Research Article

Delay-Dependent Stability Criteria of Uncertain Periodic Switched Recurrent Neural Networks with Time-Varying Delays

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This paper deals with the problem of delay-dependent stability criterion of uncertain periodic switched recurrent neural networks with time-varying delays. When uncertain discrete-time recurrent neural network is a periodic system, it is expressed as switched neural network for the finite switching state. Based on the switched quadratic Lyapunov functional approach (SQLF) and free-weighting matrix approach (FWM), some linear matrix inequality criteria are found to guarantee the delay-dependent asymptotical stability of these systems. Two examples illustrate the exactness of the proposed criteria.

1. Introduction

Recurrent neural networks (RNNs) are a very important tool for many application areas such as associative memory, pattern recognition, signal processing, model identification, and combinatorial optimization. With the development of research on RNNs in theory and application, the model is more and more complex. When the continuous-time RNNs are simulated using computer, they should be discretized into discrete-time RNNs [1–3]. Simultaneously, in implementations of artificial neural networks, time-varying delay may occur due to finite switching speeds of the amplifiers and communication time [4, 5]. Therefore, researchers have considered that discrete-time RNNs with time-varying delay are incorporated in the processing and/or transmission parts of the network architectures [6–9]. Parameter uncertainties and nonautonomous phenomena often exist in real systems due to modeling inaccuracies [4]. Particularly when we consider a long-term dynamical behaviors of the system and consider seasonality of the changing environment, the parameters of the system usually

will change with time [10–14]. In order to model those systems with neural networks, the uncertain (or switched or jumping) neural networks with time-varying delay appear in many papers [6, 15–24]. So in this paper we consider the stability of the following discrete-time recurrent neural networks with time-varying delay:

$$u(k + 1) = (A + \Delta A(k))u(k) + (W_1 + \Delta W_1(k))g(u(k))$$
$$+ (W_2 + \Delta W_2(k))f(u(k - d(k))) + (I + \Delta I(k)), \tag{1.1}$$

where $u(k) = \{u_1(k), u_2(k), \ldots, u_n(k)\} \in R^n$ is the state vector associated with n neurons, $A = \text{diag}\{a_1, a_2, \ldots, a_n\}$ is a diagonal matrix with positive entries, W_1 and W_2 are, respectively, the connection weight matrix and the delayed connection weight matrix, I is input vector, $g(u) = \{g_1(u), g_2(u), \ldots, g_n(u)\}$ and $f(u) = \{f_1(u), f_2(u), \ldots, f_n(u)\}$ are the neuron activation function vectors, and $d(k)$ is nonnegative differential time-varying functions which denote the time delays and satisfy

$$0 \le d_1 \le d(k) \le d_2. \tag{1.2}$$

In most literatures it is required that parameter uncertainty matrices, such as $\Delta A(k)$, $\Delta W_1(k)$, $\Delta W_2(k)$, and $\Delta I(k)$, should be in the form

$$[\Delta A(k) \ \Delta W_1(k) \ \Delta W_2(k) \ \Delta I(k)] = DF(k)[E_a \ E_{w1} \ E_{w2} \ E_i], \tag{1.3}$$

where E_a, E_{w1}, E_{w2}, and E_i are given constant matrices of appropriate dimensions and $F(k)$ is an uncertain matrix such that

$$F^T(k)F(k) \le I. \tag{1.4}$$

In practice, however, $\Delta A(k)$, $\Delta W_1(k)$, $\Delta W_2(k)$, and $\Delta I(k)$ are generally difficult to have the decomposition of matrices for D, E_a, E_{w1}, E_{w2}, and E_i. In addition, periodic oscillation in recurrent neural networks is an interesting dynamic behavior as many biological and cognitive activities require repetition [7, 10, 11, 25]. Simultaneously, periodic oscillations in recurrent neural networks have been found in many applications such as associative memories, pattern recognition, machine learning, and robot motion control [25]. So, if (1.1) is an uncertain periodic neural network in which the period is less than a constant b, then (1.1) can be expressed as switched neural network for the finite switching state, that is, if (1.1) is a neural network with period a $(0 < a \le b)$, then $\Delta A(k) = \Delta A(k + a)$, $\Delta W_1(k) = \Delta W_1(k + a)$, $\Delta W_2(k) = \Delta W_2(k + a)$, and $\Delta I(k) = \Delta I(k + a)$, which is corresponding to a switched neural network set: $\Omega = \bigcup_{a=1}^{b} U_a$, where $U_a = \{S_{ai} = (A_{ai}, W_{1ai}, W_{2ai}, I_{ai}) \mid 0 < i \le a\}$, $A_{ai} = A + \Delta A(k + i)$, $W_{1ai} = W_1 + \Delta W_1(k + i)$, $W_{2ai} = W_2 + \Delta W_2(k + i)$, and $I_{ai} = I + \Delta I(k + i)$. Suppose N is the number of elements of Ω; then (1.1) is actually modified by

$$u(k + 1) = A_{r(k)}u(k) + W_{1r(k)}g(u(k)) + W_{2r(k)}f(u(k - d(k))) + I_{r(k)}, \tag{1.5}$$

where $r(k)$ is a switching rule defined by $r(k) : N \rightarrow \Omega$ with $\Omega = \{S_1, S_2, \ldots, S_N\}$. Moreover, $r(k) = j$ means the sub-recurrent neural network (sub-RNN) S_j, which is corresponding to $(A_j, W_{1j}, W_{2j}, I_j)$, is active.

The dynamic behaviors of those models are foundations for applications. Under (1.5) most papers discuss the stability of uncertain neural networks with the common Lyapunov function approach [5, 6, 15–23, 25]. To the best of the authors' knowledge, up to now, there is scarcely any paper that studies the uncertain periodic neural networks using the SQLF. This situation motivates this research.

Motivated by the above discussions, the authors intend to study a problem of the delay-dependent stability criterion of uncertain discrete-time recurrent neural networks with time-varying delays that the uncertain recurrent neural networks have a finite number of sub-RNNs, and the sub-RNNs may change from one to another according to arbitrary switching and restricted switching. The contributions of this paper are the following. (1) Using a switching graph, uncertain periodic recurrent neural networks with time-varying delays are transformed into switched recurrent neural networks; (2) the derivative of the SQLF (3.7) of the literature [8] is improved in (3.11), please see Remark 4.3 and Table 3; (3) based on the switching graph, the delay-dependent stability criteria of switched recurrent neural networks are studied by FWM and SQLF. Then an effective LMI approach is developed to solve the problem.

This paper is organized as follows. In Section 2, we give some basic definitions. We analyze the stability of the system (2.2) with the SQLF and FWM in Section 3. Some examples are given in Section 4. Section 5 offers the conclusions of this paper.

2. Preliminaries

In many electronic circuits, nonmonotonic functions can be more appropriate to describe the neuron activation in designing and implementing an artificial neural network [7]; hence, we have the following assumption.

For any $j \in \{1, 2, \ldots, n\}$, there exist constants g_j^-, g_j^+, f_j^-, and f_j^+ such that

$$
\begin{aligned}
g_j^- &\le \frac{g_j(\theta_1) - g_j(\theta_2)}{\theta_1 - \theta_2} \le g_j^+, \quad \forall \theta_1, \theta_2 \in R, \\
f_j^- &\le \frac{f_j(\theta_1) - f_j(\theta_2)}{\theta_1 - \theta_2} \le f_j^+, \quad \forall \theta_1, \theta_2 \in R.
\end{aligned}
\tag{2.1}
$$

Under the assumption, the equilibrium points of UDNN (1.1) exist by the fixed point theorem [1]. In the following, let $u^\circ = \{u_1^\circ, u_2^\circ, \ldots, u_n^\circ\}$ be the equilibrium point of (1.1); then $x(\cdot) = u(\cdot) - u^\circ$. The systems (1.1) and (1.5) are, respectively, shifted to the following form:

$$
x(k+1) = (A + \Delta A(k))x(k) + (W_1 + \Delta W_1(k))g(x(k)) + (W_2 + \Delta W_2(k))f(x(k - \tau(k))),
$$

$$
x(k+1) = A_{r(k)}x(k) + W_{1r(k)}g(x(k)) + W_{2r(k)}f(x(k - d(k))), \quad r(k) \in \Omega.
\tag{2.2}
$$

For convenience, the switching graph is defined.

Definition 2.1. Let $\Gamma = (\Omega, \mathbf{W})$ be a switching graph, where Ω is the set of sub-RNNs S_i and $\mathbf{W}$ is the set of weighted arcs $\mathbf{w}_{ij} \in \{0,1\}$. $\mathbf{w}_{ij} = 1$ (or 0) represents the sub-RNN S_i switches (or does not switch) to the sub-RNN S_j.

Remark 2.2. When $\mathbf{w}_{ij} = 1$, if $\sum_{l=1}^{N} \mathbf{w}_{jl} = 0$, that is sub-RNN S_j cannot switch to any other sub-RNN, we suppose that the uncertain neural networks will always stay in the sub-RNN S_i that means $\mathbf{w}_{ii} = 1$ and $\mathbf{w}_{ij} = 0$.

Throughout this paper, the superscript T stands for the transpose of a matrix, $P > 0$ means that the matrix P is positive definite, and the symmetric terms in a symmetric matrix are denoted by $*$, for example,

$$
\begin{bmatrix} M & O \\ * & N \end{bmatrix} = \begin{bmatrix} M & O \\ O^T & N \end{bmatrix},
$$

$$
G_1 = \mathrm{diag}\{g_1^- g_1^+, g_2^- g_2^+, \ldots, g_n^- g_n^+\},
$$

$$
G_2 = \mathrm{diag}\left\{ -\frac{g_1^- + g_1^+}{2}, -\frac{g_2^- + g_2^+}{2}, \ldots, -\frac{g_n^- + g_n^+}{2} \right\}, \tag{2.3}
$$

$$
F_1 = \mathrm{diag}\{f_1^- f_1^+, f_2^- f_2^+, \ldots, f_n^- f_n^+\},
$$

$$
F_2 = \mathrm{diag}\left\{ -\frac{f_1^- + f_1^+}{2}, -\frac{f_2^- + f_2^+}{2}, \ldots, -\frac{f_n^- + f_n^+}{2} \right\}.
$$

3. Asymptotical Stability of Uncertain Periodic Switched Recurrent Neural Networks

Theorem 3.1. *Let d_1 and d_2 be positive integers such that $0 \leq d_1 \leq d_2$. Based on a switching graph Γ, the system (2.2) is asymptotical stable if, when $\mathbf{w}_{ij} = 1$ $(S_i, S_j \in \Omega)$, there exist the corresponding symmetric matrices $P_i = P_i^T > 0$, $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$, $R = R^T > 0$, $Z_1 = Z_1^T > 0$, $Z_2 = Z_2^T > 0$, $X^{ij} = (X^{ij})^T \geq 0$, $U^{ij} = (U^{ij})^T \geq 0$, $H^{ij} = \mathrm{diag}\{h_1^{ij}, h_2^{ij}, \ldots, h_n^{ij}\} \geq 0$, $O^{ij} = \mathrm{diag}\{o_1^{ij}, o_2^{ij}, \ldots, o_n^{ij}\} \geq 0$, and any appropriate dimensional matrices N^{ij}, M^{ij}, and T^{ij} such that the following LMIs hold:*

$$
\Phi^{ij} = \begin{bmatrix} \Phi_{11}^{ij} & \Phi_{12}^{ij} & \Phi_{13}^{ij} & \Phi_{14}^{ij} & \Phi_{15}^{ij} & \Phi_{16}^{ij} \\ * & \Phi_{22}^{ij} & \Phi_{23}^{ij} & \Phi_{24}^{ij} & 0 & 0 \\ * & * & \Phi_{33}^{ij} & \Phi_{34}^{ij} & 0 & 0 \\ * & * & * & \Phi_{44}^{ij} & 0 & -F_2 O^{ij} \\ * & * & * & * & \Phi_{55}^{ij} & \Phi_{56}^{ij} \\ * & * & * & * & * & \Phi_{66}^{ij} \end{bmatrix} < 0, \tag{3.1}
$$

$$
\Lambda_1^{ij} = \begin{bmatrix} X^{ij} & N^{ij} \\ * & Z_1 \end{bmatrix} > 0, \tag{3.2}
$$

$$\Lambda_2^{ij} = \begin{bmatrix} U^{ij} & T^{ij} \\ * & Z_2 \end{bmatrix} > 0, \tag{3.3}$$

$$\Lambda_3^{ij} = \begin{bmatrix} X^{ij} + U^{ij} & M^{ij} \\ * & Z_1 + Z_2 \end{bmatrix} > 0, \tag{3.4}$$

where

$$\Phi_{11}^{ij} = A_i^T P_j A_i - P_i + Q_1 + Q_2 + (d_2 - d_1 + 1)R + (A_i - E)^T (d_2 Z_{1j} + (d_2 - d_1)Z_{2j})(A_i - E)$$
$$+ \left(N_1^{ij}\right)^T + N_1^{ij} + d_2 X_{11}^{ij} + (d_2 - d_1)U_{11}^{ij} - G_1 H^{ij},$$

$$\Phi_{12}^{ij} = \left(N_2^{ij}\right)^T + T_1^{ij} + d_2 X_{12}^{ij} + (d_2 - d_1)U_{12}^{ij},$$

$$\Phi_{13}^{ij} = \left(N_3^{ij}\right)^T - M_1^{ij} + d_2 X_{13}^{ij} + (d_2 - d_1)U_{13}^{ij},$$

$$\Phi_{14}^{ij} = -N_1^{ij} + \left(N_4^{ij}\right)^T + M_1^{ij} - T_1^{ij} + d_2 X_{14}^{ij} + (d_2 - d_1)U_{14}^{ij},$$

$$\Phi_{15}^{ij} = A_i^T P_j W_{1i} + (A_i - E)^T (d_2 Z_1 + (d_2 - d_1)Z_2)W_{1i} - G_2 H^{ij},$$

$$\Phi_{16}^{ij} = A_i^T P_j W_{2i} + (A_i - E)^T (d_2 Z_1 + (d_2 - d_1)Z_2)W_{2i},$$

$$\Phi_{22}^{ij} = -Q_1 - R + \left(T_2^{ij}\right)^T + T_2^{ij} + d_2 X_{22}^{ij} + (d_2 - d_1)U_{22}^{ij},$$

$$\Phi_{23}^{ij} = -M_2^{ij} + \left(T_3^{ij}\right)^T + d_2 X_{23}^{ij} + (d_2 - d_1)U_{23}^{ij},$$

$$\Phi_{24}^{ij} = -N_2^{ij} + \left(T_4^{ij}\right)^T + M_2^{ij} - T_2^{ij} + d_2 X_{24}^{ij} + (d_2 - d_1)U_{24}^{ij},$$

$$\Phi_{33}^{ij} = -Q_2 - R - M_3^{ij} - \left(M_3^{ij}\right)^T + d_2 X_{33}^{ij} + (d_2 - d_1)U_{33}^{ij},$$

$$\Phi_{34}^{ij} = -N_3^{ij} - \left(M_4^{ij}\right)^T + M_3^{ij} - T_3^{ij} + d_2 X_{34}^{ij} + (d_2 - d_1)U_{34}^{ij},$$

$$\Phi_{44}^{ij} = -N_4^{ij} - \left(N_4^{ij}\right)^T + M_4^{ij} + \left(M_4^{ij}\right)^T - T_4^{ij} - \left(T_4^{ij}\right)^T + d_2 X_{44}^{ij} + (d_2 - d_1)U_{44}^{ij} - F_1 O^{ij},$$

$$\Phi_{55}^{ij} = W_{1i}^T P_j W_{1i} + W_{1i}^T (d_2 Z_1 + (d_2 - d_1)Z_2)W_{1i} - H^{ij},$$

$$\Phi_{56}^{ij} = W_{1i}^T P_j W_{2i} + W_{1i}^T (d_2 Z_1 + (d_2 - d_1)Z_2)W_{2i},$$

$$\Phi_{66}^{ij} = W_{2i}^T P_j W_{2i} + W_{2i}^T (d_2 Z_1 + (d_2 - d_1)Z_2)W_{2i} - O^{ij}.$$

$$\tag{3.5}$$

Proof. Suppose that $y(l) = x(l + 1) - x(l)$; then we have $x(k + 1) = x(k) + y(k)$ and $x(k) = x(k - d(k)) + \sum_{i=k-d(k)}^{k-1} y(l)$.

We consider the following SQLF:

$$V_{r(k)}(k) = V_{1r(k)}(k) + V_{2r(k)}(k) + V_{3r(k)}(k) + V_{4r(k)}(k),$$

$$V_{1r(k)}(k) = x^T(k)P_{r(k)}x(k),$$

$$V_{2r(k)}(k) = \sum_{l=k-d_1}^{k-1} x^T(l)Q_{1r(k)}x(l) + \sum_{l=k-d_2}^{k-1} x^T(l)Q_{2r(k)}x(l),$$

(3.6)

$$V_{3r(k)}(k) = \sum_{\theta=k-d_2}^{k-d_1}\sum_{l=\theta}^{k-1} x^T(l)R_{r(k)}x(l),$$

(3.7)

$$V_{4r(k)}(k) = \sum_{\theta=-d_2}^{-1}\sum_{l=k+\theta}^{k-1} y^T(l)Z_{1r(k)}y(l) + \sum_{\theta=-d_2}^{-d_1-1}\sum_{l=k+\theta}^{k-1} y^T(l)Z_{2r(k)}y(l).$$

(3.8)

It is clear that the following equations are true:

$$\sum_{l=k-d_2}^{k-1} y^T(l)Z_{1r(k)}y(l) = \sum_{l=k-d_2}^{k-d(k)-1} y^T(l)Z_{1r(k)}y(l) + \sum_{l=k-d(k)}^{k-1} y^T(l)Z_{1r(k)}y(l),$$

$$\sum_{l=k-d_2}^{k-d_1-1} y^T(l)Z_{2r(k)}y(l) = \sum_{l=k-d_2}^{k-d(k)-1} y^T(l)Z_{2r(k)}y(l) + \sum_{l=k-d(k)}^{k-d_1-1} y^T(l)Z_{2r(k)}y(l).$$

(3.9)

Firstly, we prove that under $\mathbf{w}_{ij} = 1$ the SQLF is less than 0. Suppose that $r(k) = i$ and $r(k+1) = j$, that means the sub-RNN S_i switches to the sub-RNN S_j; we obtain

$$\Delta V_{1i}(k) = V_{1j}(k+1) - V_{1i}(k) = \left(A_ix(k) + W_{1i}g(x(k)) + W_{2i}f(x(k-d(k)))\right)^T$$

$$\times P_j\left(A_ix(k) + W_{1i}g(x(k)) + W_{2i}f(x(k-d(k)))\right) - x^T(k)P_ix(k),$$

$$\Delta V_{2i}(k) = V_{2j}(k+1) - V_{2i}(k) = x^T(k)(Q_{1j}+Q_{2j})x(k) - x^T(k-d_1)Q_{1i}x(k-d_1)$$

$$- x^T(k-d_2)Q_{2i}x(k-d_2) + \sum_{l=k+1-d_1}^{k-1} x^T(l)(Q_{1j}-Q_{1i})x(l)$$

$$+ \sum_{l=k+1-d_2}^{k-1} x^T(l)(Q_{2j}-Q_{2i})x(l),$$

(3.10)

$$\Delta V_{3i}(k) = V_{3j}(k+1) - V_{3i}(k) \le (d_2-d_1+1)x^T(k)R_jx(k)$$

$$- x^T(k-d_1)R_ix(k-d_1) - x^T(k-d_2)R_ix(k-d_2) - x^T(k-d(k))R_ix(k-d(k))$$

$$+ \sum_{l=k+1-d_1}^{k-1} x^T(l)(R_j-R_i)x(l) + \sum_{\theta=k+1-d_2}^{k-d_1}\sum_{l=\theta}^{k-1} x^T(l)(R_j-R_i)x(l),$$

(3.11)

$$\Delta V_{4i}(k) = V_{4j}(k+1) - V_{4i}(k) = d_2 y^T(k) Z_{1j} y(k) + (d_2 - d_1) y^T(k) Z_{2j} y(k)$$

$$- \sum_{l=k-d(k)}^{k-1} y^T(l) Z_{1i} y(l) - \sum_{l=k-d(k)}^{k-d_1-1} y^T(l) Z_{2i} y(l)$$

$$- \sum_{l=k-d_2}^{k-d(k)-1} y^T(l)(Z_{1i} + Z_{2i}) y(l) + \sum_{\theta=-d_2}^{-1} \sum_{l=k+1+\theta}^{k-1} y^T(l)(Z_{1j} - Z_{1i}) y(l) \qquad (3.12)$$

$$+ \sum_{\theta=-d_2}^{-d_1-1} \sum_{l=k+1+\theta}^{k-1} y^T(l)(Z_{2j} - Z_{2i}) y(l).$$

In order to strictly guarantee $\Delta V_{2i}(k) < 0$, $\sum_{l=k+1-d_1}^{k-1} x^T(l)(Q_{1j} - Q_{1i}) x(l)$ should be less than 0. In the switching graph Γ if there exists sub-RNN S_i ($S_i \in \Omega$), which satisfied $\sum_{j=1}^{N} \mathbf{w}_{ij} = 0$, which means the S_i cannot switch to any other sub-RNN, the equation $Q_{1i} = Q_{1j}$ can be grounded, otherwise the switching sequence must be $\beta = \{S_{\alpha_1} \rightarrow S_{\alpha_2} \rightarrow, \dots, \rightarrow S_{\alpha_j} \rightarrow \cdots, \rightarrow S_{\alpha_L} \rightarrow S_{\alpha_{L+1}}\}$ ($S_{\alpha_j} \in \Omega$), and there exist l such that $S_{\alpha_{L+1}} = S_{\alpha_l}$ ($1 \le l < L$). Because the affection of the sub-RNNs $S_1, S_2 \dots, S_{l-1}$ and S_l on the whole system is before time α_l, after α_l the β changes to a periodic sequence $\beta' = \{S_{\alpha_l} \rightarrow \cdots, \rightarrow S_{\alpha_L} \rightarrow S_{\alpha_{L+1}}\}$. Suppose that $i = \alpha_l = \alpha_{L+1}$; then in switching sequence β' the following LMIs all hold:

$$Q_{1\alpha_l} - Q_{1\alpha_{l+1}} \ge 0 \quad (Q_{1i} - Q_{1(l+1)} \ge 0),$$

$$Q_{1\alpha_{l+1}} - Q_{1\alpha_{l+2}} \ge 0,$$

$$\vdots \qquad (3.13)$$

$$Q_{1\alpha_L} - Q_{1\alpha_{L+1}} \ge 0 \quad (Q_{1(L)} - Q_{1i} \ge 0);$$

then the solution of (3.13) is $Q_{1i} = Q_{1\alpha_l} = Q_{1\alpha_{l+1}} = \cdots = Q_{1\alpha_L}$. Thus, we suppose that

$$Q_1 = Q_{1i} = Q_{1j}. \qquad (3.14)$$

Similar to $\sum_{l=k+1-d_1}^{k-1} x^T(l)(Q_{1j} - Q_{1i}) x(l)$, together with $\sum_{l=k+1-d_2}^{k-1} x^T(l)(Q_{2j} - Q_{2i}) x(l)$, $\sum_{l=k+1-d_1}^{k-1} x^T(l)(R_j - R_i) x(l)$, $\sum_{\theta=k+1-d_2}^{k-d_1} \sum_{l=\theta}^{k-1} x^T(l)(R_j - R_i) x(l)$, $\sum_{\theta=-d_2}^{-1} \sum_{l=k+1+\theta}^{k-1} y^T(l)(Z_{1j} - Z_{1i}) y(l)$ and $\sum_{\theta=-d_2}^{-d_1-1} \sum_{l=k+1+\theta}^{k-1} y^T(l)(Z_{2j} - Z_{2i}) y(l)$, we suppose that

$$Q_2 = Q_{2i} = Q_{2j},$$
$$R = R_i = R_j, \qquad (3.15)$$

$$Z_1 = Z_{1i} = Z_{1j}, \qquad (3.16)$$

$$Z_2 = Z_{2i} = Z_{2j}. \qquad (3.17)$$

On the other hand, for any appropriately dimensioned matrices N^{ij}, M^{ij}, and T^{ij} the following equations are true:

$$2\xi^T(k)N^{ij}\left[x(k) - x(k - d(k)) - \sum_{l=k-d(k)}^{k-1} y(l)\right] = 0, \qquad (3.18)$$

$$2\xi^T(k)M^{ij}\left[x(k - d(k)) - x(k - d_2) - \sum_{l=k-d_2}^{k-d(k)-1} y(l)\right] = 0,$$

$$2\xi^T(k)T^{ij}\left[x(k - d_1) - x(k - d(k)) - \sum_{l=k-d(k)}^{k-d_1-1} y(l)\right] = 0, \qquad (3.19)$$

where $\xi(k) = [x^T(k), x^T(k - d_1), x^T(k - d_2), x^T(k - d(k))]^T$.

In addition, for any semipositive definite matrix $X^{ij} = (X^{ij})^T$ and $U^{ij} = (U^{ij})^T$, the following equations hold:

$$d_2\xi^T(k)X^{ij}\xi(k) - \sum_{l=k-d_2}^{k-d(k)-1} \xi^T(k)X^{ij}\xi(k) - \sum_{l=k-d(k)}^{k-1} \xi^T(k)X^{ij}\xi(k) = 0,$$

$$(d_2 - d_1)\xi^T(k)U^{ij}\xi(k) - \sum_{l=k-d_2}^{k-d(k)-1} \xi^T(k)U^{ij}\xi(k) - \sum_{l=k-d(k)}^{k-d_1-1} \xi^T(k)U^{ij}\xi(k) = 0. \qquad (3.20)$$

From the assumption, we have

$$\left(g_l(x(k)) - g_l^- x_l(k)\right) \times \left(g_l(x(k)) - g_l^+ x_l(k)\right) \leq 0, \quad l = 1, 2, \dots, n,$$

$$\left(f_l(x(k - d_k)) - f_l^- x_l(k - d_k)\right) \times \left(f_l(x(k - d_k)) - f_l^+ x_l(k - d_k)\right) \leq 0, \quad l = 1, 2, \dots, n. \qquad (3.21)$$

Similar to the conclusion in [8], for $H^{ij} = \text{diag}\{h_1, h_2, \dots, h_n\} \geq 0, O^{ij} = \text{diag}\{o_1, o_2, \dots, o_n\} \geq 0$ the following inequalities are also true:

$$-\begin{bmatrix} x(k) \\ g(x(k)) \end{bmatrix}^T \begin{bmatrix} G_1 H^{ij} & G_2 H^{ij} \\ \star & H^{ij} \end{bmatrix} \begin{bmatrix} x(k) \\ g(x(k)) \end{bmatrix} \geq 0, \qquad (3.22)$$

$$-\begin{bmatrix} x(k - d(k)) \\ f(x(k - d(k))) \end{bmatrix}^T \begin{bmatrix} F_1 O^{ij} & F_2 O^{ij} \\ \star & O^{ij} \end{bmatrix} \begin{bmatrix} x(k - d(k)) \\ f(x(k - d(k))) \end{bmatrix} \geq 0. \qquad (3.23)$$

Then we add the terms on the right side of (3.16)–(3.23) to yield

$$\Delta V_i(k) = \Delta V_{1i}(k) + \Delta V_{2i}(k) + \Delta V_{3i}(k) + \Delta V_{4i}(k)$$

$$\leq \eta_1^T(k)\Phi^{ij}\eta_1(k) - \sum_{l=k-d(k)}^{k-1} \eta_2^T(k,l)\Lambda_1^{ij}\eta_2(k,l)$$

$$- \sum_{l=k-d(k)}^{k-d_1-1} \eta_2^T(k,l)\Lambda_2^{ij}\eta_2(k,l) - \sum_{l=k-d_2}^{k-d(k)-1} \eta_2^T(k,l)\Lambda_3^{ij}\eta_2(k,l), \qquad (3.24)$$

where $\eta_1^T(k) = [\xi^T(k), g^T(x(k)), f^T(x(k-d(k)))]^T$, $\eta_2^T(k,l) = [\xi^T(k), y^T(l)]^T$.

And $\xi(k)$ is defined in (3.18). Φ^{ij}, Λ_1^{ij}, Λ_2^{ij}, and Λ_3^{ij} are defined in (3.1)–(3.4). Therefore, when the corresponding LMIs satisfy $\Phi^{ij} < 0$, $\Lambda_1^{ij} \geq 0$, $\Lambda_2^{ij} \geq 0$, and $\Lambda_3^{ij} \geq 0$, $S_i, S_j \in \Omega$, $\Delta V_i(k) \leq 0$.

Secondly, based on the switching graph Γ, when $\mathbf{w}_{ij} = 1$ $(S_i, S_j \in \Omega)$, all corresponding $\Delta V_i(k)$ are less than 0 that means the system (2.2) is asymptotical stable. This completes the proof of Theorem 3.1. □

Remark 3.2. Using the method in [8], it is easily to know that the system is the globally exponentially stable.

Remark 3.3. In $V_{3i}(k)$ of the literature [8], $-\sum_{l=k-d_2}^{k-d(k)} x^T(l)R_i x(l) \leq -x^T(k-d(k))R_i x(k-d(k))$ may lead to considerable conservativeness. Then, it is improved as $-\sum_{l=k-d_2}^{k-d(k)} x^T(l)R_i x(l) \leq -x^T(k-d_2)R_i x(k-d_2) - x^T(k-d(k))R_i x(k-d(k)) - x^T(k-d_1)R_i x(k-d_1)$. Please see Table 3.

Combined with Theorem 3.1, we consider the common Lyapunov function approach; then we have the following.

Corollary 3.4. *Let d_1 and d_2 be positive integers such that $0 \leq d_1 \leq d_2$. The system (2.2) is asymptotical stable if there exist symmetric matrices $P = P^T > 0$, $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$, $R = R^T > 0$, $Z_1 = Z_1^T > 0$, $Z_2 = Z_2^T > 0$, $X = X^T \geq 0$, $U = U^T \geq 0$, $H = \mathrm{diag}\{h_1, h_2, \ldots, h_n\} \geq 0$, $O = \mathrm{diag}\{o_1, o_2, \ldots, o_n\} \geq 0$, and any appropriate dimensional matrices N, M, and T such that the following LMIs hold:*

$$\Phi_i = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & \Phi_{15} & \Phi_{16} \\ * & \Phi_{22} & \Phi_{23} & \Phi_{24} & 0 & 0 \\ * & * & \Phi_{33} & \Phi_{34} & 0 & 0 \\ * & * & * & \Phi_{44} & 0 & -F_2 O \\ * & * & * & * & \Phi_{55} & \Phi_{56} \\ * & * & * & * & * & \Phi_{66} \end{bmatrix} < 0, \quad S_i \in \Omega,$$

$$\Lambda_1 = \begin{bmatrix} X & N \\ * & Z_1 \end{bmatrix} > 0, \qquad (3.25)$$

$$\Lambda_2 = \begin{bmatrix} U & T \\ * & Z_2 \end{bmatrix} > 0,$$

$$\Lambda_3 = \begin{bmatrix} X+U & M \\ * & Z_1+Z_2 \end{bmatrix} > 0,$$

where

$$
\begin{aligned}
\Phi_{11} &= A_i^T P A_i - P + Q_1 + Q_2 + (d_2 - d_1 + 1)R + (A_i - E)^T (d_2 Z_1 + (d_2 - d_1)Z_2)(A_i - E) \\
&\quad + N_1^T + N_1 + d_2 X_{11} + (d_2 - d_1)U_{11} - G_1 H, \\
\Phi_{12} &= N_2^T + T_1 + d_2 X_{12} + (d_2 - d_1)U_{12}, \\
\Phi_{13} &= N_3^T - M_1 + d_2 X_{13} + (d_2 - d_1)U_{13}, \\
\Phi_{14} &= -N_1 + N_4^T + M_1 - T_1 + d_2 X_{14} + (d_2 - d_1)U_{14}, \\
\Phi_{15} &= A_i^T P W_{1i} + (A_i - E)^T (d_2 Z_1 + (d_2 - d_1)Z_2)W_{1i} - G_2 H, \\
\Phi_{16} &= A_i^T P W_{2i} + (A_i - E)^T (d_2 Z_1 + (d_2 - d_1)Z_2)W_{2i}, \\
\Phi_{22} &= -Q_1 - R + T_2^T + T_2 + d_2 X_{22} + (d_2 - d_1)U_{22}, \\
\Phi_{23} &= -M_2 + T_3^T + d_2 X_{23} + (d_2 - d_1)U_{23}, \\
\Phi_{24} &= -N_2 + T_4^T + M_2 - T_2 + d_2 X_{24} + (d_2 - d_1)U_{24}, \\
\Phi_{33} &= -Q_2 - R - M_3 - M_3^T + d_2 X_{33} + (d_2 - d_1)U_{33}, \\[4pt]
\Phi_{34} &= -N_3 - M_4^T + M_3 - T_3 + d_2 X_{34} + (d_2 - d_1)U_{34}, \\
\Phi_{44} &= -N_4 - N_4^T + M_4 + M_4^T - T_4 - T_4^T - F_1 O + d_2 X_{44} + (d_2 - d_1)U_{44} - R, \\
\Phi_{55} &= W_{1i}^T P W_{1i} + W_{1i}^T (d_2 Z_1 + (d_2 - d_1)Z_2)W_{1i} - H, \\
\Phi_{56} &= W_{1i}^T P W_{2i} + W_{1i}^T (d_2 Z_1 + (d_2 - d_1)Z_2)W_{2i}, \\
\Phi_{66} &= W_{2i}^T P W_{2i} + W_{2i}^T (d_2 Z_1 + (d_2 - d_1)Z_2)W_{2i} - O.
\end{aligned}
$$

$$(3.26)$$

4. Examples

Example 4.1. Consider the discrete-time recurrent neural network (2.2) with

$$
A_1 = \begin{bmatrix} 0.7 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \quad
W_{11} = \begin{bmatrix} 0.1 & -0.2 & 0.1 \\ 0.3 & -0.1 & 0 \\ 0 & -0.1 & -0.4 \end{bmatrix}, \quad
W_{21} = \begin{bmatrix} -0.3 & 0.1 & -0.2 \\ 0.2 & -0.1 & 0.1 \\ 0 & -0.02 & 0.07 \end{bmatrix},
$$

$$
A_2 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}, \quad
W_{12} = \begin{bmatrix} 0.4 & -0.2 & 0 \\ 0.2 & -0.2 & 0 \\ -0.3 & -0.1 & -0.5 \end{bmatrix}, \quad
W_{22} = \begin{bmatrix} -0.3 & 0.1 & -0.2 \\ 0 & -0.03 & 0 \\ 0.5 & -0.2 & 0.5 \end{bmatrix},
$$

$$
A_3 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}, \quad
W_{13} = \begin{bmatrix} 0.2 & 0 & 0.3 \\ 0.04 & -0.3 & 0 \\ -0.1 & -0.6 & -0.1 \end{bmatrix}, \quad
W_{23} = \begin{bmatrix} -0.08 & 0.1 & -0.2 \\ 0.2 & -0.4 & -0.05 \\ 0 & -0.2 & -0.1 \end{bmatrix},
$$

$$
\Gamma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},
$$

$$g_1(x) = \tanh(0.4x), \qquad g_2(x) = \tanh(0.2x), \qquad g_3(x) = \tanh(-0.8x),$$

$$f_1(x) = \tanh(-0.6x), \qquad f_2(x) = \tanh(0.4x), \qquad f_3(x) = \tanh(0.2x).$$

(4.1)

Then we have

$$G_1 = F_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & -0.4 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & -0.2 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}.$$

(4.2)

Employing the LMIs in Theorem 3.1 yields upper bounds on d_2 that guarantee the stability of system (1.1) for various lower bounds d_1, which are listed in Table 1. When $d_1 = 1$ and $d_2 = 3$, it can be seen from Figure 1 that all the state solutions corresponding to the 10 random initial points are convergent asymptotically to the unique equilibrium $x^* = \{0, 0, 0\}$, and according to Theorem 3.1, LMIs (3.1)–(3.4) are solvable in Matlab 7.0.1.

Example 4.2. Consider the discrete-time recurrent neural network (2.2) with

$$A = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \qquad W_1 = \begin{bmatrix} 0.5 & -0.2 & 0.2 \\ 0.1 & -0.1 & 0 \\ -0.4 & -0.1 & -0.3 \end{bmatrix}, \qquad W_2 = \begin{bmatrix} -0.2 & 0.1 & 0.2 \\ 0 & -0.3 & 0.1 \\ 0 & -0.3 & 0.03 \end{bmatrix},$$

$$g_1(x) = f_1(x) = \tanh(0.6x), \qquad g_2(x) = f_2(x) = \tanh(0.4x), \qquad g_3(x) = f_3(x) = \tanh(0.2x).$$

(4.3)

Then we have

$$G_1 = F_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad G_2 = F_2 = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}.$$

(4.4)

Employing the LMIs in [8] and those in Corollary 3.4 yields upper bounds on d_2 that guarantee the stability of system for various lower bounds d_1, which are listed in Table 2. It is clear that the obtained upper bounds of this paper are better than those of [8]. It can be seen from Figure 2 that, when $d_1 = 1$ and $d_2 = 3$, all the state solutions corresponding to the 10 random initial points are convergent asymptotically to the unique equilibrium $x^* = \{0, 0, 0\}$.

Remark 4.3. Employing the LMIs in [8] and those in Corollary 3.4 yields upper bounds on d_2 that guarantee the stability of system (1.1) of the Example 1 of [8] for various lower bounds d_1, which are listed in Table 3.
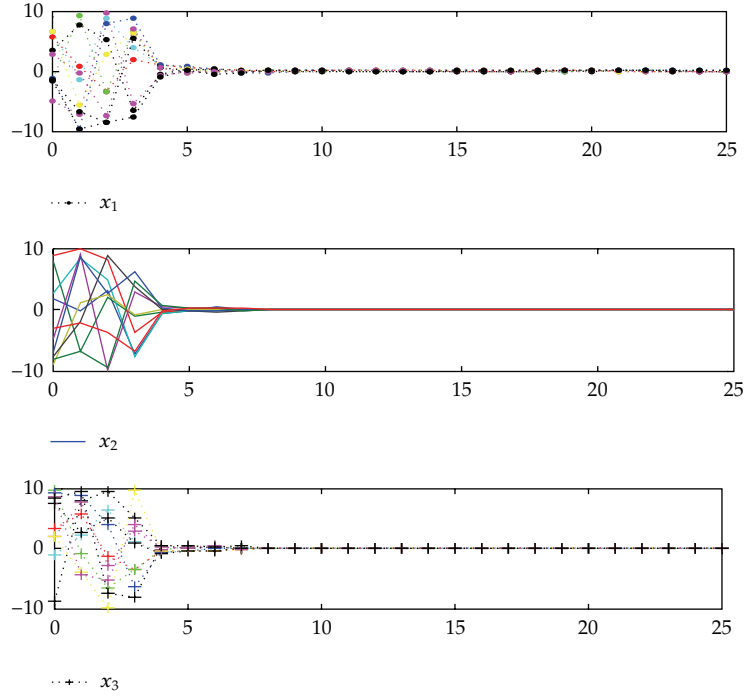
Figure 1: Global convergence of states x_1, x_2, and x_3 in Example 4.1, when $d_1 = 1$ and $d_2 = 3$.
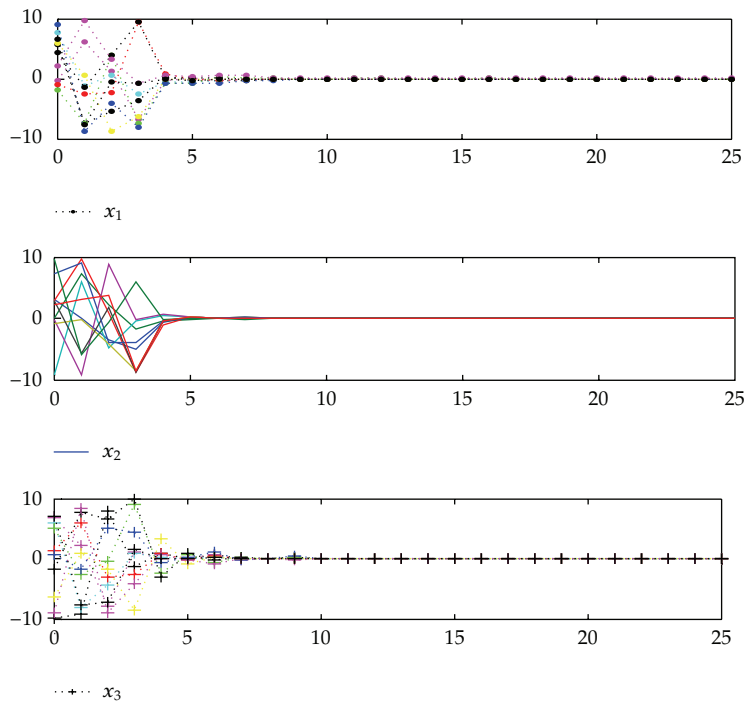

Figure 2: Global convergence of states x_1, x_2, and x_3 in Example 4.2, when $d_1 = 1$ and $d_2 = 3$.

Table 1: Allowable upper bound of d_2 with given d_1.

d_1	0	2	4	6	8	20
Theorem 3.1	9	11	13	15	17	29

Table 2: Allowable upper bound of d_2 with given d_1.

d_1	0	2	4	6	8	10	20
Reference [8]	20	22	24	26	28	30	40
Corollary 3.4	38	40	42	44	46	48	58

Table 3: Allowable upper bound of d_2 with given d_1.

d_1	0	2	4	6	10
Reference [8]	11	13	15	17	21
Corollary 3.4	12	14	16	18	22

5. Conclusions

This paper was dedicated to the delay-dependent stability of uncertain periodic switched recurrent neural networks with time-varying delay. A less conservative LMI-based globally stability criterion is obtained with the switched quadratic Lyapunov functional approach and free-weighting matrix approach for periodic uncertain discrete-time recurrent neural networks with a time-varying delay. One example illustrates the exactness of the proposed criterion. Another example demonstrates that the proposed method is an improvement over the existing one.

Acknowledgments

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