{A}R^{-1}\tilde{A}^T\right)_l < 0 \tag{3.16}$$

is equivalent to

$$p_j\delta_j I_N + \beta_1\underline{p}\tilde{\tilde{A}} - \beta_1\underline{p}\hat{D} + \frac{p_j^2}{4}\tilde{A}R^{-1}\tilde{A}^T < 0, \tag{3.17}$$

where $\hat{D} = \mathrm{diag}(\underbrace{\hat{d},\ldots,\hat{d}}_{l},\underbrace{0\ldots,0}_{N-l})$.

Differentiating the function $V(t)$ along the trajectories of (3.1), and combining with (3.8), one can obtain

$$\begin{aligned}\frac{dV(t)}{dt} \le{}& \sum_{i=1}^{N} e_i^T P\left\{ f(x_i(t)) - f(s(t)) + \sum_{j=1}^{N} a_{ij}g(e_j(t)) + \sum_{j=1}^{N} b_{ij}h(e_j(t-\tau(t)))\right\}\\ &- \beta_1\underline{p}\sum_{i=1}^{N} d_i(t)e_i^T(t)e_i(t) + \mu^2\sum_{j=1}^{n}\tilde{e}_j^T(t)Q\tilde{e}_j(t)\end{aligned}$$

$$
\begin{aligned}
&- \mu^2 \sum_{j=1}^{n} (1-\dot{\tau}(t)) \tilde{e}_j^T(t-\tau(t)) Q \tilde{e}_j(t-\tau(t)) + \beta_1 \underline{p} \sum_{i=1}^{l} \left(d_i(t) - \hat{d}\right) e_i^T(t) e_i(t) \\
&= \sum_{i=1}^{N} e_i^T P \left\{ f(x_i(t)) - f(s(t)) + \sum_{j=1}^{N} a_{ij} g\left(e_j(t)\right) + \sum_{j=1}^{N} b_{ij} h\left(e_j(t-\tau(t))\right) \right\} \\
&- \beta_1 \underline{p} \sum_{i=1}^{l} \hat{d} e_i^T(t) e_i(t) + \mu^2 \sum_{j=1}^{n} \tilde{e}_j^T(t) Q \tilde{e}_j(t) \\
&- \mu^2 \sum_{j=1}^{n} (1-\dot{\tau}(t)) \tilde{e}_j^T(t-\tau(t)) Q \tilde{e}_j(t-\tau(t)).
\end{aligned} \tag{3.18}
$$

The remaining part of the proof is similar to that of Theorem 3.1, hence we omit it. □

To make Theorem 3.3 more applicable, we let $P = I_N, R = \|\tilde{A}\|_2 I_N, Q = \|B\|_2 I_N$, then we can easily obtain the following corollary.

Corollary 3.4. *Under the Assumptions 1–4, the pinning controlled directed network* (3.1) *is globally asymptotically stable at the homogenous trajectory* $s(t)$ *if the following conditions are satisfied:*

$$
\begin{gathered}
\max_{l+1 \le i \le N} \{a_{ii}\} < -\frac{4\delta^* + \left\|\tilde{A}\right\|_2}{4\beta_1}, \\
\left(\mu^2 + \frac{1-\alpha}{4}\right) \|B\|_2 + \beta_2^2 \left\|\tilde{A}\right\|_2 < \eta,
\end{gathered} \tag{3.19}
$$

where $\delta^* = \max_{1 \le j \le n}\{\delta_j\}$.

Remark 3.5. According to this corollary, we can rearrange network nodes in ascending order based on their in-degrees $-a_{ii}$ [7] and choose the first l network nodes as pinned candidates to satisfy the first pinning condition of this corollary.

4. Numerical Simulation

In this section, a simple example is used to explain the effectiveness of the proposed network synchronization criteria.

Example 4.1. We consider a directed network consisting of three identical Hindmarsh-Rose (HR) neuron systems [19], which is described by

$$
\dot{x}_i(t) = f(x_i(t), t) + \sum_{j=1}^{N} a_{ij} g\left(x_j(t)\right) + \sum_{j=1}^{N} b_{ij} h\left(x_j(t-\tau(t))\right), \tag{4.1}
$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^T$ is the state variable of the ith node,

$$f(x(t),t) = \begin{pmatrix} f_1(x(t)) \\ f_2(x(t)) \\ f_3(x(t)) \end{pmatrix} = \begin{pmatrix} a_1x_1^2(t) - x_1^3(t) - x_2(t) - x_3(t) \\ (a_1+b_1)x_1^2(t) - x_2(t) \\ b_2(a_2x_1(t) - x_3(t) + a_3) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -1 & -1 \\ 0 & -1 & 0 \\ b_2a_2 & 0 & -b_2 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} a_1x_1^2(t) - x_1^3(t) \\ (a_1+b_1)x_1^2(t) \\ b_2a_3 \end{pmatrix}. \tag{4.2}$$

According to the discussion in [18], $g(x_j(t))$ and $h(x_j(t-\tau(t)))$ are realistically specified as $g(x_j(t)) = (g_1(x_{j1}(t)), 0, 0))^T$ and $h(x_j(t-\tau(t))) = (h_1(x_{j1}(t-\tau(t))), 0, 0)^T$, so we can select $g(x_j(t)) = (x_{j1}(t), 0, 0)^T$ and $h(x_j(t-\tau(t))) = ((1/10)\sin x_{j1}(t-\tau(t)), 0, 0)^T$, then we can easily drive that $\beta_1 = \beta_2 = 1$, $\mu = 1/10$. We let

$$A = (a_{ij}) = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 2 & -4 \end{pmatrix}, \qquad B = (b_{ij}) = \begin{pmatrix} -0.1 & 0.1 & 0 \\ 0.03 & -0.03 & 0 \\ 0.1 & 0 & -0.1 \end{pmatrix}, \tag{4.3}$$

and $\tau(t) = 1$, $a_1 = 0.1$, $a_2 = 1.5$, $a_3 = 0.1$, $b_1 = 1.1$, $b_2 = 0.1$. In this case, $\alpha = 0$ and the bound M of the first variable in the HR equation is 0.5. If we let $P = I_3$, $\Delta = 3I_3$, then we have

$$(x-y)^T P(f(x) - f(y) - \Delta(x-y))$$
$$= (x-y)^T \begin{pmatrix} -3 & -1 & -1 \\ 0 & -4 & 0 \\ 0.15 & 0 & -3.1 \end{pmatrix}(x-y) + (x-y)^T \begin{pmatrix} 0.1(x_1^2 - y_1^2) - (x_1^3 - y_1^3) \\ 1.2(x_1^2 - y_1^2) \\ 0 \end{pmatrix}. \tag{4.4}$$

Note that

$$(x-y)^T \begin{pmatrix} 0.1(x_1^2 - y_1^2) - (x_1^3 - y_1^3) \\ 1.2(x_1^2 - y_1^2) \\ 0 \end{pmatrix}$$
$$= (x_1 - y_1)^2\left[0.1(x_1+y_1) - \left(x_1^2 + x_1y_1 + y_1^2\right)\right] + 1.2(x_1+y_1)(x_1-y_1)(x_2-y_2)$$
$$\leq 0.2M(x_1-y_1)^2 + 2.4M|x_1-y_1||x_2-y_2|$$
$$= |x-y|^T \begin{pmatrix} 0.1 & 0.6 & 0 \\ 0.6 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}|x-y|. \tag{4.5}$$

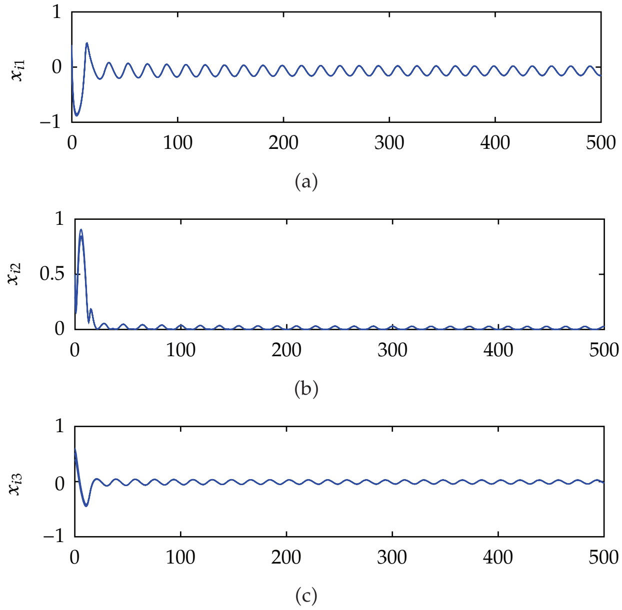

Figure 1: State trajectories $x_{ir}(t)(1 \le i \le 3, 1 \le r \le 3)$ of network (4.1) without control.

It follows that

$$
\begin{aligned}
&(x-y)^T P(f(x) - f(y) - \Delta(x-y)) \\
&\le |x-y|^T \begin{pmatrix} -2.9 & 1 & 1 \\ 0 & -4 & 0 \\ 0.15 & 0 & -3.1 \end{pmatrix} |x-y| + |x-y|^T \begin{pmatrix} -0.0875 & 0.6 & 0 \\ 0.6 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} |x-y| \\
&= |x-y|^T \begin{pmatrix} -2.9875 & 1.1 & 0.575 \\ 1.1 & -4 & 0 \\ 0.575 & 0 & -3.1 \end{pmatrix} |x-y| \le -2.0513(x-y)^T(x-y).
\end{aligned}
\tag{4.6}
$$

So, we can select $\eta = 2.0513$. Moreover, if we let $R = 2.01 I_3$, $Q = I_3, l = 2$, then, by simple computation, we can get

$$
\begin{aligned}
&\left(p_j \delta_j I_N + \beta_1 \underline{p} \tilde{\tilde{A}} + \frac{p_j^2}{4} \tilde{A} R^{-1} \tilde{A}^T \right)_l = \begin{pmatrix} 1.2488 & 0 & 0.2488 \\ 0 & 2.1244 & 0.2488 \\ 0.2488 & 0.2488 & -0.0048 \end{pmatrix}_2 = -0.0048 < 0, \\
&-\eta I_N + \mu^2 Q + \beta_2^2 R + \frac{(1-\alpha) p_j^2}{4} B Q^{-1} B^T = \begin{pmatrix} -0.0263 & -0.0015 & -0.0025 \\ -0.0015 & -0.03085 & 0.00075 \\ -0.0025 & 0.00075 & -0.0263 \end{pmatrix} < 0
\end{aligned}
\tag{4.7}
$$

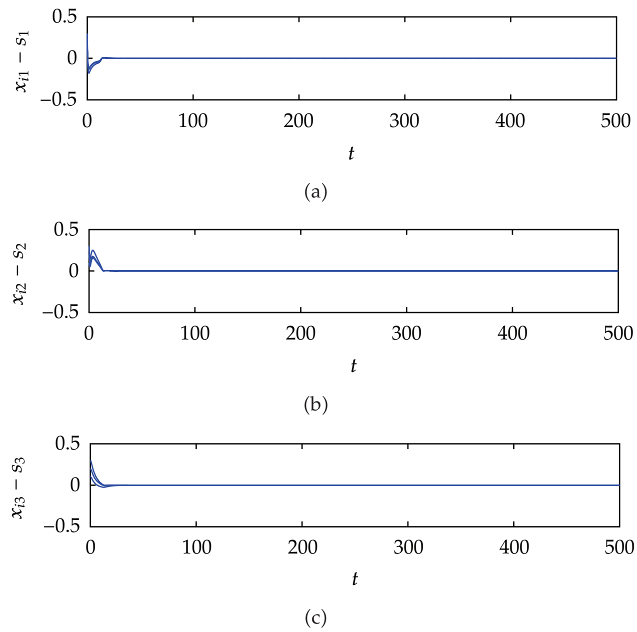

Figure 2: Synchronization errors $x_{ir} - s_r (1 \leq i \leq 3, 1 \leq r \leq 3)$ of network (4.1) for pinning the first two neurons.

for $j = 1, 2, 3$. So, according to Theorem 3.3, when the first two neurons are pinned, the directed network (4.1) with adaptive pinning controllers is globally asymptotically stable at the homogenous trajectory $s(t)$. Figure 1 shows the state variables $x_{ir}(t)$ $(1 \leq i \leq 3, 1 \leq r \leq 3)$ of network (4.1) with initial values as $x_i(0) = (0.1+0.1i, 0.2+0.1i, 0.3+0.1i)^T$, $1 \leq i \leq 3$ without control. Figure 2 shows the synchronization errors $e_{ir}(t)(1 \leq i \leq 3, 1 \leq r \leq 3)$ for pinning the first two neurons. From Figure 2, it is easy to see that the errors between the synchronized states converge to zero under the given conditions.

Remark 4.2. Since the nondelayed and delayed coupling matrices A and B are asymmetric, the theorem in [18] fails to conclude whether the dynamical system (4.1) can be synchronized. However, one can obtain the global synchronization using our results.

5. Conclusion

In this paper, on the basis of the previous work, we investigated the stabilization problem of delayed complex dynamical networks with nonlinear inner-coupling by pinning a small fraction of nodes. By using Lyapunov stability theory and linear matrix inequality (LMI) approach, some sufficient conditions ensuring the pinning synchronization are obtained. A comparison between our results and the previous results implies that our results establish a new set of pinning synchronization criteria for delayed complex dynamical networks with nonlinear inner-coupling. As an example, a network consisting of 3 identical Hindmarsh-Rose neuron systems is studied.

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