Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2010, Article ID 891564, 6 pages
doi:10.1155/2010/891564

Research Article

On a Higher-Order Difference Equation

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Received 20 May 2010; Accepted 23 June 2010

Academic Editor: Leonid Berezansky

We describe in an elegant and short way the behaviour of positive solutions of the higher-order difference equation $x_n = cx_{n-p}x_{n-p-q}/x_{n-q}$, $n \in \mathbb{N}_0$, where $p, q \in \mathbb{N}$ and $c > 0$, extending some recent results in the literature.

1. Introduction

Studying difference equations has attracted a considerable interest recently, see, for example, [1–39] and the references listed therein. The study of positive solutions of the following higher-order difference equations:

$$x_n = \max\left\{A, B\frac{x_{n-p_1}^{r_1}x_{n-p_2}^{r_2}\cdots x_{n-p_k}^{r_k}}{x_{n-q_1}^{s_1}x_{n-q_2}^{s_2}\cdots x_{n-q_l}^{s_l}}\right\}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

and

$$x_n = A + B\frac{x_{n-p_1}^{r_1}x_{n-p_2}^{r_2}\cdots x_{n-p_k}^{r_k}}{x_{n-q_1}^{s_1}x_{n-q_2}^{s_2}\cdots x_{n-q_l}^{s_l}}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $A, B > 0, p_i,\ q_i$ are natural numbers such that $p_1 < p_2 < \cdots < p_k,\ q_1 < q_2 < \cdots < q_l$, $r_i, s_i \in \mathbb{R}_+$, and $k \in \mathbb{N}$ was proposed by Stević in several talks, see, for example, [21, 26]. For some results concerning equations related to (1.1) see, for example, [6, 7, 10, 29, 31, 32, 34, 38], while some results on equations related to (1.2) can be found, for example, in [3, 8, 9, 11–14, 18–20, 22, 25, 29, 32, 33, 35] (see also related references cited therein).

Case $A = 0$ is of some less interest, since in this case positive solutions of (1.1) and (1.2), by using the change $y_n = \ln x_n$, become solutions of a linear difference equation with constant coefficients. However, some particular results for the case recently appeared in the literature, see [16, 17, 39].

Nevertheless, motivated by the above-mentioned papers, we will describe the behaviour of positive solutions of the higher-order difference equation

$$x_n = \frac{cx_{n-p}x_{n-p-q}}{x_{n-q}}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

where $p, q \in \mathbb{N}$ and $c > 0$, in, let us say, an elegant and short way.

Let us introduce the following.

Definition 1.1. A solution $(x_n)_{n=-(p+q)}^{\infty}$ of (1.3) is said to be *eventually periodic* with period τ if there is $n_0 \in \{-(p+q), \ldots, -1, 0, 1, \ldots\}$ such that $x_{n+\tau} = x_n$ for all $n \geq n_0$. If $n_0 = -(p+q)$, then we say that the sequence $(x_n)_{n=-(p+q)}^{\infty}$ is *periodic* with period τ.

For some results on equations all solutions of which are eventually periodic see, for example, [2, 4, 8, 15, 28, 37] and the references therein.

Definition 1.2. One says that a solution $(x_n)_{n=n_0}^{\infty}$ of a difference equation *converges geometrically* to x^* if there exist $L \in \mathbb{R}_+$ and $\theta \in [0, 1)$ such that

$$|x_n - x^*| \leq L\theta^n, \quad \forall n \geq n_0. \tag{1.4}$$

Now we return to (1.3).

First, note that if $p = q$, then (1.3) becomes

$$x_n = cx_{n-2p}, \quad n \in \mathbb{N}_0, \tag{1.5}$$

from which easily follow the following results:

(a) if $c = 1$, then all positive solutions of (1.5) are periodic with period $2p$;

(b) if $c \in (0, 1)$, then each positive solution of (1.5) converges to zero. Moreover, its subsequences $(x_{2pm-i})_{m\in\mathbb{N}_0}, i = 1, 2, \ldots, 2p$, converges decreasingly to zero as $m \to \infty$;

(c) if $c \in (1, \infty)$, then each positive solution of (1.5) tends to infinity as $n \to \infty$. Moreover, its subsequences $(x_{2pm-i})_{m\in\mathbb{N}_0}, i = 1, 2, \ldots, 2p$, tend increasingly to infinity as $m \to \infty$.

We may assume that p and q are relatively prime integers, that is, $\gcd(p, q) = 1$ (the greatest common divisor of numbers p and q). Namely, if $\gcd(p, q) = r > 1$, then by using the changes $z_m^{(i)} = x_{mr+i}, i = 0, 1, \ldots, r-1$, (1.3) is reduced to r copies of the following equation:

$$z_n = \frac{cz_{n-p_1}z_{n-p_1-q_1}}{z_{n-q_1}}, \quad n \in \mathbb{N}_0, \tag{1.6}$$

where $p_1 = p/r, q_1 = q/r, c > 0$, and $\gcd(p_1, q_1) = 1$.

Further, note that from (1.3), we have that

$$x_n x_{n-q} = c x_{n-p} x_{n-p-q}, \quad n \in \mathbb{N}_0, \tag{1.7}$$

which implies that the sequence $u_n = x_n x_{n-q}$, $n \geq -p$, satisfies the following simple difference equation:

$$u_n = c u_{n-p}, \quad n \in \mathbb{N}_0. \tag{1.8}$$

2. Main Results

Here we formulate and prove our main results.

Theorem 2.1. *Assume that $c = 1$, $\gcd(p, q) = 1$, and p is odd, then all positive solutions of* (1.3) *are eventually periodic with period $\tau = 2pq$.*

Proof. By using repeatedly relation (1.7) p-times, we obtain

$$x_n = \frac{u_n}{x_{n-q}} = \frac{u_n}{u_{n-q}} x_{n-2q} = \cdots = \frac{u_n}{u_{n-q}} \frac{u_{n-2q}}{u_{n-3q}} \cdots \frac{u_{n-2q(p-1)}}{u_{n-q(2p-1)}} x_{n-2pq}. \tag{2.1}$$

Now, note that from (1.8), it follows that in this case u_n is periodic with period p. On the other hand, since $\gcd(p, q) = 1$ for each $i, j \in \{0, 1, \ldots, p-1\}$, $i \neq j$, we have that

$$\begin{aligned} (n - (2i+1)q) - (n - (2j+1)q) &= (j - i) 2q \not\equiv 0 \pmod{p}, \\ (n - (2i+2)q) - (n - (2j+2)q) &= (j - i) 2q \not\equiv 0 \pmod{p}. \end{aligned} \tag{2.2}$$

Hence, the indices $(n - (2i+1)q)$, $i \in \{0, 1, \ldots, p-1\}$, and $(n - (2i+2)q)$, $i \in \{0, 1, \ldots, p-1\}$, belong to p different subsequences. From this and the periodicity of u_n, it follows that

$$u_n u_{n-2q} \cdots u_{n-2q(p-1)} = u_{n-q} u_{n-3q} \cdots u_{n-q(2p-1)}, \tag{2.3}$$

from which the theorem follows. □

By taking the logarithm of (1.3) and using the change $v_n = \ln x_n$, we get

$$v_n + v_{n-q} - v_{n-p} - v_{n-p-q} = \ln c, \quad n \in \mathbb{N}_0. \tag{2.4}$$

The characteristic polynomial of the homogeneous part of (2.4) is

$$\lambda^{p+q} + \lambda^p - \lambda^q - 1 = (\lambda^q + 1)(\lambda^p - 1) = 0, \tag{2.5}$$

from which it follows that all its roots are expressed by

$$\exp\left(\frac{(2k+1)\pi i}{q}\right), \quad k = 0,1,\ldots,q-1, \qquad \exp\left(\frac{2l\pi i}{p}\right), \quad l = 0,1,\ldots,p-1. \tag{2.6}$$

These roots are simple if and only if

$$\frac{2k+1}{q} \neq \frac{2l}{p}, \quad \text{for each } k,l \in \mathbb{N}_0. \tag{2.7}$$

Clearly, if p is odd, inequality (2.7) holds. If p is even, that is, $p = 2^s r$, for some $s,r \in \mathbb{N}$, then, since $\gcd(p,q) = 1$, q must be odd. Then, for $k = (q-1)/2$ and $l = r$, we will get that inequality (2.7) does not hold.

From the above consideration and Theorem 2.1, we get the next corollary.

Corollary 2.2. *Assume that $c = 1$ and $\gcd(p,q) = 1$. Then all positive solutions of (1.3) are eventually periodic if and only if p is odd. Moreover, if p is odd, then the period is $\tau = 2pq$.*

Since the root $\lambda = 1$ of characteristic polynomial (2.5) is a simple one, a particular solution of nonhomogeneous (2.4) has the form

$$v_n^P = An, \tag{2.8}$$

from which, by a direct calculation, we easily get that $A = \ln c/2p$.

Hence, if p is odd, the general solution of (1.3) is

$$\begin{aligned} x_n = e^{v_n} = c^{n/2p} \exp\Bigg(\sum_{k=0}^{q-1} \left(c_{k,1} \cos \frac{(2k+1)\pi n}{q} + c_{k,2} \sin \frac{(2k+1)\pi n}{q} \right) \\ + \sum_{l=1}^{p-1} \left(d_{k,1} \cos \frac{2l\pi n}{p} + d_{k,2} \sin \frac{2l\pi n}{p} \right) \Bigg). \end{aligned} \tag{2.9}$$

Note that from (2.9), it follows that

$$x_n = c^{n/2p} \widehat{x}_n, \tag{2.10}$$

and that $\widehat{x}_n$ is a positive solution of (1.3) with $c = 1$.

From (2.9), (2.10), and Theorem 2.1 the following results directly follow.

Theorem 2.3. *Assume that $c \in (0,1)$, $\gcd(p,q) = 1$, and p is odd, then every positive solution of (1.3) converges geometrically to zero. Moreover, for each $s \in \{0,1,\ldots,2pq-1\}$, the subsequence $(x_{2pqm+s})_{m\in\mathbb{N}_0}$ converges monotonically to zero as $m \to \infty$.*

Theorem 2.4. *Assume that $c > 1$, $\gcd(p,q) = 1$, and p is odd, then every positive solution of (1.3) tends to infinity. Moreover, for each $s \in \{0,1,\ldots,2pq-1\}$, the subsequence $(x_{2pqm+s})_{m\in\mathbb{N}_0}$ converges increasingly to infinity as $m \to \infty$.*

Finally, there are two concluding interesting remarks.

Remark 2.5. Note that, since the functions $\cos((2k+1)\pi n/q)$ and $\sin((2k+1)\pi n/q)$ are periodic with period $2q$ and the functions $\cos(2l\pi n/p)$ and $\sin(2l\pi n/p)$ are periodic with period p, from the representation (2.9) we also obtain Theorem 2.1.

Remark 2.6. The results in papers [16, 17, 39], which are obtained in much complicated ways, are particular cases of our results. Namely, in [16] Özban studied a system which is transformed into (1.3) with $p = 1$, $q = m + k + 1$ and $c = 1$, in [17] he studied a system which is transformed into (1.3) with $p = 3$, and $c = b/a$, while in [39] the authors considered a system which is transformed into (1.3) with $c = b/a$, but they only considered the case when $p \leq q$.

Acknowledgments

The authors are indebted to the anonymous referees for their advice resulting in numerous improvements of the text. The research of the first author was partly supported by the Serbian Ministry of Science, through The Mathematical Institute of SASA, Belgrade, Project no. 144013.

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