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Research Article

Global Stability of a Rational Difference Equation

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We consider the higher-order nonlinear difference equation $x_{n+1} = (p + qx_{n-k})/(1 + x_n + rx_{n-k})$, $n = 0, 1, \ldots$ with the parameters, and the initial conditions $x_{-k}, \ldots, x_0$ are nonnegative real numbers. We investigate the periodic character, invariant intervals, and the global asymptotic stability of all positive solutions of the above-mentioned equation. In particular, our results solve the open problem introduced by Kulenović and Ladas in their monograph (see Kulenović and Ladas, 2002).

1. Introduction and Preliminaries

Our aim in this paper is to investigate the global behavior of solutions of the following nonlinear difference equation:

$$x_{n+1} = \frac{p + qx_{n-k}}{1 + x_n + rx_{n-k}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where the parameters p, q, r and the initial conditions $x_{-k}, \ldots, x_0$ are nonnegative real numbers, $k \in \{1, 2, \ldots\}$.

In 2002, Kulenović and Ladas [1] proposed the following open problem.

Open Problem 1. Assume that $p, q, r \in [0, \infty)$ and $k \in \{2, 3, \ldots\}$. Investigate the global behavior of all positive solutions of (1.1).

Consider the difference equation

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \ldots \tag{1.2}$$

with $\alpha, \gamma, A, B, C \in (0, \infty)$, and the initial conditions x_{-1}, x_0 are nonnegative real numbers. Note that, the authors [1, 2] investigated this equation and studied (1.2).

In this paper, we will consider the above open problem. Actually, we will investigate the global asymptotic stability and the invariant interval for all positive solutions of (1.1).

For the global behavior of solutions of some related equations, see [3–9]. Other related results can be found in [10–19]. For the sake of convenience, we recall some definitions and theorems which will be useful in the sequel.

Definition 1.1. Let I be some interval of real numbers and let

$$f : I^{m+1} \longrightarrow I \tag{1.3}$$

be a continuously differential function. Then for every set of initial conditions $y_{-k}, \ldots, y_{-1}$, $y_0 \in I$, the difference equation

$$y_{n+1} = f(y_n, y_{n-1}, \ldots, y_{n-k}), \quad n = 0, 1, \ldots, \tag{1.4}$$

has a unique solution $\{y_n\}_{n=-k}^{\infty}$.

A point $\overline{y}$ is called an equilibrium point of (1.4) if

$$\overline{y} = f(\overline{y}, \overline{y}, \ldots, \overline{y}), \tag{1.5}$$

that is,

$$y_n = \overline{y} \quad \text{for } n \geq 0 \tag{1.6}$$

is a solution of (1.4), or equivalently $\overline{y}$ is a fixed point of f.

Definition 1.2. Let $\overline{y}$ be an equilibrium point of (1.4).

(i) The equilibrium $\overline{y}$ is called locally stable (or stable) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y_{-k}, \ldots, y_{-1}, y_0 \in I$ with $\sum_{i=-k}^{i=0} |y_i - \overline{y}| < \delta$, we have $|y_n - \overline{y}| < \varepsilon$ for all $n \geq k$.

(ii) The equilibrium $\overline{y}$ of (1.4) is called locally asymptotically stable (asymptotic stable) if it is locally stable, and if there exists $\gamma > 0$ such that for all $y_{-k}, \ldots, y_{-1}, y_0 \in I$ with $\sum_{i=-k}^{i=0} |y_i - \overline{y}| < \gamma$, we have $\lim_{n\to\infty} y_n = \overline{y}$.

(iii) The equilibrium $\overline{y}$ of (1.4) is called a global attractor if for every $y_{-k}, \ldots, y_{-1}, y_0 \in I$, we have $\lim_{n\to\infty} y_n = \overline{y}$.

(iv) The equilibrium $\overline{y}$ of (1.4) is globally asymptotically stable if it is locally stable and is a global attractor.

(v) The equilibrium $\overline{y}$ of (1.4) is called unstable if it is not stable.

(vi) The equilibrium $\overline{y}$ of (1.4) is called a source, or a repeller, if there exists $r > 0$ such that for all $y_{-k}, \ldots, y_{-1}, y_0 \in I$ with $\sum_{i=-k}^{i=0} |y_i - \overline{y}| < \gamma$, there exists $N \geq 1$ such that $|y_N - \overline{y}| \geq r$.

An interval $J \subseteq I$ is called an invariant interval for (1.4) if

$$y_{-k}, \ldots, y_0 \in J \Longrightarrow y_n \in J \quad \forall n > 0. \tag{1.7}$$

That is, every solution of (1.4) with initial conditions in J remains in J.

The linearized equation associated with (1.4) about the equilibrium $\overline{y}$ is

$$y_{n+1} = \sum_{i=o}^{k} \frac{\partial f}{\partial u_i}(\overline{y}, \ldots, \overline{y}) y_{n-i}, \quad n = 0, 1, \ldots. \tag{1.8}$$

Its characteristic equation is

$$\lambda^{k+1} = \sum_{i=o}^{k} \frac{\partial f}{\partial u_i}(\overline{y}, \ldots, \overline{y}) \lambda^{k-i}. \tag{1.9}$$

Theorem 1.3 (see [20]). *Assume that f is a C^1 function and let $\overline{y}$ be an equilibrium of* (1.4). *Then the following statements are true:*

(i) *If all the roots of* (1.9) *lie in the open unit disk $|\lambda| < 1$, then the equilibrium $\overline{y}$ of* (1.4) *is asymptotically stable.*

(ii) *If at least one root of* (1.9) *has absolute value greater than one, then the equilibrium $\overline{y}$ of* (1.4) *is unstable.*

Theorem 1.4 (see [20]). *Assume that $P, Q \in R$ and $k \in \{1, 2, \ldots\}$. Then*

$$|P| + |Q| < 1 \tag{1.10}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+1} = P y_n + Q y_{n-k}, \quad n = 0, 1, \ldots. \tag{1.11}$$

Lemma 1.5 (see [4]). *Let $p \geq 2$ be a positive integer and assume that every positive solution of equation*

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{A + B x_n + C n_{n-k}}, \quad n = 0, 1, \ldots \tag{1.12}$$

is periodic with period p. If $C > 0$, then $A = B = 0$.

Lemma 1.6 (see [21]). *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), \quad n = 0, 1, \ldots, \tag{1.13}$$

where $k \in \{1, 2, \ldots\}$. Let $I = [a, b]$ be some interval of real numbers and assume that

$$f : [a, b] \times [a, b] \longrightarrow [a, b] \tag{1.14}$$

is a continuous function satisfying the following properties:

(a) *$f(u, v)$ is nonincreasing in u and nondecreasing in v,*

(b) *if $(m, M) \in [a, b] \times [a, b]$ is a solution of the system*

$$m = f(M, m), \qquad M = f(m, M), \tag{1.15}$$

then $m = M$.

Then (1.13) *has a unique equilibrium $\overline{y} \in [a, b]$, and every solution of* (1.13) *converges to $\overline{y}$.*

Lemma 1.7 (see [21]). *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), \quad n = 0, 1, \ldots, \tag{1.16}$$

where $k \in \{1, 2, \ldots\}$. Let $I = [a, b]$ be some interval of real numbers, and assume that

$$f : [a, b] \times [a, b] \longrightarrow [a, b] \tag{1.17}$$

is a continuous function satisfying the following properties:

(a) *$f(u, v)$ is nonincreasing in each of its arguments,*

(b) *if $(m, M) \in [a, b] \times [a, b]$ is a solution of the system*

$$m = f(M, M), \qquad M = f(m, m), \tag{1.18}$$

then $m = M$.

Then (1.16) *has a unique equilibrium $\overline{y} \in [a, b]$, and every solution of* (1.16) *converges to $\overline{y}$.*

2. The Special Case $pqr = 0$

If the parameters $pqr = 0$, then (1.1) contains the following several equations. We now assume that all their parameters are positive

$$x_{n+1} = \frac{qx_{n-k}}{1 + x_n + rx_{n-k}}, \quad n = 0, 1, \ldots, \tag{2.1}$$

$$x_{n+1} = \frac{p}{1 + x_n + rx_{n-k}}, \quad n = 0, 1, \ldots, \tag{2.2}$$

$$x_{n+1} = 0, \quad n = 0, 1, \ldots, \tag{2.3}$$

$$x_{n+1} = \frac{p}{1 + x_n}, \quad n = 0, 1, \ldots, \tag{2.4}$$

$$x_{n+1} = \frac{qx_{n-k}}{1 + x_n}, \quad n = 0, 1, \ldots, \tag{2.5}$$

$$x_{n+1} = \frac{p + qx_{n-k}}{1 + x_n}, \quad n = 0, 1, \ldots. \tag{2.6}$$

Equation (2.2) was studied in [19], where it is shown that the unique positive equilibrium is a global attractor. Equation (2.3) is trivial. Equation (2.4) is the Riccati equation [1]. Equation (2.5) can be reduces to (2.6), which was discussed in [22], and they showed that the unique positive equilibrium of (2.6) is globally asymptotically stable when $q < 1$. So, here we only consider (2.1).

Clearly, $\overline{x} = 0$ is always an equilibrium of (2.1) and when $q > 1$, (2.1) also possesses the unique positive equilibrium $\overline{x} = (q - 1)/(1 + r)$.

The linearized equation associated with (2.1) about the zero equilibrium is

$$z_{n+1} - qz_{n-k} = 0. \tag{2.7}$$

The linearized equation associated with (2.1) about the positive equilibrium is

$$z_{n+1} + \frac{\overline{x}}{1 + \overline{x} + r\overline{x}} z_n - \frac{q - r\overline{x}}{1 + \overline{x} + r\overline{x}} z_{n-k} = 0. \tag{2.8}$$

From this and by Theorem 1.4, we have the following result.

Theorem 2.1. (a) *Assume that $q < 1$. Then $\overline{x} = 0$ of* (2.1) *is locally asymptotically stable.*

(b) *Assume that $q > 1$ and $r > 1$. Then the unique positive equilibrium $\overline{x} = (q - 1)/(1 + r)$ of* (2.1) *is locally asymptotically stable.*

Theorem 2.2. *Assume that $q > 1$ and $r > 1$. Then the unique positive equilibrium $\overline{x}$ of* (2.1) *is globally asymptotically stable.*

Proof. By Theorem 2.1, the positive equilibrium of (2.1) is locally asymptotically stable. It suffices to show that $\overline{x}$ is a global attractor.

Let

$$f(x, y) = \frac{qy}{1 + x + ry} \quad \text{for } x, y \in (0, \infty), \tag{2.9}$$

then $f(x, y)$ is nonincreasing in x and nondecreasing in y. So

$$0 \le f(x, y) \le \frac{q}{r} \quad \text{for } x, y \in (0, \infty). \tag{2.10}$$

From

$$\frac{qm}{1 + M + rm} = m, \qquad \frac{qM}{1 + m + rM} = M, \tag{2.11}$$

we have $m = M$.

Hence by Lemma 1.6 the proof is complete. □

In the following sections we assume that all parameters in (1.1) are positive.

3. Local Stability and Period-Two Solutions

The equilibria of (1.1) are the solutions of the equation

$$\overline{x} = \frac{p + q\overline{x}}{1 + \overline{x} + r\overline{x}}. \tag{3.1}$$

So (1.1) possesses the unique positive equilibrium

$$\overline{x} = \frac{q - 1 + \sqrt{(q-1)^2 + 4p(r+1)}}{2(r+1)}. \tag{3.2}$$

The linearized equation associated with (1.1) about the positive equilibrium is

$$z_{n+1} + \frac{\overline{x}}{1 + \overline{x} + r\overline{x}} z_n - \frac{q - r\overline{x}}{1 + \overline{x} + r\overline{x}} z_{n-k} = 0. \tag{3.3}$$

By Theorem 1.4, it is sufficient to show that in

$$\left| \frac{\overline{x}}{1 + \overline{x} + r\overline{x}} \right| + \left| \frac{q - r\overline{x}}{1 + \overline{x} + r\overline{x}} \right| < 1. \tag{3.4}$$

Thus

$$|q - r\overline{x}| < 1 + r\overline{x}. \tag{3.5}$$

If $q - r\overline{x} < 0$, then we have $r\overline{x} - q < 1 + r\overline{x}$, and it clearly holds.

If $q - r\overline{x} \geq 0$, then we have $q - r\overline{x} < 1 + r\overline{x}$, and

$$q - 1 < 2r\overline{x}. \tag{3.6}$$

If $q \leq 1$, the inequality (3.6) obviously holds. Suppose $q > 1$, then we can get

$$\begin{gathered} q - 1 < r\sqrt{(q-1)^2 + 4p(r+1)}, \\ (q-1)^2 < r^2(q-1)^2 + 4pr^2(r+1), \end{gathered} \tag{3.7}$$

from which it follows that

$$(r-1)(q-1)^2 + 4pr^2 > 0. \tag{3.8}$$

So we have the following result.

Theorem 3.1. *Assume that*

$$\text{either } q \leq 1 \text{ or } q > 1, \quad (r-1)(q-1)^2 + 4pr^2 > 0. \tag{3.9}$$

Then the positive equilibrium $\overline{x}$ *of* (1.1) *is locally asymptotically stable.*

Theorem 3.2. (a) *Assume that* k *is odd. Then* (1.1) *has a nonnegative prime period-two solution if and only if*

$$q > 1, \quad (r-1)(q-1)^2 + 4pr^2 < 0. \tag{3.10}$$

Further when (3.10) *holds, the period-two solution is "unique" and the value of* ϕ_1 *and* ϕ_2 *are the positive roots of the quadratic equation*

$$t^2 - \frac{q-1}{r}t + \frac{p}{1-r} = 0. \tag{3.11}$$

(b) *Assume that* k *is even. Equation* (1.1) *has no nonnegative prime period-two solution.*

Proof. (a) Assume that k is odd, then $x_{n+1} = x_{n-k}$. Let

$$\ldots, \phi_1, \phi_2, \phi_1, \phi_2, \ldots \tag{3.12}$$

be a nonnegative prime period-two solution of (1.1). Then ϕ_1, ϕ_2 satisfy the following system:

$$\phi_1 = \frac{p + q\phi_1}{1 + \phi_2 + r\phi_1}, \qquad \phi_2 = \frac{p + q\phi_2}{1 + \phi_1 + r\phi_2}. \tag{3.13}$$

Substituting the above two equations, we obtain

$$(\phi_1 - \phi_2)\left(\phi_1 + \phi_2 - \frac{q-1}{r}\right) = 0. \tag{3.14}$$

Thus

$$\phi_1 + \phi_2 = \frac{q-1}{r}. \tag{3.15}$$

Adding them and using the above equations, we can get

$$\phi_1\phi_2 = \frac{p}{1-r}. \tag{3.16}$$

Clearly, in this case the discriminant of (3.11) is positive, that is

$$\Delta = \frac{(r-1)(q-1)^2 + 4pr^2}{r^2(r-1)} > 0, \tag{3.17}$$

and so ϕ_1, ϕ_2 are the positive roots of (3.11).

(b) Assume that k is even, then $x_n = x_{n-k}$. If there exist distinctive nonnegative real number ϕ_1 and ϕ_2, such that

$$\ldots, \phi_1, \phi_2, \phi_1, \phi_2, \ldots \tag{3.18}$$

is a prime period-two solution of (1.1) and ϕ_1, ϕ_2 satisfy the following system:

$$\phi_1 = \frac{p + q\phi_2}{1 + \phi_2 + r\phi_2}, \qquad \phi_2 = \frac{p + q\phi_1}{1 + \phi_1 + r\phi_1}, \tag{3.19}$$

which is equivalent to

$$\phi_1 + \phi_1\phi_2 + r\phi_1\phi_2 = p + q\phi_2, \qquad \phi_2 + \phi_1\phi_2 + r\phi_1\phi_2 = p + q\phi_1. \tag{3.20}$$

Subtracting these two equation, we can get

$$(\phi_1 - \phi_2)(q+1) = 0. \tag{3.21}$$

Then $\phi_1 = \phi_2$. This contradicts the hypothesis that $\phi_1 \neq \phi_2$.

The proof is complete. □

Further, applying Lemma 1.5, we have the following result about the period solutions.

Theorem 3.3. *There is no integer $p \geq 2$ such that every positive solution of (1.1) is periodic with period p.*

4. Boundedness and Invariant Interval

In this section, we will investigate the boundedness and invariant interval of (1.1).

Theorem 4.1. *Every solution of* (1.1) *is bounded from above and from below by positive constants.*

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (1.1). Clearly, if the solution is bounded from above by a constant M, then

$$x_{n+1} \geq \frac{p}{1+(r+1)M} \quad \forall n \geq -k, \tag{4.1}$$

and so it is also bounded from below. Now for the sake of contradiction assume that the solution is not bounded from above. Then there exists a subsequences $\{x_{n_m+1}\}_{m=0}^{\infty}$ such that

$$\lim_{m\to\infty} n_m = \infty, \qquad \lim_{m\to\infty} x_{n_m+1} = \infty, \tag{4.2}$$

and also

$$x_{n_m+1} = \max\{x_n : n \leq n_m\} \quad \text{for } m \geq 0. \tag{4.3}$$

From (1.1) we see that

$$x_{n+1} < p + qx_{n-k} \quad \text{for } n \geq 0, \tag{4.4}$$

and so

$$\lim_{m\to\infty} x_{n_m+1} = \lim_{m\to\infty} x_{n_m-k} = \infty. \tag{4.5}$$

Hence, for sufficiently large m,

$$0 \leq x_{n_m+1} - x_{n_m-k} = \frac{p+qx_{n_m-k}}{1+x_{n_m}+rx_{n_m-k}} - x_{n_m-k} = \frac{p+\left[(q-1)-x_{n_m}-rx_{n_m-k}\right]x_{n_m-k}}{1+x_{n_m}+rx_{n_m-k}} < 0, \tag{4.6}$$

which is a contradiction. The proof is complete. □

Let

$$f(u,v) = \frac{p+qv}{1+u+rv}. \tag{4.7}$$

Then the following statements are true.

Lemma 4.2. (a) *Assume that $pr \leq q$. Then $f(u,v)$ is decreasing in u for each v and increasing in v for each u.*

(b) *Assume that* $pr > q$. *Then* $f(u,v)$ *is decreasing in* u *for each* v *and decreasing in* v *for* $u \in [0,(pr-q)/q]$, *and increasing in* v *for* $u \in [(pr-q)/q,\infty)$.

Proof. The proofs of (a) and (b) are simple and will be omitted. □

Lemma 4.3. *Equation* (1.1) *possesses the following invariant intervals:*

(a) $[0,q/r]$, *when* $pr \le q$;

(b) $[(pr-q)/q,q/r]$, *when* $q < pr < q+q^2/r$;

(c) $[0,p]$, *when* $pr = q+q^2/r$;

(d) $[q/r,(pr-q)/q]$, *when* $q+q^2/r < pr < pq+q$;

(e) $[q/r,p]$, *when* $pr \ge pq+q$.

Proof. (a) Set $g(x) = (p+qx)/(1+rx)$, so $g(x)$ is increasing for x and $g(q/r) \le q/r$. Using (1.1) we see that when $x_{-k},\ldots,x_{-1},x_0 \in [0,q/r]$, then

$$x_1 = \frac{p+qx_{-k}}{1+x_0+rx_{-k}} \le \frac{p+qx_{-k}}{1+rx_{-k}} \le g\left(\frac{q}{r}\right) \le \frac{q}{r}. \tag{4.8}$$

The proof follows by induction.

(b) Using the monotonic character of the function $f(u,v)$ which is described by Lemma 4.2(b) and the condition that $q < pr < q+q^2/r$, when $x_{-k},\ldots,x_{-1},x_0 \in [(pr-q)/q,q/r]$, we can get

$$\frac{pr-q}{q} \le f\left(\frac{q}{r},\frac{pr-q}{q}\right) \le x_1 = \frac{p+qx_{-k}}{1+x_0+rx_{-k}} = f(x_o,x_{-k}) \le f\left(\frac{pr-q}{q},\frac{q}{r}\right) = \frac{q}{r}. \tag{4.9}$$

The proof follows by induction.

(c) Set $h(x) = (p+qx)/(1+p+rx)$, $g(x) = (p+qx)/(1+rx)$, then $h(x)$ is increasing for x and $g(x)$ is decreasing for x when $pr = q+q^2/r$, we see that when $x_{-k},\ldots,x_{-1},x_0 \in [0,p]$, then

$$\begin{aligned} x_1 &= \frac{p+qx_{-k}}{1+x_0+rx_{-k}} \ge \frac{p+qx_{-k}}{1+p+rx_{-k}} \ge h(0) = \frac{p}{1+p} > 0, \\ x_1 &= \frac{p+qx_{-k}}{1+x_0+rx_{-k}} \le \frac{p+qx_{-k}}{1+rx_{-k}} \le g(0) = p. \end{aligned} \tag{4.10}$$

The proof follows by induction.

(d) Using the monotonic character of the function $f(u,v)$ which is described by Lemma 4.2(b) and the condition $q+q^2/r < pr < pq+q$, when $x_{-k},\ldots,x_{-1},x_0 \in [q/r,(pr-q)/q]$, we obtain

$$\frac{q}{r} = f\left(\frac{pr-q}{q},\frac{pr-q}{q}\right) \le x_1 = \frac{p+qx_{-k}}{1+x_0+rx_{-k}} = f(x_o,x_{-k}) \le f\left(\frac{q}{r},\frac{q}{r}\right) = \frac{pr+q^2}{r+(r+1)q} < \frac{pr-q}{q}. \tag{4.11}$$

The proof follows by induction.

(e) In view of the condition $pr \geq pq + q$, we can get $(pr - q)/q > p > q/r$. By using the monotonic character of the function $f(u, v)$ which is described by Lemma 4.2(b) and the condition $pr \geq pq + q$, when $x_{-k}, \ldots, x_{-1}, x_0 \in [q/r, p]$, we have

$$\frac{q}{r} \leq \frac{p + qp}{1 + p + pr} = f(p, p) \leq x_1 = \frac{p + qx_{-k}}{1 + x_0 + rx_{-k}} = f(x_o, x_{-k}) \leq f\left(\frac{q}{r}, \frac{q}{r}\right) = \frac{pr + q^2}{r + q + qr} < p. \tag{4.12}$$

The proof follows by induction.

The proof is complete. □

5. Semicycles Analysis

Let $\{x_n\}_{n=-k}^{\infty}$ be a positive solution of (1.1). Then we have the following equations:

$$x_{n+1} - \frac{q}{r} = \frac{q}{r} \frac{(pr - q)/q - x_n}{1 + x_n + rx_{n-k}}, \quad \text{for } n \geq 0, \tag{5.1}$$

$$x_{n+1} - p = -\frac{px_n + (pr - q)x_{n-k}}{1 + x_n + rx_{n-k}}, \quad \text{for } n \geq 0, \tag{5.2}$$

$$x_{n+1} - \frac{pr - q}{q} = \frac{(1/q)(q + q^2/r - pr) + ((pr - q)/q)(q/r - x_n) + (r/q)(q + q^2/r - pr)x_{n-k}}{1 + x_n + rx_{n-k}}, \quad \text{for } n \geq 0, \tag{5.3}$$

$$x_{n+1} - \overline{x} = \frac{r(\overline{x} - q/r)(\overline{x} - x_{n-k}) + \overline{x}(\overline{x} - x_n)}{1 + x_n + rx_{n-k}}, \quad \text{for } n \geq 0, \tag{5.4}$$

$$x_{n+2} - x_n = \frac{rx_{n+1-k}(q/r - x_n)(rx_{n-k} + x_n + 1) + (q + r)x_{n-k}(pr/(q + r) - x_n) + (p - x_n - x_n^2)}{(1 + x_n + rx_{n-k})(1 + rx_{n+1-k}) + p + qx_{n-k}}, \quad \text{for } n \geq 0. \tag{5.5}$$

If $pr = q + q^2/r$, then the unique positive equilibrium is $\overline{x} = q/r$, and (5.1) and (5.5) change into

$$x_{n+1} - \frac{q}{r} = \frac{q}{r} \frac{q/r - x_n}{1 + x_n + rx_{n-k}}, \quad \text{for } n \geq 0, \tag{5.6}$$

$$x_{n+2} - x_n = \frac{(q/r - x_n)(r^2 x_{n+1-k} x_{n-k} + rx_n x_{n+1-k} + rx_{n+1-k} + (pr^2/q)x_{n-k} + x_n + q/r + 1)}{(1 + x_n + rx_{n-k})(1 + rx_{n+1-k}) + p + qx_{n-k}}, \quad \text{for } n \geq 0. \tag{5.7}$$

The following lemma is straightforward consequences of identities (5.1)–(5.7).

Lemma 5.1. *Assume that* $pr \leq q$ *and let* $\{x_n\}_{n=-k}^{\infty}$ *be a solution of* (1.1). *Then the following statements are true:*

(i) $x_n \leq q/r$ *for all* $n \geq 1$;

(ii) *if for some* $N \geq 0$, $x_{N-k} \leq \overline{x}$ *and* $x_N \geq \overline{x}$, *then* $x_{N+1} \leq \overline{x}$;

(iii) *if for some* $N \geq 0$, $x_{N-k} > \overline{x}$ *and* $x_N < \overline{x}$, *then* $x_{N+1} > \overline{x}$;

(iv) $0 \leq \overline{x} \leq q/r$.

Lemma 5.2. *Assume that* $q < pr < q + q^2/r$ *and let* $\{x_n\}_{n=-k}^{\infty}$ *be a solution of* (1.1). *Then the following statements are true:*

(i) *if for some* $N \geq 0$, $x_N < q/r$, *then* $x_{N+1} > (pr - q)/q$;

(ii) *if for some* $N \geq 0$, $x_N < (pr - q)/q$, *then* $x_{N+1} > q/r$;

(iii) *if for some* $N \geq 0$, $x_N = (pr - q)/q$, *then* $x_{N+1} = q/r$;

(iv) *if for some* $N \geq 0$, $x_N > (pr - q)/q$, *then* $x_{N+1} < q/r$;

(v) *if for some* $N \geq 0$, $(pr - q)/q \leq x_N \leq q/r$, *then* $(pr - q)/q \leq x_n \leq q/r$ *for* $n \geq N$;

(vi) *if for some* $N \geq 0$, $x_{N-k} \leq \overline{x}$, *and* $x_N \geq \overline{x}$, *then* $x_{N+1} \leq \overline{x}$;

(vii) *if for some* $N \geq 0$, $x_{N-k} > \overline{x}$, *and* $x_N < \overline{x}$, *then* $x_{N+1} > \overline{x}$;

(viii) *if for some* $N \geq 0$, $x_N < (pr - q)/q$, *then* $x_{N+2} > x_N$;

(ix) *if for some* $N \geq 0$, $x_N > q/r$, *then* $x_{N+2} < x_N$;

(x) $(pr - q)/q \leq \overline{x} \leq q/r$.

Lemma 5.3. *Assume that* $pr = q + q^2/r$ *and let* $\{x_n\}_{n=-k}^{\infty}$ *be a solution of* (1.1). *Then the following statements are true:*

(i) *if for some* $N \geq 0$, $x_N > q/r$, *then* $x_{N+1} < q/r$;

(ii) *if for some* $N \geq 0$, $x_N = q/r$, *then* $x_{N+1} = q/r$;

(iii) *if for some* $N \geq 0$, $x_N < q/r$, *then* $x_{N+1} > q/r$;

(iv) *if for some* $N \geq 0$, $x_N \geq \overline{x}$, *then* $x_{N+1} \leq \overline{x}$;

(v) *if for some* $N \geq 0$, $x_N < \overline{x}$, *then* $x_{N+1} > \overline{x}$;

(vi) *if for some* $N \geq 0$, $x_N > q/r$, *then* $x_{N+2} < x_N$;

(vii) *if for some* $N \geq 0$, $x_N < q/r$, *then* $x_{N+2} > x_N$;

(viii) $\overline{x} = q/r$.

Lemma 5.4. *Assume that* $q + q^2/r < pr < pq + q$ *and let* $\{x_n\}_{n=-k}^{\infty}$ *be a solution of* (1.1). *Then the following statements are true:*

(i) *if for some* $N \geq 0$, $x_N > q/r$, *then* $x_{N+1} < (pr - q)/q$;

(ii) *if for some* $N \geq 0$, $x_N < (pr - q)/q$, *then* $x_{N+1} > q/r$;

(iii) *if for some* $N \geq 0$, $x_N = (pr - q)/q$, *then* $x_{N+1} = q/r$;

(iv) *if for some* $N \geq 0$, $x_N > (pr - q)/q$, *then* $x_{N+1} < q/r$;

(v) *if for some* $N \geq 0$, $q/r \leq x_N \leq (pr - q)/q$, *then* $q/r \leq x_n \leq (pr - q)/q$ *for* $n \geq N$;

(vi) *if for some* $N \geq 0$, $x_{N-k} \leq \overline{x}$ *and* $x_N \leq \overline{x}$, *then* $x_{N+1} \geq \overline{x}$;

(vii) *if for some* $N \geq 0$, $x_{N-k} > \overline{x}$ *and* $x_N > \overline{x}$, *then* $x_{N+1} < \overline{x}$;

(viii) *if for some* $N \geq 0$, $x_N < (pr - q)/q$, *then* $x_{N+1} > q/r$;

(ix) *if for some* $N \geq 0$, $x_N < q/r$, *then* $x_{N+2} > x_N$;

(x) *if for some* $N \geq 0$, $x_N > (pr - q)/q$, *then* $x_{N+2} < x_N$;

(xi) $q/r \leq \overline{x} \leq (pr - q)/q$.

Lemma 5.5. *Assume that* $pr \geq pq + q$ *and let* $\{x_n\}_{n=-k}^{\infty}$ *be a solution of* (1.1). *Then the following statements are true:*

(i) $x_n \leq p$ *for all* $n \geq 1$;

(ii) *if for some* $N \geq 0$, $x_N \leq p$, *then* $x_{N+1} \geq q/r$;

(iii) *if for some* $N \geq 0$, $q/r \leq x_N \leq p$, *then* $q/r \leq x_n \leq p$ *for* $n \geq N$;

(iv) *if for some* $N \geq 0$, $x_{N-k} \leq \overline{x}$ *and* $x_N \leq \overline{x}$, *then* $x_{N+1} \geq \overline{x}$;

(v) *if for some* $N \geq 0$, $x_{N-k} > \overline{x}$ *and* $x_N > \overline{x}$, *then* $x_{N+1} < \overline{x}$;

(vi) *if for some* $N \geq 0$, $x_N < q/r$, *then* $x_{N+2} > x_N$;

(vii) *if for some* $N \geq 0$, $x_N > p$, *then* $x_{N+2} < x_N$;

(viii) $q/r \leq \overline{x} \leq p$.

The following results are consequences of Lemmas 5.1–5.5.

Theorem 5.6. *Let* $\{x_n\}_{n=-k}^{\infty}$ *be a nontrivial solution of* (1.1) *and* $\overline{x}$ *is the unique positive equilibrium point of* (1.1). *Then the following statements are true.*

(a) *Assume that* $pr \leq q$. *Then except possibly for the first semicycle, every oscillatory solution of* (1.1) *has semicycles of length at most* k.

(b) *Assume that* $q < pr < q + q^2/r$. *Then, except possibly for the first semicycle, every oscillatory solution of* (1.1) *which lies in the invariant interval* $[(pr - q)/q, q/r]$ *has semicycles of length at most* k.

(c) *Assume that* $pr = q + q^2/r$. *Then after the first semicycle, every oscillatory solution of* (1.1) *about the equilibrium point* $\overline{x}$ *with semicycle of length one.*

(d) *Assume that* $q + q^2/r < pr < pq + q$. *Then, except possibly for the first semicycle, every oscillatory solution of* (1.1) *which lies in the invariant interval* $[q/r, (pr - q)/q]$ *has semicycles that is either of length at least* $k - 1$, *or of length at most* $k + 1$.

(e) *Assume that* $pr \geq pq + q$. *Then, except possibly for the first semicycle, every oscillatory solution of* (1.1) *which lies in the invariant interval* $[q/r, p]$ *has semicycles that is either of length at least* $k - 1$, *or of length at most* $k + 1$.

6. Global Stability

In this section, we will investigate global stability of the positive equilibrium point $\overline{x}$ of (1.1).

Theorem 6.1. *Assume that* (3.9) *holds and let* $\{x_n\}_{n=-k}^{\infty}$ *be a positive solution of* (1.1). *Then every solution of* (1.1) *eventually enters the invariant interval*

(a) $[0, q/r]$ *if* $pr \leq q$;

(b) $[(pr-q)/q, q/r]$ *if* $q < pr < q + q^2/r$;

(c) $[q/r, (pr-q)/q]$ *if* $q + q^2/r < pr < pq + q$;

(d) $[q/r, p]$ *if* $pr \geq pq + q$.

Proof. (a) The proof is a direct consequence of Lemma 5.1.

(b) From Lemma 5.2(v) we know that if there exist an integer N such that $x_N \in [(pr-q)/q, q/r]$, then $x_n \in [(pr-q)/q, q/r]$ for $n \geq N$ and the result follows. Now assume for the sake of contradiction that all terms of $\{x_n\}$ never enter the invariant interval $[(pr-q)/q, q/r]$ for $n \geq 0$. Notice that Lemma 5.2(ii) implies that $x_{n+1} > q/r$ for $x_n < (pr-q)/q$. Further using Lemma 5.2(viii) and (ix), we obtain that the subsequence $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=0}^{\infty}$ are both monotonous. If one of them is decreasing, then it is bounded above by $(pr-q)/q$, and if one of them is decreasing, then it is bounded below by q/r. Thus $\lim_{n\to\infty} x_{2n}$ and $\lim_{n\to\infty} x_{2n+1}$ exist. Set

$$\lim_{n\to\infty} x_{2n} = L, \qquad \lim_{n\to\infty} x_{2n+1} = M, \tag{6.1}$$

then $L \leq (pr-q)/q$ and $M \geq q/r$, or vice versa. From which it follows that

$$\ldots, L, M, L, M, \ldots \tag{6.2}$$

is a period-two solution of (1.1), which is a contradiction, since when (2.3) holds, (1.1) has no period-two solution.

(c) The proof is similar to (b), so will be omitted.

(d) In view of Lemma 5.5(i) and (iii), we know that $x_n \leq p$ for all $n \geq 1$ and $[q/r, p]$ is an invariant interval of (1.1). If there exist an integer N such that $x_N \in [q/r, p]$, then $x_n \in [q/r, p]$ for $n \geq N$, from which it follows that the result is true. Now assume for the sake of contradiction that terms of $\{x_n\}$ never enter the invariant interval $[q/r, p]$, then they all lie in the interval $[0, q/r]$. Noticing that $x_1 \leq q/r$ and $pr \geq pq + q$ hold, we get

$$x_2 - x_1 = \frac{p - x_1 + r x_{1-k}(q/r - x_1) - x_1^2}{1 + x_1 + r x_{1-k}} \geq \frac{p - q/r - (q/r)^2}{1 + x_1 + r x_{1-k}} > 0, \tag{6.3}$$

from which it follows by induction that the sequence $\{x_n\}$ is increasing in the interval $[0, q/r]$. Hence, $\lim_{n\to\infty} x_n$ exists and $\lim_{n\to\infty} x_n \leq q/r$, which is a contradiction because (1.1) has no equilibrium point in the interval $[0, q/r]$.

The proof is complete. □

Theorem 6.2. *Assume that* (3.9) *holds. Then the positive equilibrium* $\overline{x}$ *is a global attractor of* (1.1).

Proof. The proof is finished by considering the following five cases.

Case 1 (when $pr \le q$). By Lemma 4.3(a) and Theorem 6.1(a), we know that (1.1) possesses an invariant interval $[0, q/r]$ and every solution of (1.1) eventually enters the interval $[0, q/r]$. Further, it is easy to see that $f(u, v)$ decreases in u and increases in v in $[0, q/r]$.

Finally observe that when (3.9) holds, the only solution of the system

$$\frac{p + qm}{1 + M + rm} = m, \qquad \frac{p + qM}{1 + m + rM} = M, \tag{6.4}$$

is $m = M$. Further, Lemma 5.1 implies that (1.1) has a unique equilibrium $\overline{x} \in [0, q/r]$. Thus, in view of Lemma 1.6, every solution of (1.1) converges to $\overline{x}$. So the unique positive equilibrium $\overline{x}$ is a global attractor of (1.1).

Case 2 (when $q < pr < q + q^2/r$). By Lemma 4.3(b) and Theorem 6.1(b), we know that (1.1) possesses an invariant interval $[(pr - q)/q, q/r]$ and every solution of (1.1) eventually enters the interval $[(pr - q)/q, q/r]$. Further, it is easy to see that $f(u, v)$ decreases in u and increases in v in $[(pr - q)/q, q/r]$. Then using the same argument in Case 1, (1.1) has a unique equilibrium $\overline{x} \in [(pr - q)/q, q/r]$ and every solution of (1.1) converges to $\overline{x}$. So the unique positive equilibrium $\overline{x}$ is a global attractor of (1.1).

Case 3 (when $pr = q + q^2/r$). In view of part (c) of Theorem 5.6, we know that, after the first semicycle, the nontrivial solution oscillates about $\overline{x}$ with semicycles of length one. Considering the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=0}^{\infty}$, we have

$$x_{2n} > \frac{q}{r}, \quad x_{2n+1} < \frac{q}{r} \quad \text{for } n \ge 0, \tag{6.5}$$

or

$$x_{2n} < \frac{q}{r}, \quad x_{2n+1} > \frac{q}{r} \quad \text{for } n \ge 0. \tag{6.6}$$

Let us consider Case 1. Case 2 can be handled in a similar way. In view of Theorem 4.1 and Lemma 5.3, we know that $\{x_n\}_{n=-k}^{\infty}$ is bounded and $x_{n+1} = q/r$ if $x_n = q/r$ and

$$x_1 < x_3 < x_5 < \cdots < \frac{q}{r} < \cdots < x_4 < x_2 < x_0. \tag{6.7}$$

So $\lim_{n\to\infty} x_{2n}$ and $\lim_{n\to\infty} x_{2n+1}$ exist.

Let

$$L = \lim_{n\to\infty} x_{2n}, \qquad M = \lim_{n\to\infty} x_{2n+1}, \tag{6.8}$$

then $L \ge q/r, M \le q/r$. If $L \ne M$, then L, M is a period-two solution of (1.1). Furthermore, the condition (3.9) holds. This contradicts Theorem 3.2. Thus $L = M$ and $\lim_{n\to\infty} x_n = q/r$. So $\overline{x} = q/r$ is a global attractor of (1.1).

Case 4 (when $q + q^2/r < pr < pq + q$). By Theorem 6.1(c), we know that every solution of (1.1) eventually enters the interval $[q/r, (pr - q)/q]$. Furthermore, it is easy to see that the function $f(u, v)$ decreases in each of its arguments in the interval $[q/r, (pr - q)/q]$. Let $m, M \in [q/r, (pr - q)/q]$ is a solution of the system

$$\frac{p + qm}{1 + m + rm} = M, \qquad \frac{p + qM}{1 + M + rM} = m, \tag{6.9}$$

that is, the solution of the system

$$p + qm = M + mM + rmM, \qquad p + qM = m + mM + rmM. \tag{6.10}$$

Then $(m - M)(q + 1) = 0$, which implies that $m = M$. Employing Lemma 1.7, we see that (1.1) has a unique equilibrium $\overline{x} \in [q/r, (pr - q)/q]$ and every solution of (1.1) converges to $\overline{x}$. Thus the unique positive equilibrium $\overline{x}$ is a global attractor of (1.1).

Case 5. When $pr \geq pq + q$. By Theorem 6.1(d), we know that every solution of (1.1) eventually enters the interval $[q/r, p]$. Further, it is clear to see that the function $f(u, v)$ decreases in each of its arguments in the interval $[q/r, p]$. Then, using the same argument as in Case 4, (1.1) has a unique equilibrium $\overline{x} \in [q/r, p]$ and every solution of (1.1) converges to $\overline{x}$. Thus the unique positive equilibrium $\overline{x}$ is a global attractor of (1.1).

The proof is complete. □

In view of Theorems 3.1 and 6.2, we have the following result, which solves Open Problem 1 when conditions (3.9) holds.

Theorem 6.3. *Assumed that* (3.9) *holds. Then the positive equilibrium of* (1.1) *is globally asymptotically stable.*

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