Research Article

Iterative Approximation of Common Fixed Points of Two Nonself Asymptotically Nonexpansive Mappings

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Suppose that K is nonempty closed convex subset of a uniformly convex and smooth Banach space E with P as a sunny nonexpansive retraction and $F := F(T_1) \cap F(T_2) = \{x \in K : T_1 x = T_2 x = x\} \neq \emptyset$. Let $T_1, T_2 : K \to E$ be two weakly inward nonself asymptotically nonexpansive mappings with respect to P with two sequences $\{k_n^{(i)}\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty (i = 1, 2)$, respectively. For any given $x_1 \in K$, suppose that $\{x_n\}$ is a sequence generated iteratively by $x_{n+1} = (1 - \alpha_n)(PT_1)^n y_n + \alpha_n (PT_2)^n y_n$, $y_n = (1 - \beta_n) x_n + \beta_n (PT_1)^n x_n$, $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[a, 1 - a]$ for some $a \in (0, 1)$. Under some suitable conditions, the strong and weak convergence theorems of $\{x_n\}$ to a common fixed point of T_1 and T_2 are obtained.

1. Introduction

Let E be a real Banach space with K, its nonempty subset. Let $T : K \to K$ be a mapping. A point $x \in K$ is called a fixed point of T if and only if $Tx = x$. In this paper, $\mathbb{N}$ stands for the set of natural numbers. We will also denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$ and by $F := F(T_1) \cap F(T_2)$, the set of common fixed points of two mappings T_1 and T_2. T is called asymptotically nonexpansive if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$, $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in K$ and all $n \in \mathbb{N}$. T is called uniformly L-Lipschitzian if for some $L > 0$, $\|T^n x - T^n y\| \leq L \|x - y\|$ for all $n \in \mathbb{N}$ and all $x, y \in K$. T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. Let $P : E \to K$ be a nonexpansive retraction of E into K. A nonself-mapping $T : K \to E$ is called asymptotically nonexpansive (according to Chidume et al. [1]) if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$, we have $\|T(PT)^{n-1} x - T(PT)^{n-1} y\| \leq k_n \|x - y\|$ for all $x, y \in K$ and $n \in \mathbb{N}$. T is called uniformly

L-Lipschitzian if for some $L > 0$, $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|$ for all $n \in \mathbb{N}$ and all $x, y \in K$.

In what follows, we fix $x_1 \in K$ as a starting point of the process under consideration, and take $\{\alpha_n\}$, $\{\beta_n\}$ sequences in $(0, 1)$.

Agarwal et al. [2] recently introduced the iteration process

$$
\begin{aligned}
x_{n+1} &= (1 - \alpha_n)T^n x_n + \alpha_n T^n y_n, \\
y_n &= (1 - \beta_n)x_n + \beta_n T^n x_n, \quad n \in \mathbb{N}.
\end{aligned}
\tag{1.1}
$$

They showed that their process is independent of Mann and Ishikawa and converges faster than both of these. See Proposition 3.1 [2].

Obviously the above process deals with one self-mapping only. The case of two mappings in iteration processes has also remained under study since Das and Debata [3] gave and studied a two mappings scheme. Also see, for example, Takahashi and Tamura [4] and Khan and Takahashi [5]. Note that two mappings case, that is, approximating the common fixed points, has its own importance as it has a direct link with the minimization problem, see, for example, Takahashi [6].

Being an important generalization of the class of nonexpansive self-mappings, the class of asymptotically nonexpansive self-mappings was introduced by Goebel and Kirk [7] whereas the concept of asymptotically nonexpansive nonself-mappings was introduced by Chidume et al. [1] in 2003 as a generalization of asymptotically nonexpansive self-mappings. Actually they studied the iteration process

$$
x_{n+1} = P\Big((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n\Big), \quad n \in \mathbb{N}.
\tag{1.2}
$$

Nonself asymptotically nonexpansive mappings have been studied by many authors [8–11]. Wang [10] studied the process

$$
\begin{aligned}
x_{n+1} &= P\Big((1 - \alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n\Big), \\
y_n &= P\Big((1 - \beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n\Big), \quad n \in \mathbb{N}.
\end{aligned}
\tag{1.3}
$$

Very recently, Thianwan [12] considered a new iterative scheme (called projection type Ishikawa iteration) as follows:

$$
\begin{aligned}
x_{n+1} &= P\Big((1 - \alpha_n)y_n + \alpha_n T_1(PT_1)^{n-1}y_n\Big), \\
y_n &= P\Big((1 - \beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n\Big), \quad n \in \mathbb{N}.
\end{aligned}
\tag{1.4}
$$

As a matter of fact, if T is a self-mapping, then P is an identity mapping. In addition, if $T : K \to E$ is asymptotically nonexpansive and $P : E \to K$ is a nonexpansive retraction, then

$PT : K \rightarrow K$ is asymptotically nonexpansive. Indeed, for all $x, y \in K$ and $n \in \mathbb{N}$, it follows that

$$
\begin{aligned}
\left\| (PT)^n x - (PT)^n y \right\| &= \left\| PT(PT)^{n-1} x - PT(PT)^{n-1} y \right\| \\
&\leq \left\| T(PT)^{n-1} x - T(PT)^{n-1} y \right\| \\
&\leq k_n \left\| x - y \right\|.
\end{aligned}
\tag{1.5}
$$

The converse, however, may not be true. Therefore, Zhou et al. [13] introduced the following generalized definition recently.

Definition 1.1 (see [13]). Let K be a nonempty subset of real normed linear space E. Let $P : E \rightarrow K$ be the nonexpansive retraction of E into K.

   (i) A nonself-mapping $T : \mathrm{K} \rightarrow E$ is called asymptotically nonexpansive with respect to P if there exists sequences $\{k_n\} \in [1, \infty)$ with $k_n \rightarrow 1$ as $n \rightarrow \infty$ such that

$$
\left\| (PT)^n x - (PT)^n y \right\| \leq k_n \left\| x - y \right\|, \quad \forall x, y \in K, n \in \mathbb{N}.
\tag{1.6}
$$

   (ii) A nonself-mapping $T : K \rightarrow E$ is said to be uniformly L-Lipschitzian with respect to P if there exists a constant $L \geq 0$ such that

$$
\left\| (PT)^n x - (PT)^n y \right\| \leq L \left\| x - y \right\|, \quad \forall x, y \in K, n \in \mathbb{N}.
\tag{1.7}
$$

Futhermore, by studying the following iterative process

$$
x_1 \in K, \quad x_{n+1} = \alpha_n x_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n, \quad n \in \mathbb{N},
\tag{1.8}
$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[a, 1 - a]$ for some $a \in (0, 1)$, satisfying $\alpha_n + \beta_n + \gamma_n = 1$, Zhou et al. [13] obtained some strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings with respect to P in uniformly convex Banach spaces. As a consequence, the main results of Chidume et al. [1] were deduced.

Incorporating the ideas of Agarwal et al. [2], Thianwan [12], and Zhou et al. [13], a new two-step iterative scheme for two nonself asymptotically nonexpansive mappings is introduced and studied in this paper. Our process reads as follows.

Let K be a nonempty closed convex subset of a real normed linear space E with retraction P. Let $T_1, T_2 : K \rightarrow E$ be two nonself asymptotically nonexpansive mappings with respect to P:

$$
\begin{aligned}
x_1 &\in K, \\
x_{n+1} &= (1 - \alpha_n) \, (PT_1)^n y_n + \alpha_n (PT_2)^n y_n, \\
y_n &= (1 - \beta_n) x_n + \beta_n (PT_1)^n x_n, \quad n \in \mathbb{N},
\end{aligned}
\tag{1.9}
$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. Following the method of Agarwal et al. [2], it is not difficult to see that our process is able to compute common fixed points at a rate better than (1.3) and (1.4).

Under suitable conditions, the sequence $\{x_n\}$ defined by (1.9) can also be generalized to iterative sequence with errors. Thus all the results proved in this paper can also be proved for the iterative process with errors. In this case our main iterative process (1.9) looks like

$$x_1 \in K,$$

$$x_{n+1} = \alpha_n(PT_1)^n y_n + \beta_n(PT_2)^n y_n + \gamma_n u_n, \tag{1.10}$$

$$y_n = \alpha'_n x_n + \beta'_n(PT_1)^n x_n + \gamma'_n v_n, \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are real sequences in $[0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K. Observe that the iterative process (1.10) with errors reduces to the iterative process (1.9) when $\gamma_n = \gamma'_n = 0$.

2. Preliminaries

For the sake of convenience, we restate the following concepts and results.

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of E is the function $\delta_E(\varepsilon) : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : \|x\| = 1, \|y\| = 1, \varepsilon = \|x-y\|\right\}. \tag{2.1}$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

Let E be a Banach space and $S(E) = \{x \in E : \|x\| = 1\}$. The space E is said to be smooth if

$$\lim_{t \to 0}\frac{\|x+ty\| - \|x\|}{t} \tag{2.2}$$

exists for all $x, y \in S(E)$.

A subset K of E is said to be a retract if there exists a continuous mapping $P : E \to K$ such that $Px = x$ for all $x \in K$. A mapping $P : E \to E$ is said to be a retraction if $P^2 = P$. Let C and K be subsets of a Banach space E. A mapping P from C into K is called sunny if $P(Px + t(x - Px)) = Px$ for $x \in C$ with $Px + t(x - Px) \in C$ and $t \geq 0$.

Note that, if P is a retraction, then $Pz = z$ for every $z \in R(P)$ (the range of P). It is well-known that every closed convex subset of a uniformly convex Banach space is a retract.

For any $x \in K$, the inward set $I_K(x)$ is defined as follows:

$$I_K(x) = \{y \in E : y = x + \lambda(z - x), z \in K, \lambda \geq 0\}. \tag{2.3}$$

A mapping $T : K \to E$ is said to satisfy the inward condition if $Tx \in I_K(x)$ for all $x \in K$. T is said to be weakly inward if $Tx \in \operatorname{cl} I_K(x)$ for each $x \in K$, where $\operatorname{cl} I_K(x)$ is the closure of $I_K(x)$.

A Banach space E is said to satisfy Opial's condition if, for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\| \tag{2.4}$$

for all $y \in E$ with $y \neq x$, where $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x.

Recall that the mapping $T : K \to K$ with $F(T) \neq \emptyset$ is said to satisfy condition (A) [14] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in K$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$. Khan and Fukhar-ud-din [15] modified condition (A) for two mappings as follows: Two mappings $T_1, T_2 : K \to K$ are said to satisfy condition (A') [15] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$\frac{1}{2}(\|x - T_1 x\| + \|x - T_2 x\|) \geq f(d(x, F)) \tag{2.5}$$

for all $x \in K$, where $d(x, F) = \inf\{\|x - p\| : p \in F := F(T_1) \cap F(T_2)\}$.

Note that condition (A') reduces to condition (A) when $T_1 = T_2$. It is also well-known that condition (A) is weaker than demicompactness or semicompactness, see [14].

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx^* = p$.

We need the following lemmas for our main results.

Lemma 2.1 (see [16]). *If $\{r_n\}$, $\{t_n\}$ are two sequences of nonnegative real numbers such that*

$$r_{n+1} \leq (1 + t_n) r_n, \quad n \in \mathbb{N} \tag{2.6}$$

and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n \to \infty} r_n$ exists.

Lemma 2.2 (see [17]). *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Also, suppose that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that*

$$\limsup_{n \to \infty} \|x_n\| \leq r, \qquad \limsup_{n \to \infty} \|y_n\| \leq r, \qquad \lim_{n \to \infty} \|(1 - t_n)x_n + t_n y_n\| = r \tag{2.7}$$

hold for some $r \geq 0$. Then $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

Lemma 2.3 (see [18]). *Let E be real smooth Banach space, let K be nonempty closed convex subset of E with P as a sunny nonexpansive retraction, and let $T : K \to E$ be a mapping satisfying weakly inward condition. Then $F(PT) = F(T)$.*

Lemma 2.4 (see [1]). *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let T be a nonself asymptotically nonexpansive mapping. Then $I - T$ is demiclosed with respect to zero, that is, $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ imply that $Tx = x$.*

3. Main Results

3.1. Convergence Theorems in Real Banach Spaces

In this section, we prove the strong convergence of the iteration scheme (1.9) to a common fixed point of nonself asymptotically nonexpansive mappings T_1 and T_2 with respect to P in real Banach spaces. Let $T_1, T_2 : K \to E$ be two nonself asymptotically nonexpansive mappings with respect to P with sequences $\{k_n^{(i)}\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty} (k_n^{(i)} - 1) < \infty (i = 1, 2)$, respectively. Put $k_n = \max\{k_n^{(1)}, k_n^{(2)}\}$, then obviously $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. From now on we will take this sequence $\{k_n\}$ for both T_1 and T_2.

We first prove the following lemmas.

Lemma 3.1. *Let E be a real normed linear space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_1, T_2 : K \to E$ be two nonself asymptotically nonexpansive mappings with respect to P with sequence $\{k_n\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Suppose that $\{x_n\}$ is defined by (1.9) and $F \neq \emptyset$. Then,*

   (i) $\lim_{n \to \infty} \|x_n - p\|$ *exists for all $p \in F$;*

   (ii) *there exists a constant $M > 0$ such that $\|x_{n+m} - p\| \leq M\|x_n - p\|$ for all $m, n \in \mathbb{N}$ and $p \in F$.*

Proof. (i) Let $p \in F$. From (1.9), we have

$$
\begin{aligned}
\|y_n - p\| &= \|(1 - \beta_n)x_n + \beta_n (PT_1)^n x_n - p\| \\
&\leq (1 - \beta_n)\|x_n - p\| + \beta_n \|(PT_1)^n x_n - p\| \\
&\leq (1 - \beta_n)\|x_n - p\| + \beta_n k_n \|x_n - p\| \\
&= (1 + \beta_n (k_n - 1))\|x_n - p\| \\
&\leq (1 + (k_n - 1))\|x_n - p\| \\
&= k_n \|x_n - p\|.
\end{aligned}
\tag{3.1}
$$

By (3.1) and (1.9), we obtain

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - \alpha_n)(PT_1)^n y_n - \alpha_n (PT_2)^n y_n - p\| \\
&= \|(1 - \alpha_n)((PT_1)^n y_n - p) - \alpha_n ((PT_2)^n y_n - p)\| \\
&\leq (1 - \alpha_n)k_n \|y_n - p\| + \alpha_n k_n \|y_n - p\| \\
&= k_n \|y_n - p\| \\
&\leq k_n^2 \|x_n - p\| \\
&= \left(1 + \left(k_n^2 - 1\right)\right)\|x_n - p\|.
\end{aligned}
\tag{3.2}
$$

Note that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ is equivalent to $\sum_{n=1}^{\infty} (k_n^2 - 1) < \infty$. Thus, by (3.2) and Lemma 2.1, $\lim_{n \to \infty} \|x_n - p\|$ exists for all $p \in F(T)$.

(ii) From (3.2), we have

$$\|x_{n+1} - p\| \leq k_n^2 \|x_n - p\|. \tag{3.3}$$

It is well known that $1 + x \leq e^x$ for all $x \geq 0$. Using it for the above inequality, we have

$$
\begin{aligned}
\|x_{n+m} - p\| &\leq \left(1 + \left(k_{n+m-1}^2 - 1\right)\right)\|x_{n+m-1} - p\| \\
&\leq e^{k_{n+m-1}^2 - 1}\|x_{n+m-1} - p\| \\
&\leq e^{k_{n+m-1}^2 - 1}\left[\left(1 + \left(k_{n+m-2}^2 - 1\right)\right)\|x_{n+m-2} - p\|\right] \\
&\leq e^{k_{n+m-1}^2 - 1 + k_{n+m-2}^2 - 1}\|x_{n+m-2} - p\| \\
&\quad\vdots \\
&\leq e^{\sum_{j=n}^{n+m-1}(k_j^2 - 1)}\|x_n - p\| \\
&\leq M\|x_n - p\|,
\end{aligned}
\tag{3.4}
$$

where $M = e^{\sum_{j=n}^{n+m-1}(k_j^2 - 1)}$. That is, $\|x_{n+m} - p\| \leq M\|x_n - p\|$ for all $m, n \in \mathbb{N}$ and $p \in F$. $\qquad\square$

Theorem 3.2. *Let E be a real Banach space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_1, T_2 : K \to E$ be two nonself asymptotically nonexpansive mappings with respect to P with sequence $\{k_n\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose that $\{x_n\}$ is defined by (1.9) and $F \neq \emptyset$. Then, $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2 if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x_n, F) = \inf\{\|x - p\| : p \in F\}$.*

Proof. The necessity of the conditions is obvious. Thus, we need only prove the sufficiency. Suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. From (3.2), we have

$$d(x_{n+1}, F) \leq \left(1 + \left(k_n^2 - 1\right)\right)d(x_n, F). \tag{3.5}$$

As $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, therefore $\lim_{n\to\infty} d(x_n, F)$ exists by Lemma 2.1. But by hypothesis $\liminf_{n\to\infty} d(x_n, F) = 0$, therefore we must have $\lim_{n\to\infty} d(x_n, F) = 0$.

Next we show that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, therefore there exists a constant n_0 such that for all $n \geq n_0$, we have

$$d(x_n, F) < \frac{\varepsilon}{4M}, \tag{3.6}$$

where $M > 0$ is the constant in Lemma 3.1(ii). So we can find $p' \in F$ such that

$$\|x_{n_0} - p'\| < \frac{\varepsilon}{2M}. \tag{3.7}$$

Using Lemma 3.1(ii), we have for all $n \geq n_0$ and $m \in \mathbb{N}$ that

$$
\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p'\| + \|x_n - p'\| \\
&\leq M\|x_{n_0} - p'\| + M\|x_{n_0} - p'\| \\
&= 2M\|x_{n_0} - p'\| \\
&< \varepsilon.
\end{aligned}
\tag{3.8}
$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset K of a Banach space E, therefore it must converge to a point in K. Let $\lim_{n \to \infty} x_n = q$. Now, $\lim_{n \to \infty} d(x_n, F) = 0$ gives that $d(q, F) = 0$. Since the set of fixed points of asymptotically nonexpansive mappings is closed, we have $q \in F$. This completes the proof of the theorem. $\square$

On the lines similar to this theorem, we can also prove the following theorem which addresses the error terms.

Theorem 3.3. *Let E be a real Banach space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_1, T_2 : K \to E$ be two asymptotically nonexpansive mappings with respect to P with sequence $\{k_n\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose that $\{x_n\}$ is defined by (1.10) with $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$ and $F \neq \emptyset$, Then, $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2 if and only if $\liminf_{n \to \infty} d(x_n, F) = 0$, where $d(x_n, F) = \inf\{\|x - p\| : p \in F\}$.*

3.2. Convergence Theorems in Real Uniformly Convex Banach Spaces

In this section, we prove the strong and weak convergence of the sequence defined by the iteration scheme (1.9) to a common fixed point of nonself asymptotically nonexpansive mappings T_1 and T_2 with respect to P in real uniformly convex and smooth Banach space. We first prove the following lemma.

Lemma 3.4. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. Let $T_1, T_2 : K \to E$ be two nonself asymptotically nonexpansive mappings with respect to P with sequence $\{k_n\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose that $\{x_n\}$ is defined by (1.9), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[a, 1 - a]$ for some $a \in (0, 1)$. If $F \neq \emptyset$, then*

$$
\lim_{n \to \infty} \|x_n - (PT_1)x_n\| = \lim_{n \to \infty} \|x_n - (PT_2)x_n\| = 0.
\tag{3.9}
$$

Proof. By Lemma 3.1(i), $\lim_{n \to \infty} \|x_n - p\|$ exists. Assume that $\lim_{n \to \infty} \|x_n - p\| = c$. If $c = 0$, the conclusion is obvious. Suppose $c > 0$. Taking $\lim \sup$ on both sides in that inequality (3.1), we have

$$
\limsup_{n \to \infty} \|y_n - p\| \leq \limsup_{n \to \infty} \|x_n - p\| = \lim_{n \to \infty} \|x_n - p\| = c.
\tag{3.10}
$$

Thus $\|(PT_1)^n y_n - p\| \le k_n \|y_n - p\|$ for all $n \in \mathbb{N}$ implies that

$$\limsup_{n \to \infty} \|(PT_1)^n y_n - p\| \le c. \tag{3.11}$$

Similarly,

$$\limsup_{n \to \infty} \|(PT_2)^n y_n - p\| \le c. \tag{3.12}$$

Further,

$$
\begin{aligned}
c &= \lim_{n \to \infty} \|x_{n+1} - p\| \\
&= \lim_{n \to \infty} \|(1 - \alpha_n)(PT_1)^n y_n + \alpha_n (PT_2)^n y_n - p\| \\
&= \lim_{n \to \infty} \|(1 - \alpha_n)((PT_1)^n y_n - p) + \alpha_n ((PT_2)^n y_n - p)\| \\
&\le \lim_{n \to \infty} \left[(1 - \alpha_n) \left\| \limsup_{n \to \infty} ((PT_1)^n y_n - p) \right\| + \alpha_n \left\| \limsup_{n \to \infty} ((PT_2)^n y_n - p) \right\| \right] \\
&\le \lim_{n \to \infty} [(1 - \alpha_n)c + \alpha_n c] \\
&= c
\end{aligned}
\tag{3.13}
$$

gives that

$$\lim_{n \to \infty} \|(1 - \alpha_n)((PT_1)^n y_n - p) + \alpha_n ((PT_2)^n y_n - p)\| = c. \tag{3.14}$$

Hence, using (3.11), (3.12), (3.14), and Lemma 2.2, we obtain

$$\lim_{n \to \infty} \|(PT_2)^n y_n - (PT_1)^n y_n\| = 0. \tag{3.15}$$

Noting that

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - \alpha_n)(PT_1)^n y_n + \alpha_n (PT_2)^n y_n - p\| \\
&\le \|(PT_1)^n y_n - p\| + \alpha_n \|(PT_2)^n y_n - (PT_1)^n y_n\| \\
&\le k_n \|y_n - p\| + \alpha_n \|(PT_2)^n y_n - (PT_1)^n y_n\|
\end{aligned}
\tag{3.16}
$$

which yields that

$$c \le \liminf_{n \to \infty} \|y_n - p\|. \tag{3.17}$$

By (3.10) and (3.17), we obtain

$$\lim_{n \to \infty} \|y_n - p\| = c. \tag{3.18}$$

Moreover, $\|(PT_1)^n x_n - p\| \le k_n \|x_n - p\|$ for all $n \in \mathbb{N}$ implies that

$$\limsup_{n \to \infty} \|(PT_1)^n x_n - p\| \le c. \tag{3.19}$$

Hence

$$
\begin{aligned}
c &= \lim_{n \to \infty} \|y_n - p\| \\
&= \lim_{n \to \infty} \|(1 - \beta_n) x_n + \beta_n (PT_1)^n x_n - p\| \\
&= \lim_{n \to \infty} \|(1 - \beta_n)(x_n - p) + \beta_n ((PT_1)^n x_n - p)\| \\
&\le \lim_{n \to \infty} \left[(1 - \beta_n) \left\| \limsup_{n \to \infty} (x_n - p) \right\| + \beta_n \left\| \limsup_{n \to \infty} ((PT_1)^n x_n - p) \right\| \right] \\
&\le \lim_{n \to \infty} \left[(1 - \beta_n) c + \beta_n c \right] \\
&= c
\end{aligned}
\tag{3.20}
$$

gives that

$$\lim_{n \to \infty} \|(1 - \beta_n)(x_n - p) + \beta_n ((PT_1)^n x_n - p)\| = c. \tag{3.21}$$

Again by Lemma 2.2, we obtain

$$\lim_{n \to \infty} \|(PT_1)^n x_n - x_n\| = 0. \tag{3.22}$$

In addition, from $y_n = (1 - \beta_n) x_n + \beta_n (PT_1)^n x_n$, we have

$$\|y_n - x_n\| = \beta_n \|(PT_1)^n x_n - x_n\|. \tag{3.23}$$

Hence by (3.22),

$$\lim_{n \to \infty} \|y_n - x_n\| = 0. \tag{3.24}$$

Also

$$
\begin{aligned}
\|(PT_2)^n y_n - x_n\| &\le \|(PT_2)^n y_n - (PT_1)^n y_n\| + \|(PT_1)^n y_n - (PT_1)^n x_n\| + \|(PT_1)^n x_n - x_n\| \\
&\le \|(PT_2)^n y_n - (PT_1)^n y_n\| + k_n \|y_n - x_n\| + \|(PT_1)^n x_n - x_n\|
\end{aligned}
\tag{3.25}
$$

implies by (3.15), (3.22), and (3.24) that

$$\lim_{n \to \infty} \|(PT_2)^n y_n - x_n\| = 0. \tag{3.26}$$

Using (3.24) and (3.26), we obtain

$$\begin{aligned}
\|(PT_2)^n x_n - x_n\| &\leq \|(PT_2)^n x_n - (PT_2)^n y_n\| + \|(PT_2)^n y_n - x_n\| \\
&\leq k_n \|x_n - y_n\| + \|(PT_2)^n y_n - x_n\|,
\end{aligned} \tag{3.27}$$

so that

$$\lim_{n \to \infty} \|(PT_2)^n x_n - x_n\| = 0. \tag{3.28}$$

Then

$$\begin{aligned}
\|(PT_1)^n y_n - x_n\| &\leq \|(PT_1)^n y_n - (PT_1)^n x_n\| + \|(PT_1)^n x_n - x_n\| \\
&\leq k_n \|y_n - x_n\| + \|(PT_1)^n x_n - x_n\|
\end{aligned} \tag{3.29}$$

gives

$$\lim_{n \to \infty} \|(PT_1)^n y_n - x_n\| = 0. \tag{3.30}$$

From (3.15), and (3.30), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(1 - \alpha_n)(PT_1)^n y_n + \alpha_n (PT_2)^n y_n - x_n\| \\
&\leq \|(PT_1)^n y_n - x_n\| + \alpha_n \|(PT_2)^n y_n - (PT_1)^n y_n\| \\
&\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.31}$$

Thus from $\|x_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + \|x_n - y_n\|$, we get

$$\lim_{n \to \infty} \|x_{n+1} - y_n\| = 0. \tag{3.32}$$

From (3.30), (3.31) and

$$\|x_{n+1} - (PT_1)^n y_n\| \leq \|x_{n+1} - x_n\| + \|x_n - (PT_1)^n y_n\|, \tag{3.33}$$

we have

$$\lim_{n \to \infty} \|x_{n+1} - (PT_1)^n y_n\| = 0. \tag{3.34}$$

Now we make use of the fact that every nonself asymptotically nonexpansive mapping with respect to P must be uniformly L-Lipschitzian with respect to P combined with (3.22), (3.32), and (3.34), where $L = \sup_{n \in \mathbb{N}} \{k_n\} \geq 1$, to reach at

$$
\begin{aligned}
\left\| x_n - (PT_1)x_n \right\| &\leq \left\| x_n - (PT_1)^n x_n \right\| + \left\| (PT_1)^n x_n - (PT_1)^n y_{n-1} \right\| + \left\| (PT_1)^n y_{n-1} - (PT_1)x_n \right\| \\
&\leq \left\| x_n - (PT_1)^n x_n \right\| + k_n \left\| x_n - y_{n-1} \right\| + L \left\| (PT_1)^{n-1} y_{n-1} - x_n \right\|.
\end{aligned}
$$

(3.35)

Thus

$$
\lim_{n \to \infty} \left\| x_n - (PT_1)x_n \right\| = 0.
$$

(3.36)

From (3.24), (3.26), and (3.31), we have

$$
\begin{aligned}
\left\| x_{n+1} - (PT_2)^n x_n \right\| &\leq \left\| x_{n+1} - x_n \right\| + \left\| x_n - (PT_2)^n y_n \right\| + \left\| (PT_2)^n y_n - (PT_2)^n x_n \right\| \\
&\leq \left\| x_{n+1} - x_n \right\| + \left\| x_n - (PT_2)^n y_n \right\| + k_n \left\| y_n - x_n \right\| \\
&\longrightarrow 0 \quad \text{as } n \to \infty,
\end{aligned}
$$

(3.37)

and so

$$
\lim_{n \to \infty} \left\| x_{n+1} - (PT_2)^n x_n \right\| = 0.
$$

(3.38)

Again making use of the fact that every nonself asymptotically nonexpansive mapping with respect to P must be uniformly L-Lipschitzian with respect to P and (3.28), (3.31) and (3.38), we have

$$
\begin{aligned}
\left\| x_{n+1} - (PT_2)x_{n+1} \right\| &\leq \left\| x_{n+1} - (PT_2)^{n+1} x_{n+1} \right\| + \left\| (PT_2)^{n+1} x_{n+1} - (PT_2)^{n+1} x_n \right\| \\
&\quad + \left\| (PT_2)^{n+1} x_n - (PT_2)x_{n+1} \right\| \\
&\leq \left\| x_{n+1} - (PT_2)^{n+1} x_{n+1} \right\| + k_{n+1} \left\| x_{n+1} - x_n \right\| + L \left\| (PT_2)^n x_n - x_{n+1} \right\|.
\end{aligned}
$$

(3.39)

This gives,

$$
\lim_{n \to \infty} \left\| x_n - (PT_2)x_n \right\| = 0.
$$

(3.40)

This completes the proof of the lemma.   □

Theorem 3.5. *Let K be a nonempty closed convex subset of a real uniformly convex and smooth Banach space E with P as a sunny nonexpansive retraction. Let $T_1, T_2 : K \to E$ be two weakly inward and nonself asymptotically nonexpansive mappings with respect to P with sequence $\{k_n\} \subset [1, \infty)$*

satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose that $\{x_n\}$ is defined by (1.9), where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[a, 1 - a]$ for some $a \in (0, 1)$. If one of T_1 and T_2 is completely continuous and $F \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.

Proof. By Lemma 3.1(i), $\lim_{n \to \infty} \|x_n - p\|$ exists for any $p \in F$. It is sufficient to show that $\{x_n\}$ has a subsequence which converges strongly to a common fixed point of T_1 and T_2. By Lemma 3.4, $\lim_{n \to \infty} \|x_n - (PT_1)x_n\| = \lim_{n \to \infty} \|x_n - (PT_2)x_n\| = 0$. Suppose that T_1 is completely continuous. Noting that P is nonexpansive, we conclude that there exists subsequence $\{PT_1 x_{n_j}\}$ of $\{PT_1 x_n\}$ such that $PT_1 x_{n_j} \to p$. Thus $\|x_{n_j} - p\| \leq \|x_{n_j} - PT_1 x_{n_j}\| + \|PT_1 x_{n_j} - p\|$ implies $x_{n_j} \to p$ as $j \to \infty$. Again $\lim_{j \to \infty} \|x_{n_j} - (PT_1)x_{n_j}\| = 0$ yields by continuity of P and T_1 that $p = PT_1 p$. Similarly $p = PT_2 p$. By Lemma 2.3, $p = T_1 p = T_2 p$. Since F is closed, so $p \in F$. Thus $\{x_n\}$ converges strongly to a common fixed point p of T_1 and T_2. This completes the proof. □

Theorem 3.6. *Let K be a nonempty closed convex subset of a real uniformly convex and smooth Banach space E with P as a sunny nonexpansive retraction. Let $T_1, T_2 : K \to E$ be two weakly inward and nonself asymptotically nonexpansive mappings with respect to P with sequence $\{k_n\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose that $\{x_n\}$ is defined by (1.9), where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[a, 1 - a]$ for some $a \in (0, 1)$. If T_1 and T_2 satisfy condition (A') and $F \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.*

Proof. By Lemma 3.1(i), $\lim_{n \to \infty} \|x_n - p\|$ exists, and so, $\lim_{n \to \infty} d(x_n, F)$ exists for all $p \in F$. Also, by Lemma 3.4 $\lim_{n \to \infty} \|x_n - (PT_1)x_n\| = \lim_{n \to \infty} \|x_n - (PT_2)x_n\| = 0$. It follows from condition (A') and Lemma 2.3 that

$$\lim_{n \to \infty} f(d(x_n, F)) \leq \lim_{n \to \infty} \left(\frac{1}{2}(\|x_n - (PT_1)x_n\| + \|x_n - (PT_2)x_n\|) \right) = 0. \tag{3.41}$$

That is,

$$\lim_{n \to \infty} f(d(x_n, F)) = 0. \tag{3.42}$$

Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$, therefore we have

$$\lim_{n \to \infty} d(x_n, F) = 0. \tag{3.43}$$

From Theorem 3.2, we obtain that $\{x_n\}$ is a Cauchy sequence in K. Since K is a closed subset of a complete space, there exists a $q \in K$ such that $x_n \to q$ as $n \to \infty$. Then, $\lim_{n \to \infty} d(x_n, F) = 0$ yields that $d(q, F) = 0$. Further, it follows from the closedness of F that $q \in F$. This completes the proof. □

Theorem 3.7. *Let K be a nonempty closed convex subset of a real uniformly convex and smooth Banach space E satisfying Opial's condition with P as a sunny nonexpansive retraction. Let $T_1, T_2 : K \to E$ be two weakly inward and nonself asymptotically nonexpansive mappings with respect to P with sequence $\{k_n\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose that $\{x_n\}$ is defined by (1.9),*

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[a, 1-a]$ for some $a \in (0,1)$. If $F \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2.

Proof. Let $p \in F$. By Lemma 3.1(i), $\lim_{n\to\infty}\|x_n - p\|$ exists, and $\{x_n\}$ is bounded. Note that PT_1 and PT_2 are self-mappings from K into itself. We now prove that $\{x_n\}$ has a unique weak subsequential limit in F. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ converge weakly to p_1 and p_2, respectively. By Lemma 3.4, we have $\lim_{n\to\infty}\|x_{n_k} - (PT_i)x_{n_k}\| = 0$, $(i = 1, 2)$. Lemma 2.4 guarantees that $(I - PT_1)p_1 = 0$, that is, $(PT_1)p_1 = p_1$. Similarly, $(PT_2)p_1 = p_1$. Again in the same way, we can prove that $p_2 \in F$. Lemma 2.3 now assures that $p_1, p_2 \in F$. For uniqueness, assume that $p_1 \neq p_2$, then by Opial's condition, we have

$$
\begin{aligned}
\lim_{n\to\infty}\|x_n - p_1\| &= \lim_{k\to\infty}\|x_{n_k} - p_1\| \\
&< \lim_{k\to\infty}\|x_{n_k} - p_2\| \\
&= \lim_{j\to\infty}\|x_{n_j} - p_2\| \\
&< \lim_{j\to\infty}\|x_{n_j} - p_1\| \\
&= \lim_{n\to\infty}\|x_n - p_1\|,
\end{aligned}
\tag{3.44}
$$

which is a contradiction and hence $p_1 = p_2$. As a result, $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2. $\qquad\square$

In a way similar to the above, we can also prove the results involving error terms as follows.

Theorem 3.8. *Let K be a nonempty closed convex subset of a real uniformly convex and smooth Banach space E with P as a sunny nonexpansive retraction. Let $T_1, T_2 : K \to E$ be two weakly inward and nonself asymptotically nonexpansive mappings with respect to P with sequence $\{k_n\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose that $\{x_n\}$ is the sequence defined by (1.10) satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$;

(ii) $\{\alpha_n\}$ and $\{\alpha_n'\}$ are two sequences in $[a, 1-a]$ for some $a \in (0,1)$.

If one of T_1 and T_2 is completely continuous and $F \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.

Theorem 3.9. *Let K be a nonempty closed convex subset of a real uniformly convex and smooth Banach space E with P as a sunny nonexpansive retraction. Let $T_1, T_2 : K \to E$ be two weakly inward and nonself asymptotically nonexpansive mappings with respect to P with sequence $\{k_n\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose that $\{x_n\}$ is the sequence defined by (1.10) satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$;

(ii) $\{\alpha_n\}$ and $\{\alpha_n'\}$ are two sequences in $[a, 1-a]$ for some $a \in (0,1)$.

If T_1 and T_2 satisfy condition (A') and $F \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.

Theorem 3.10. *Let K be a nonempty closed convex subset of a real uniformly convex and smooth Banach space E satisfying Opial's condition with P as a sunny nonexpansive retraction. Let $T_1, T_2 : K \to E$ be two weakly inward and nonself asymptotically nonexpansive mappings with respect to P with sequence $\{k_n\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose that $\{x_n\}$ is the sequence defined by (1.10) satisfying the following conditions:*

(i) *$\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$;*

(ii) *$\{\alpha_n\}$ and $\{\alpha_n'\}$ are two sequences in $[a, 1-a]$ for some $a \in (0, 1)$.*

If $F \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2.

Acknowledgments

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