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Research Article

Arithmetic Identities Involving Genocchi and Stirling Numbers

Guodong Liu

Department of Mathematics, Huizhou University, Huizhou, Guangdong 516015, China

Correspondence should be addressed to Guodong Liu, gdliu@pub.huizhou.gd.cn

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An explicit formula, the generalized Genocchi numbers, was established and some identities and congruences involving the Genocchi numbers, the Bernoulli numbers, and the Stirling numbers were obtained.

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1. Introduction

The Genocchi numbers G_n and the Bernoulli numbers B_n ($n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$) are defined by the following generating functions (see [1]):

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} \quad (|t| < \pi), \tag{1.1}$$

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad (|t| < 2\pi), \tag{1.2}$$

respectively. By (1.1) and (1.2), we have

$$G_{2n+1} = B_{2n+1} = 0, \quad (n \in \mathbb{N}) \quad G_n = 2(1 - 2^n) B_n, \tag{1.3}$$

with $\mathbb{N}$ being the set of positive integers.

The Genocchi numbers G_n satisfy the recurrence relation

$$G_0 = 0, \quad G_1 = 1, \quad G_n = -\frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}G_k \quad (n \geq 2), \tag{1.4}$$

so we find $G_2 = -1,\ G_4 = 1,\ \ G_6 = -3,\ G_8 = 17,\ G_{10} = -155,\ G_{12} = 2073,\ G_{14} = -38227,\ldots$.

The Stirling numbers of the first kind $s(n,k)$ can be defined by means of (see [2])

$$(x)_n = x(x-1)\ldots(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k, \tag{1.5}$$

or by the generating function

$$\left(\log(1+x)\right)^k = k!\sum_{n=k}^{\infty} s(n,k)\frac{x^n}{n!}. \tag{1.6}$$

It follows from (1.5) or (1.6) that

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k), \tag{1.7}$$

with $s(n,0) = 0\ (n > 0)$, $s(n,n) = 1\ (n \geq 0)$, $s(n,1) = (-1)^{n-1}(n-1)!\ (n > 0)$, $s(n,k) = 0\ (k > n$ or $k < 0)$.

Stirling numbers of the second kind $S(n,k)$ can be defined by (see [2])

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k \tag{1.8}$$

or by the generating function

$$(e^x - 1)^k = k!\sum_{n=k}^{\infty} S(n,k)\frac{x^n}{n!}. \tag{1.9}$$

It follows from (1.8) or (1.9) that

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \tag{1.10}$$

with $S(n,0) = 0\ (n > 0)$, $S(n,n) = 1\ (n \geq 0)$, $S(n,1) = 1\ (n > 0)$, $S(n,k) = 0\ (k > n$ or $k < 0)$.

The study of Genocchi numbers and polynomials has received much attention; numerous interesting (and useful) properties for Genocchi numbers can be found in many books (see [1, 3–16]). The main purpose of this paper is to prove an explicit formula for the generalized Genocchi numbers (cf. Section 2). We also obtain some identities congruences involving the Genocchi numbers, the Bernoulli numbers, and the Stirling numbers. That is, we will prove the following main conclusion.

Theorem 1.1. *Let* $n \geq k \quad (n, k \in \mathbb{N})$*, then*

$$\sum_{\substack{v_1,\ldots,v_k\in\mathbb{N}\\ v_1+\cdots+v_k=n}} \frac{G_{v_1}\cdots G_{v_k}}{(v_1\cdots v_k)v_1!\cdots v_k!} = (-1)^{n-k}\frac{2^k k!}{n!}\sum_{j=k}^{n}\frac{1}{2^j}S(n,j)s(j,k). \tag{1.11}$$

Remark 1.2. Setting $k = 1$ in (1.11), and noting that $s(j,1) = (-1)^{j-1}(j-1)!$, we obtain

$$G_n = 2n\sum_{j=1}^{n}(-1)^{n-j}\frac{(j-1)!}{2^j}S(n,j) \quad (n \in \mathbb{N}). \tag{1.12}$$

Remark 1.3. By (1.11) and (1.3), we have

$$\sum_{\substack{v_1,\ldots,v_k\in\mathbb{N}\\ v_1+\cdots+v_k=n}} \frac{(2^{v_1}-1)B_{v_1}\cdots(2^{v_k}-1)B_{v_k}}{(v_1\cdots v_k)v_1!\cdots v_k!} = (-1)^{n}\frac{k!}{n!}\sum_{j=k}^{n}\frac{1}{2^j}S(n,j)s(j,k). \tag{1.13}$$

Theorem 1.4. *Let* $n, k \in \mathbb{N}$*, then*

$$\sum_{j=0}^{n}\frac{(-1)^j(k+j-1)!}{2^j}S(n,j) = 2^{k-1}\sum_{j=0}^{k-1}(-1)^j s(k,k-j)\frac{G_{n+k-j}}{n+k-j}. \tag{1.14}$$

Remark 1.5. Setting $k = 1,2,3,4$ in (1.14), we get

$$\begin{aligned}
\sum_{j=0}^{n}\frac{(-1)^j j!}{2^j}S(n,j) &= \frac{1}{n+1}G_{n+1},\\
\sum_{j=0}^{n}\frac{(-1)^j (j+1)!}{2^j}S(n,j) &= \frac{2}{n+1}G_{n+1}+\frac{2}{n+2}G_{n+2},\\
\sum_{j=0}^{n}\frac{(-1)^j (j+2)!}{2^j}S(n,j) &= \frac{8}{n+1}G_{n+1}+\frac{12}{n+2}G_{n+2}+\frac{4}{n+3}G_{n+3},\\
\sum_{j=0}^{n}\frac{(-1)^j (j+3)!}{2^j}S(n,j) &= \frac{48}{n+1}G_{n+1}+\frac{88}{n+2}G_{n+2}+\frac{48}{n+3}G_{n+3}+\frac{8}{n+4}G_{n+4}.
\end{aligned} \tag{1.15}$$

Theorem 1.6. *Let* $n \in \mathbb{N}$*,* $m \in \mathbb{N}_0$*, then*

$$\frac{2^n}{n+1}G_{n+1} \equiv 2^{n-m}\sum_{j=0}^{m}\binom{m}{j}j^n \pmod{m+1}. \tag{1.16}$$

Remark 1.7. Setting $m = p - 1$ in (1.16), we have

$$\frac{1}{n+1} G_{n+1} \equiv \sum_{j=0}^{p-1} (-1)^j j^n \pmod{p}, \tag{1.17}$$

where p is any odd prime.

2. Definition and Lemma

Definition 2.1. For a real or complex parameter x, we have the generalized Genocchi numbers $G_n^{(x)}$, which are defined by

$$\left(\frac{2}{e^{2t}+1}\right)^x = \sum_{n=0}^{\infty} G_n^{(x)} \frac{t^n}{n!} \quad \left(|t| < \frac{\pi}{2};\ 1^x := 1\right). \tag{2.1}$$

By (1.1) and (2.1), we have

$$nG_{n-1}^{(1)} = 2^{n-1} G_n. \tag{2.2}$$

Remark 2.2. For an integer x, the higher-order Euler numbers $E_{2n}^{(x)}$ are defined by the following generating functions (see [17]):

$$\left(\frac{2}{e^{t}+e^{-t}}\right)^x = \sum_{n=0}^{\infty} E_{2n}^{(x)} \frac{t^{2n}}{(2n)!} \quad \left(|t| < \frac{\pi}{2}\right). \tag{2.3}$$

Then we have

$$G_n^{(x)} = (-1)^n \sum_{k=0}^{[n/2]} \binom{n}{2k} E_{2k}^{(x)} x^{n-2k}, \tag{2.4}$$

where $[n/2]$ denotes the greatest integer not exceeding $n/2$.

Lemma 2.3. *Let $n \geq k$ $(n, k \in \mathbb{N})$, then*

$$G_n^{(x)} = \sum_{k=1}^{n} \omega(n,k) x^k, \tag{2.5}$$

where

$$\omega(n,k) = (-1)^k \sum_{j=k}^{n} 2^{n-j} S(n,j) s(j,k). \tag{2.6}$$

Proof. By (2.1), (1.5), and (1.9) we have

$$\begin{aligned}
\sum_{n=0}^{\infty} G_n^{(x)} \frac{t^n}{n!} &= \left(\frac{2}{e^{2t}+1}\right)^x = \left(\frac{1}{1+(1/2)(e^{2t}-1)}\right)^x \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j}{2^j} \binom{x+j-1}{j} \left(e^{2t}-1\right)^j \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j}{2^j} \binom{x+j-1}{j} j! \sum_{n=j}^{\infty} 2^n S(n,j) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^j j! 2^{n-j} \binom{x+j-1}{j} S(n,j) \frac{t^n}{n!},
\end{aligned} \tag{2.7}$$

which readily yields

$$\begin{aligned}
G_n^{(x)} &= \sum_{j=0}^{n} (-1)^j j! 2^{n-j} \binom{x+j-1}{j} S(n,j) \\
&= \sum_{j=0}^{n} (-1)^j 2^{n-j} S(n,j) (x+j-1)(x+j-2)\cdots(x+1)x \\
&= \sum_{j=0}^{n} (-1)^j 2^{n-j} S(n,j) \sum_{k=1}^{j} (-1)^{j-k} s(j,k) x^k \\
&= \sum_{k=1}^{n} (-1)^k \sum_{j=k}^{n} 2^{n-j} S(n,j) s(j,k) x^k = \sum_{k=1}^{n} \omega(n,k) x^k.
\end{aligned} \tag{2.8}$$

This completes the proof of Lemma 2.3 . □

Remark 2.4. From (1.7), (1.10), and Lemma 2.3 we know that $G_n^{(x)}$ is a polynomial of x with integral coefficients. For example, setting $n = 1, 2, 3, 4$ in Lemma 2.3, we get

$$\begin{gathered}
G_1^{(x)} = -x, \qquad G_2^{(x)} = -x + x^2, \qquad G_3^{(x)} = 3x^2 - x^3, \\
G_4^{(x)} = 2x + 3x^2 - 6x^3 + x^4.
\end{gathered} \tag{2.9}$$

Remark 2.5. Let $n, m \in \mathbb{N}$, then by (2.5), we have

$$\sum_{k=1}^{n} \omega(n,k) = \frac{2^n}{n+1} G_{n+1}. \tag{2.10}$$

Therefore, if $q \in \mathbb{N}$ is odd, then by (2.10) we get

$$G_{2^k q} \equiv 0 (\bmod\ q), \tag{2.11}$$

where $k \in \mathbb{N}$.

3. Proof of the Theorems

Proof of Theorem 1.1. By applying Lemma 2.3, we have

$$k!\omega(n,k) = \frac{d^k}{dx^k} G_n^{(x)}|_{x=0}. \tag{3.1}$$

On the other hand, it follows from (2.1) that

$$\sum_{n=k}^{\infty} \frac{d^k}{dx^k} G_n^{(x)}|_{x=0} \frac{t^n}{n!} = \left(\log \frac{2}{e^{2t}+1}\right)^k, \tag{3.2}$$

where $\log(2/(e^{2t}+1))$ is the principal branch of logarithm of $2/(e^{2t}+1)$.

Thus, by (3.1) and (3.2), we have

$$k! \sum_{n=k}^{\infty} \omega(n,k) \frac{t^n}{n!} = \left(\log \frac{2}{e^{2t}+1}\right)^k. \tag{3.3}$$

Now note that

$$\frac{d}{dt} \log \frac{2}{e^{2t}+1} = \frac{2}{e^{2t}+1} - 2 = \sum_{n=0}^{\infty} G_n^{(1)} \frac{t^n}{n!} - 2 = \sum_{n=0}^{\infty} \frac{2^n G_{n+1}}{n+1} \frac{t^n}{n!} - 2, \tag{3.4}$$

whence by integrating from 0 to t, we deduce that

$$\log \frac{2}{e^{2t}+1} = \sum_{n=1}^{\infty} \frac{2^{n-1} G_n}{n} \frac{t^n}{n!} - 2t = \sum_{n=1}^{\infty} (-1)^n \frac{2^{n-1} G_n}{n} \frac{t^n}{n!}. \tag{3.5}$$

Since $G_{2n+1} = 0 \ (n \in \mathbb{N})$. Substituting (3.5) in (3.3) we get

$$\omega(n,k) = (-1)^n \frac{n! 2^{n-k}}{k!} \sum_{\substack{v_1,\ldots,v_k \in \mathbb{N} \\ v_1+\cdots+v_k=n}} \frac{G_{v_1} \cdots G_{v_k}}{(v_1 \cdots v_k) v_1! \cdots v_k!}. \tag{3.6}$$

By (3.6) and (2.6), we may immediately obtain Theorem 1.1. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.4. By (2.1) and note the identity

$$2\left(\frac{2}{e^{2t}+1}\right)^{x}+\frac{1}{x}\frac{d}{dt}\left(\frac{2}{e^{2t}+1}\right)^{x}=\left(\frac{2}{e^{2t}+1}\right)^{x+1}, \tag{3.7}$$

we have

$$G_n^{(x+1)}=2G_n^{(x)}+\frac{1}{x}G_{n+1}^{(x)}. \tag{3.8}$$

By (3.8), (1.7), and note that $G_n^{(1)}=2^n/(n+1)G_{n+1}$, we obtain

$$\begin{aligned}G_n^{(k)}&=\frac{1}{(k-1)!}\sum_{j=0}^{k-1}(-1)^j2^js(k,k-j)G_{n+k-1-j}^{(1)}\\&=\frac{2^{n+k-1}}{(k-1)!}\sum_{j=0}^{k-1}(-1)^js(k,k-j)\frac{G_{n+k-j}}{n+k-j}.\end{aligned} \tag{3.9}$$

Comparing (3.9) and (2.8), we immediately obtain Theorem 1.4. This completes the proof of Theorem 1.4. □

Proof of Theorem 1.6. By Lemma 2.3, we have

$$G_n^{(m+x)}=\sum_{j=1}^{n}\omega(n,j)(m+x)^j\equiv\sum_{j=1}^{n}\omega(n,j)x^j=G_n^{(x)}\ (\mathrm{mod}\ \ m). \tag{3.10}$$

Therefore

$$G_n^{(k)}=G_n^{(m+k-m)}\equiv G_n^{(-m)}=2^{n-m}\sum_{j=0}^{m}\binom{m}{j}j^n\ (\mathrm{mod}\ \ m+k). \tag{3.11}$$

Taking $k=1$ in (3.11) and note that $G_n^{(1)}=2^n/(n+1)G_{n+1}$, we immediately obtain Theorem 1.6. This completes the proof of Theorem 1.6. □

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