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Research Article

Multiple Positive Periodic Solutions for Delay Differential System

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We obtain some existence results for multiple positive periodic solutions of some delay differential systems. Examples are presented as applications. For a general positive integer $m \geq 2$, main results of this paper do not appear in former literatures as we know. Comparing with the existing results, our results are new also when $m = 1$.

1. Introduction

It is known that multiple delay Logistic equations

$$\begin{aligned} y'(t) &= -a(t)y(t) + f\left(t, y(t-\tau_0(t)), y(t-\tau_1(t)), \dots, y(t-\tau_n(t))\right), \\ y'(t) &= a(t)y(t) - f\left(t, y(t-\tau_0(t)), y(t-\tau_1(t)), \dots, y(t-\tau_n(t))\right), \end{aligned} \tag{1.1}$$

are generalizations of many biological models, such as Logistic models of Single-species growth (see [1–3]),

$$\begin{aligned} y'(t) &= a(t)y(t)\left[1 - \frac{y(t-\tau(t))}{K(t)}\right], \\ y'(t) &= y(t)\left[a(t) - \sum_{i=1}^{n} b_i(t)y(t-\tau_i(t))\right], \\ y'(t) &= a(t)y(t)\left[1 - \prod_{i=1}^{n} \frac{y(t-\tau_i(t))}{K(t)}\right], \end{aligned} \tag{1.2}$$

and red blood cell models (see [4–7]),

$$y'(t) = -a(t)y(t) + b(t)e^{-y(t-\tau(t))},$$
$$y'(t) = -a(t)y(t) + \frac{b(t)}{1 + y^n(t-\tau(t))}. \tag{1.3}$$

For biological models, positive periodic solutions are often important and many results have been achieved in this direction, for instance, [8–10].

To the best of our knowledge, few papers concerning the existence of multiple positive solutions of (1.1) can be found in literature. Furthermore, no papers have yet deal with the more general nonautonomous delay differential systems

$$\begin{aligned} y_1'(t) &= -a_1(t)y_1(t) + f_1(t, Y_1(t), Y_2(t), \dots, Y_m(t)), \\ y_2'(t) &= -a_2(t)y_2(t) + f_2(t, Y_1(t), Y_2(t), \dots, Y_m(t)), \\ &\vdots \\ y_m'(t) &= -a_m(t)y_m(t) + f_m(t, Y_1(t), Y_2(t), \dots, Y_m(t)), \end{aligned} \tag{1.4}$$

$$\begin{aligned} y_1'(t) &= a_1(t)y_1(t) - f_1(t, Y_1(t), Y_2(t), \dots, Y_n(t)), \\ y_2'(t) &= a_2(t)y_2(t) - f_2(t, Y_1(t), Y_2(t), \dots, Y_n(t)), \\ &\vdots \\ y_n'(t) &= a_n(t)y_n(t) - f_m(t, Y_1(t), Y_2(t), \dots, Y_n(t)), \end{aligned} \tag{1.5}$$

where m, n are all positive integer and

$$\begin{gathered} a_i(\cdot), \quad i \in \Lambda_1 := \{1, 2, \dots, m\}, \\ f_i(\cdot, \dots, \cdot), \quad i \in \Lambda_1, \\ \tau_j(\cdot), \quad j \in \Lambda_2 := \{1, 2, \dots, n\} \end{gathered} \tag{1.6}$$

are given functions and signs Y_i, $i \in \Lambda_1$ are given as follows:

$$Y_i(t) := \big(y_i(t-\tau_1(t)), y_i(t-\tau_2(t)), \dots, y_i(t-\tau_n(t))\big), \quad t \in R,\ i \in \Lambda_1. \tag{1.7}$$

The extension to systems is a natural one; for example, many occurrences in nature involve two or more populations coexisting in an environment, with the model being best described by a system of differential equations (see [11]).

The aim of this paper is to investigate systems (1.4) and (1.5). In what follows we only discuss the existence of positive periodic solutions of system (1.4); similar results can be obtained for system (1.5). By using multiple fixed-point theorems (see Lemmas 2.1 and 2.2), which are different from those used in [8–10], we obtain the existence of multiple positive periodic solutions of system (1.4) (see Theorems 3.1, 4.1, and 4.3). Some examples are given also to illustrate our main theorems. Main results of this paper are new also even if $m = 1$ (see Remark 4.5).

This paper is organized as follows. In Section 2, we make some preliminaries. In Section 3, we derive existence result (see Theorem 3.1) for two positive periodic solutions of system (1.4). Example 3.2 is given below Theorem 3.1. The existence of three positive periodic solutions of system (1.4) is presented in Section 4 (see Theorems 4.1 and 4.3). Applications of Theorems 4.1 and 4.3 may be seen from Examples 4.2 and 4.4.

2. Preliminaries

We make the basic assumption throughout this paper that

$$\begin{gathered} T > 0 \quad \text{is a fixed constant;} \\ a_i \in C(R, [0,\infty)), \quad a(t) \not\equiv 0, \quad a_i(t) = a_i(t+T), \quad t \in R,\ i \in \Lambda_1; \\ f_i \in C\big(R \times [0,\infty)^{m\times n}, [0,\infty)\big), \quad i \in \Lambda_1; \\ f_i \quad \text{is } T\text{-periodic function in relative to } t,\ i \in \Lambda_1; \\ \tau_j \in C(R, [0,\infty)), \quad \tau_j(t+T) = \tau_j(t), \quad t \in R,\ j \in \Lambda_2. \end{gathered} \tag{2.1}$$

Let us now provide some preparations. Let S be a real Banach space and let P be a cone in S. A map α is said to be a nonnegative continuous concave functional on cone P if $\alpha : P \to [0,\infty)$ is continuous and

$$\alpha\big(tx + (1-t)y\big) \geq t\alpha(x) + (1-t)\alpha\big(y\big) \quad \forall x, y \in P,\ t \in [0,1]. \tag{2.2}$$

For numbers M, N such that $0 < M < N$, and a nonnegative continuous concave functional α on cone P, we define

$$\begin{gathered} P_M := \{x \in P : \|x\| < M\}, \\ P(\alpha, M) := \{x \in P : \alpha(x) < M\}, \\ P(\alpha, M, N) := \{x \in P : M \leq \alpha(x), \|x\| \leq N\}. \end{gathered} \tag{2.3}$$

Setting $u = (u_1, u_2, \ldots, u_n) \in [0,\infty)^n$, we define

$$|u|_0 := \max_{j\in\Lambda_2}\{u_j\}. \tag{2.4}$$

Write

$$D := \{y(t) : y \in C(R,R), y(t+T) = y(t)\},$$
$$E := \underbrace{D \times D \times \cdots \times D}_{m},$$
$$\|y\|_0 := \sup_{t\in[0,T]} |y(t)| \quad \text{for } y \in D, \tag{2.5}$$
$$\|y\| := \sum_{i\in\Lambda_1} \|y_i\|_0 \quad \text{for } y = (y_1, y_2, \ldots, y_m) \in E,$$
$$P := \{y = (y_1, y_2, \ldots, y_m) \in E : y_i(t) \geq \delta_i \|y_i\|_0, i \in \Lambda_1\},$$

where

$$\delta_i = e^{-\int_0^T a_i(s)ds}, \quad i \in \Lambda_1. \tag{2.6}$$

Then $(D, \|\cdot\|_0)$ and $(E, \|\cdot\|)$ are all Banach spaces and P is a cone in E. Set

$$M_1^i := \frac{1}{e^{\int_0^T a_i(s)ds} - 1}, \quad M_2^i := \frac{e^{\int_0^T a_i(s)ds}}{e^{\int_0^T a_i(s)ds} - 1}, \quad i \in \Lambda_1,$$
$$G_i(t,s) := \frac{e^{\int_t^s a_i(\xi)d\xi}}{e^{\int_0^T a_i(\xi)d\xi} - 1}, \quad (t,s) \in R \times [t, t+T], \ i \in \Lambda_1. \tag{2.7}$$

It is easy to see that for any $(t,s) \in R \times [t, t+T]$, functions $G_i(t,s)$, $i \in \Lambda_1$ have properties

$$M_1^i := G_i(t,t) \leq G_i(t,s) \leq G_i(t,t+T) := M_2^i, \quad i \in \Lambda_1,$$
$$\delta_i = \frac{M_1^i}{M_2^i} \leq \frac{G_i(t,s)}{G_i(t,t+T)} \leq 1, \quad i \in \Lambda_1. \tag{2.8}$$

Now we define an operator $A : E \to E$ as follows:

$$Ay(t) := (A_1(t), A_2(t), \ldots, A_m(t)), \quad t \in R, \ y = (y_1, y_2, \ldots, y_m) \in E, \tag{2.9}$$

where

$$A_i(t) := \int_t^{t+T} G_i(t,s) f_i(s, Y_1(s), Y_2(s), \ldots, Y_n(s)) ds, \quad t \in R, \ i \in \Lambda_1, \tag{2.10}$$

signs Y_i, $i \in \Lambda_1$ are given in (1.7) and we often use them in the remainder of this paper. It is easy to say that a T-periodic solution of operator equation

$$y = Ay, \tag{2.11}$$

on P, that is, a fixed point of operator A, is a T-positive periodic solution of system (1.4). So, our main results concerning multiple positive solutions of system (1.4) will arise as application of the following fixed-point theorem.

Lemma 2.1 (see [12]). *Let P be a cone in a real Banach space B. Let α and γ be increasing, nonnegative, continuous functionals on P, and let θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some $c > 0$ and $M > 0$,*

$$\gamma(x) \leq \theta(x) \leq \alpha(x), \quad \|x\| \leq M\gamma(x), \quad \forall x \in \overline{P(\gamma, c)}. \tag{2.12}$$

Suppose there exists a completely continuous operator $A : \overline{P(\gamma, c)} \to P$ and $0 < a < b < c$ such that

$$\theta(\lambda x) \leq \lambda\theta(x), \quad \text{for } 0 \leq \lambda \leq 1, \ x \in \partial P(\theta, b), \tag{2.13}$$

and

(i) *$\gamma(Ax) > c$, for all $x \in \partial P(\gamma, c)$;*

(ii) *$\theta(Ax) < b$, for all $x \in \partial P(\theta, b)$;*

(iii) *$P(\alpha, a) \neq \phi$, and $\alpha(Ax) > a$, for all $x \in \partial P(\alpha, a)$.*

Then A has at least two fixed points x_1 and x_2 belonging to $\overline{P(\gamma, c)}$ such that

$$\begin{aligned} &a < \alpha(x_1), \quad \text{with } \theta(x_1) < b, \\ &b < \theta(x_2), \quad \text{with } \gamma(x_2) < c. \end{aligned} \tag{2.14}$$

Lemma 2.2 (see [13]). *Let P be a cone in a real Banach space E, let $A : \overline{P_c} \to \overline{P_c}$ be completely continuous, and let α be a nonnegative continuous concave functional on P with $\alpha(x) \leq \|x\|$ for all $x \in \overline{P_c}$. Suppose that there exists $0 < d < M < N \leq c$ such that*

(i) *$\{x \in P(\alpha, M, N) : \alpha(x) > M\} \neq \phi$ and $\alpha(Ax) > M$ for $x \in P(\alpha, M, N)$;*

(ii) *$\|Ax\| < d$ for all $\|x\| \leq d$;*

(iii) *$\alpha(Ax) > M$ for $x \in P(\alpha, M, c)$ with $\|Ax\| > N$.*

Then A has at least three fixed points x_1, x_2, x_3 satisfying

$$\|x_1\| < d, \qquad M < \alpha(x_2), \qquad \|x_3\| > d, \qquad \alpha(x_3) < M. \tag{2.15}$$

3. Existence of Two Positive Solutions of System (1.4)

In this section, we apply Lemma 2.1 to establish Theorem 3.1, the existence result of two positive solutions of system (1.4). Example 3.2 will be given as an application of Theorem 3.1.

Theorem 3.1. *Assume that there exist numbers $0 < a < b < c$ such that the following three assumptions are satisfied.*

(H_1) *One has*

$$f_{k_0}(t, U_1, U_2, \ldots, U_m) > \frac{c+1}{TM_1}, \quad t \in R, \ \varphi \in \left[c, c\delta^{-1}\right], \tag{3.1}$$

where $k_0 \in \Lambda_1$ is fixed and

$$U_i := \left(U_i^1, U_i^2, \ldots, U_i^n\right), \quad U_i^j \in [0, \infty), \ i \in \Lambda_1, \ j \in \Lambda_2,$$
$$M_1 := \min\left\{M_1^1, M_1^2, \ldots, M_1^m\right\}, \qquad \delta := \min\{\delta_1, \delta_2, \ldots, \delta_m\}, \qquad \varphi := |U_1|_0 + |U_2|_0 + \cdots + |U_m|_0, \tag{3.2}$$

(H_2) *$t \in R$, $U_i \in [0, \infty)^n$, $i \in \Lambda_1$, and $\varphi \in [b, b\delta^{-1}]$ imply*

$$\sum_{i=1}^{m} f_i(t, U_1, U_2, \ldots, U_m) < \frac{b}{(1+\delta)TM_2}, \tag{3.3}$$

where

$$M_2 := \max\left\{M_2^1, M_2^2, \ldots, M_2^m\right\}, \tag{3.4}$$

(H_3) *$t \in R$, $U_i \in [0, \infty)^n$, $i \in \Lambda_1$, and $\varphi \in [\delta a, a]$ imply*

$$\sum_{i=1}^{m} f_i(t, U_1, U_2, \ldots, U_m) > \frac{a+1}{TM_1}. \tag{3.5}$$

Then system (1.4) *has at least two T-positive periodic solutions.*

Proof. We begin by defining

$$\gamma(y) := \theta(y) := \sum_{i=1}^{m} \min_{t \in [0,T]} y_i(t), \quad y = (y_1, y_2, \ldots, y_m) \in P,$$
$$\alpha(y) := \|y\|, \quad y \in P. \tag{3.6}$$

Clearly, α and γ are increasing, nonnegative, continuous functionals on P, and θ is nonnegative a continuous functional on P with $\theta(0) = 0$. Moreover, we observe that

$$\gamma(y) = \theta(y) \le \alpha(y), \quad \forall y \in P, \tag{3.7}$$

$$\|y\| \le \delta^{-1}\gamma(y), \quad \forall y \in P, \tag{3.8}$$

$$\theta(\lambda x) = \lambda\theta(x), \quad \text{for } 0 \le \lambda \le 1,\ x \in \partial P(\theta, b). \tag{3.9}$$

Now, we proceed to show that other conditions of Lemma 2.1 are also satisfied.

Firstly, we will show that

$$A : \overline{P(\gamma, c)} \longrightarrow P \quad \text{is completely continuous.} \tag{3.10}$$

In fact, we have from (2.3), for any $y = (y_1, y_2, \dots, y_m) \in \overline{P(\gamma, c)}$,

$$\|A_i\|_0 \le M_2^i \int_0^T f_i(s, Y_1(s), Y_2(s), \dots, Y_n(s))ds, \quad i \in \Lambda_1, \tag{3.11}$$

which yields

$$\begin{aligned} A_i(t) &\ge M_1^i \int_0^T f_i(s, Y_1(s), Y_2(s), \dots, Y_n(s))ds \\ &\ge \frac{M_1^i}{M_2^i}\|A_i\|_0 \\ &= \delta_i\|A_i\|_0, \quad t \in R,\ y \in \overline{P(\gamma, c)},\ i \in \Lambda_1. \end{aligned} \tag{3.12}$$

Hence $Ay \in P$ for all $y \in \overline{P(\gamma, c)}$. Furthermore, we know from the continuity of functions $f_i(\cdot, \dots, \cdot)$, $a_i(\cdot)$, $\Gamma_i(\cdot, \cdot)$, $i \in \Lambda_1$ that the operator A is completely continuous. Hence, we conclude that (3.10) holds.

Secondly, let us prove

$$\gamma(Ay) > c, \quad \forall y \in \partial P(\gamma, c). \tag{3.13}$$

For any $y = (y_1, y_2, \dots, y_m) \in \partial P(\gamma, c)$, so that $\gamma(y) = c$, we get, in view of (1.7), (2.4) and (3.8),

$$\begin{aligned} c = \sum_{i=1}^m \min_{t\in[0,T]} y_i(t) &\le \sum_{i=1}^m |Y_i(t)|_0, \quad t \in R, \\ \sum_{i=1}^m |Y_i(t)|_0 \le \sum_{i=1}^m \|y_i\|_0 &\le \delta^{-1}\gamma(y) = c\delta^{-1}, \quad t \in R. \end{aligned} \tag{3.14}$$

Consequently, for any $y \in \partial P(\gamma, c)$, condition (H_1) and (3.14) imply that

$$\begin{aligned}\gamma(Ay) &= \sum_{i=1}^{m} \min_{t\in[0,T]} \int_t^{t+T} G_i(t,s) f_i(s, Y_1(s), Y_2(s), \dots, Y_m(s))ds \\ &\geq \min_{t\in[0,T]} \int_t^{t+T} G_{k_0}(t,s) f_{k_0}(s, Y_1(s), Y_2(s), \dots, Y_m(s))ds \\ &\geq \min_{t\in[0,T]} \int_t^{t+T} M_1^{k_0} \frac{c+1}{TM_1} ds \\ &> c,\end{aligned} \tag{3.15}$$

which gives (3.13).

Thirdly, we verify

$$\theta(Ay) < b, \quad \forall y \in \partial P(\theta, b). \tag{3.16}$$

As before, $\theta(y) = b$ and (1.7), (2.4), and (3.8) also tell us that

$$b \leq \sum_{i=1}^{m} |Y_i(t)|_0 \leq b\delta^{-1}, \quad t \in R. \tag{3.17}$$

Then condition (H_2), (3.17), and the fact that the function min is concave imply

$$\begin{aligned}\theta(Ay) &= \sum_{i=1}^{m} \min_{t\in[0,T]} \int_t^{t+T} G_i(t,s) f_i(s, Y_1(s), Y_2(s), \dots, Y_m(s))ds \\ &\leq \min_{t\in[0,T]} \int_t^{t+T} \sum_{i=1}^{m} M_2^i f_i(s, Y_1(s), Y_2(s), \dots, Y_m(s))ds \\ &\leq \min_{t\in[0,T]} \int_t^{t+T} M_2 \frac{b}{(1+\delta)TM_2} ds \\ &< b.\end{aligned} \tag{3.18}$$

Thus (3.16) holds.

Finally, let us prove

$$P(\alpha, a) \neq \phi, \qquad \alpha(Ay) > a, \quad \forall y \in \partial P(\alpha, a). \tag{3.19}$$

Obviously,

$$y = (y_1, y_2, \dots, y_m) = \left(\frac{a}{m+1}, \frac{a}{m+1}, \dots, \frac{a}{m+1}\right) \in P(\alpha, a). \tag{3.20}$$

In addition, for any $y \in \partial P(\alpha, a)$, we get

$$\delta a \leq \sum_{i=1}^{m} |Y_i(t)|_0 \leq a, \quad t \in R \tag{3.21}$$

since $\sum_{i=1}^{m} \|y_i\|_0 = a$ and $y = (y_1, y_2, \dots, y_m) \in P$. So we have from condition (H_3) that

$$\begin{aligned} \alpha(Ay) &= \sum_{i=1}^{m} \max_{t\in[0,T]} \int_t^{t+T} G_i(t,s) f_i(s, Y_1(s), Y_2(s), \dots, Y_m(s)) ds \\ &\geq \max_{t\in[0,T]} \int_t^{t+T} \sum_{i=1}^{m} M_1^i f_i(s, Y_1(s), Y_2(s), \dots, Y_m(s)) ds \\ &\geq \max_{t\in[0,T]} \int_t^{t+T} M_1 \frac{a+1}{TM_1} ds \\ &> a. \end{aligned} \tag{3.22}$$

Hence (3.19) holds.

To sum up, (3.6)–(3.10), (3.13), (3.16), and (3.19) tell us that conditions of Lemma 2.1 all hold here. Consequently, system (1.4) has at least two T-positive periodic solutions $y^1 = (y_1^1, y_2^1, \dots, y_m^1)$ and $y^2 = (y_1^2, y_2^2, \dots, y_m^2)$ belonging to $\overline{P(\gamma, c)}$ such that

$$\begin{gathered} a < \left\| y^1 \right\|, \quad \text{with } \sum_{i=1}^{m} \min_{t\in[0,T]} y_i^1 < b, \\ b < \sum_{i=1}^{m} \min_{t\in[0,T]} y_i^2 < c. \end{gathered} \tag{3.23}$$

□

As an application of Theorem 3.1, we provide the following example. For convenience, all examples in this paper are given when $n = 1, m = 2$.

Example 3.2. Assume that $\tau_1 > 0$ is a fixed constant. Consider the following system:

$$\begin{aligned} y_1'(t) &= -(\ln 2)|\sin t| y_1(t) + \frac{1}{4\pi} e^{|\cos t|-1} g_1(t, y_1(t-\tau_1), y_2(t-\tau_1)), \\ y_2'(t) &= -\frac{1}{2}(\ln 3)|\cos t| y_2(t) + \frac{1}{4\pi} e^{|\sin t|-1} g_2(t, y_1(t-\tau_1), y_2(t-\tau_1)), \end{aligned} \tag{3.24}$$

where

$$g_1(t, u_1, u_2) := \begin{cases} 12e, & t \in R, \ u_1 + u_2 \in [0, 160], \\ \dfrac{2401e(u_1 + u_2 - 160) + 12e(199 - u_1 - u_2)}{39}, & t \in R, \ u_1 + u_2 \in [160, 199], \\ 2401e, & t \in R, \ u_1 + u_2 \in [199, \infty). \end{cases}$$

$$g_2(t,u_1,u_2) := \begin{cases} 42\dfrac{2}{3}, & t \in R,\ u_1+u_2 \in [0,160], \\ \dfrac{(u_1+u_2-160)+42(2/3)(199-u_1-u_2)}{39}, & t \in R,\ u_1+u_2 \in [160,199], \\ 1, & t \in R,\ u_1+u_2 \in [199,\infty). \end{cases} \tag{3.25}$$

We set

$$\begin{gathered} a = 1, \qquad b = 40, \qquad c = 199, \\ a_1(t) := (\ln 2)|\sin t|, \quad a_2(t) := \frac{1}{2}(\ln 3)|\cos t|, \quad t \in R, \\ f_1(t, y_1(t-\tau), y_2(t-\tau)) = \frac{1}{4\pi}e^{|\cos t|-1}g_1(t, y_1(t-\tau), y_2(t-\tau)), \quad t \in R, \\ f_2(t, y_1(t-\tau), y_2(t-\tau)) = \frac{1}{4\pi}e^{|\sin t|-1}g_2(t, y_1(t-\tau), y_2(t-\tau)), \quad t \in R \end{gathered} \tag{3.26}$$

Then

$$T = \pi, \qquad \delta = \frac{1}{4}, \qquad M_1 = \frac{1}{3}, \qquad M_2 = \frac{3}{2}. \tag{3.27}$$

We may verify that conditions (H_1), (H_2), and (H_3) are all satisfied. Hence, Theorem 3.1 tells us that system (3.24) has at least two π-positive periodic solutions $y^1 = (y_1^1, y_2^1)$ and $y^2 = (y_1^2, y_2^2)$ such that

$$\begin{gathered} 1 < \max_{t\in[0,\pi]} y_1^1 + \max_{t\in[0,\pi]} y_2^1,\ \min_{t\in[0,\pi]} y_1^1 + \min_{t\in[0,\pi]} y_2^1 < 40, \\ 40 < \min_{t\in[0,\pi]} y_1^2 + \min_{t\in[0,\pi]} y_2^2 < 199. \end{gathered} \tag{3.28}$$

4. Existence of Three Positive Solutions of System (1.4)

For the sake of convenience we list the assumptions to be used in this section as follows.

(H_4) There exists a number $C_1 > 0$ such that

$$f_i(t, U_1, U_2, \ldots, U_m) < \frac{C_1}{mTM_2}, \quad t \in R,\ U_i \in [0,\infty)^n,\ \varphi \le C_1,\ i \in \Lambda_1, \tag{4.1}$$

where φ and U_i, $i \in \Lambda_1$ are given in (H_1).

(H_5) There exist numbers $C_2 > C_1$ and $i_0 \in \Lambda_1$ such that

$$f_{i_0}(t, U_1, U_2, \ldots, U_m) > \frac{C_3}{TM_1}, \quad t \in R,\ U_i \in [0,\infty)^n,\ C_2 \le \varphi \le C_3, \tag{4.2}$$

where

$$C_3 := \frac{C_2(1+\delta)}{\delta}. \tag{4.3}$$

(H_6) One has

$$\lim_{\varphi \to \infty} \max_{t \in [0,T]} \frac{f_i(t, U_1, U_2, \ldots, U_m)}{\varphi} < \frac{1}{mTM_2}, \quad U_i \in [0, \infty)^n, \ i \in \Lambda_1. \tag{4.4}$$

(H_7) There exists a number $C_4 > C_3$ such that,

$$f_i(t, U_1, U_2, \ldots, U_m) \le \frac{C_4}{mTM_2}, \quad t \in R, \ U_i \in [0, \infty)^n, \ \varphi \le C_4, \ i \in \Lambda_1. \tag{4.5}$$

Let us now state the first existence result of three positive solutions of system (1.4).

Theorem 4.1. *Assume that conditions (H_4), (H_5), and (H_6) hold. Then system (1.4) has at least three T-positive periodic solutions.*

Proof. Firstly, we set

$$\beta(y) := \sum_{i=1}^{m} \min_{t \in [0,T]} y_i(t), \quad y = (y_1, y_2, \ldots, y_m) \in P. \tag{4.6}$$

Obviously, β is a nonnegative continuous concave functional on P and

$$\beta(y) \le \|y\|, \quad y = (y_1, y_2, \ldots, y_m) \in \overline{P_M} \text{ for any } M > 0. \tag{4.7}$$

Secondly, condition (H_6) implies that there exists a number $C_5 \ge C_3$ such that

$$A : \overline{P_{C_5}} \longrightarrow \overline{P_{C_5}} \quad \text{is completely continuous.} \tag{4.8}$$

In fact, we know from condition (H_6) that there exist numbers

$$\tau_i > 0, \quad 0 < \sigma_i < \frac{1}{mTM_2}, \quad i \in \Lambda_1 \tag{4.9}$$

satisfying

$$\max_{t \in [0,T]} \frac{f_i(t, Y_1(t), Y_2(t), \ldots, Y_m(t))}{\|y\|} \le \sigma_i, \quad \|y\| \ge \tau_i, \ i \in \Lambda_1. \tag{4.10}$$

So

$$f_i(t, Y_1(t), Y_2(t), \ldots, Y_m(t)) \le \sigma_i \|y\|, \quad t \in [0,T],\ \|y\| \ge \tau_i,\ i \in \Lambda_1. \tag{4.11}$$

Set

$$C_6^i := \max_{t\in[0,T]} \{f_i(t, Y_1(t), Y_2(t), \ldots, Y_m(t))\}, \quad t \in [0,T],\ \|y\| \le \tau_i,\ i \in \Lambda_1. \tag{4.12}$$

Then

$$f_i(t, Y_1(t), Y_2(t), \ldots, Y_m(t)) \le C_6^i + \sigma_i \|y\|, \quad t \in [0,T],\ y \in P,\ i \in \Lambda_1. \tag{4.13}$$

Let us choose

$$C_5 \ge \max\left\{C_3, \frac{mC_6^1 T M_2^1}{1 - mTM_2^1\sigma_1}, \frac{mC_6^2 T M_2^2}{1 - mTM_2^2\sigma_2}, \ldots, \frac{mC_6^m T M_2^m}{1 - mTM_2^m\sigma_m}\right\}. \tag{4.14}$$

Then for any $y = (y_1, y_2, \ldots, y_m) \in \overline{P_{C_5}}$, we have

$$\begin{aligned} \|Ay\| &= \sum_{i=1}^{m} \max_{t\in[0,T]} \int_t^{t+T} G_i(t,s) f_i(t, Y_1(s), Y_2(s)), \ldots, Y_m(s)) ds \\ &\le \sum_{i=1}^{m} \max_{t\in[0,T]} \int_t^{t+T} G_i(t, t+T)\left[\sigma_i \|y\| + C_6^i\right] ds \\ &\le \sum_{i=1}^{m} \left(\sigma_i C_5 + C_6^i\right) T M_2^i \\ &\le C_5, \end{aligned} \tag{4.15}$$

which implies $A(y) \in \overline{P_{C_5}}$ for all $y \in \overline{P_{C_5}}$. Moreover, we know from the proof of (3.10) that $A : \overline{P_{C_5}} \to \overline{P_{C_5}}$ is completely continuous.

Thirdly, let us show that numbers

$$0 < C_1 < C_2 < C_3 \le C_5 \tag{4.16}$$

satisfy conditions (i), (ii), and (iii) of Lemma 2.2.

Step 1. We prove that

$$\{y \in P(\beta, C_2, C_3) : \beta(y) > C_2\} \ne \phi, \quad \beta(Ay) > C_2, \quad y \in P(\beta, C_2, C_3). \tag{4.17}$$

Clearly, $y = (y_1, y_2, \ldots, y_m) = (C_3/m, C_3/m, \ldots, C_3/m) \in \{P(\beta, C_2, C_3) : \beta(y) > C_2\}$. Moreover, for any $y = (y_1, y_2, \ldots, y_m) \in P(\beta, C_2, C_3)$, we have

$$C_2 \leq \sum_{i=1}^{m} \min_{t\in[0,T]} y_i(t) \leq \sum_{i=1}^{m} |Y_i(t)|_0 \leq \|y\| \leq C_3. \tag{4.18}$$

Then condition (H_5), (4.6), and (4.18) imply that

$$\begin{aligned} \beta(Ay) &= \sum_{i=1}^{m} \min_{t\in[0,T]} \int_t^{t+T} G_i(t,s) f_i(s, Y_1(s), Y_2(s), \ldots, Y_m(s)) ds \\ &\geq \min_{t\in[0,T]} \int_t^{t+T} G_{i_0}(t,s) f_{i_0}(s, Y_1(s), Y_2(s), \ldots, Y_m(s)) ds \\ &\geq \min_{t\in[0,T]} \int_t^{t+T} M_1^{i_0} \frac{C_3}{TM_1} ds \\ &\geq C_3 \\ &> C_2, \end{aligned} \tag{4.19}$$

which gives $\beta(Ay) > C_2$ for $y \in P(\beta, C_2, C_3)$. And then we arrive at (4.17).

Step 2. Condition (H_4) implies

$$\|Ay\| < C_1 \quad \text{for } \|y\| \leq C_1. \tag{4.20}$$

In fact, for any $y = (y_1, y_2, \ldots, y_m) \in \overline{P_{C_1}}$, that is, $\|y\| \leq C_1$, from

$$|Y_i(t)|_0 \leq \|y_i\|_0, \quad t \in [0,T], \ i \in \Lambda_1, \tag{4.21}$$

and condition (H_4) we have

$$\begin{aligned} \|Ay\| &= \sum_{i=1}^{m} \max_{t\in[0,T]} \int_t^{t+T} G_i(t,s) f_i(s, Y_1(s), Y_2(s), \ldots, Y_m(s)) ds \\ &\leq \sum_{i=1}^{m} \max_{t\in[0,T]} \int_t^{t+T} M_2^i f_i(s, Y_1(s), Y_2(s), \ldots, Y_m(s)) ds \\ &< m \max_{t\in[0,T]} \int_t^{t+T} M_2 \frac{C_1}{mTM_2} ds \\ &= C_1, \end{aligned} \tag{4.22}$$

which yields (4.20).

Step 3. $\beta(Ay) > C_2$ for $y \in P(\beta, C_2, C_5)$ with $\|Ay\| > C_3$. This is the case because $Ay \in P$ implies

$$\beta(Ay) = \sum_{i=1}^{m} \min_{t\in[0,T]} A_i(t) \geq \delta \sum_{i=1}^{m} \|A_i\|_0 = \delta \|Ay\| > \delta C_3 > C_2. \tag{4.23}$$

At present, we may say that hypotheses of Lemma 2.2 (the Leggett-Willaims theorem) are satisfied. Hence system (1.4) has at least three T-positive periodic solutions:

$$u = (u_1, u_2, \ldots, u_m), \qquad v = (v_1, v_2, \ldots, v_m), \qquad w = (w_1, w_2, \ldots, w_m), \tag{4.24}$$

such that

$$\begin{aligned} &\|u\| < C_1, \qquad C_2 < \sum_{i=1}^{m} \min_{t\in[0,T]} v_i, \\ &\|w\| > C_1, \qquad \sum_{i=1}^{m} \min_{t\in[0,T]} w_i < C_2. \end{aligned} \tag{4.25}$$

We give the following example to illustrate Theorem 4.1. □

Example 4.2. Consider the following system:

$$\begin{aligned} y_1'(t) &= -\frac{1}{4\pi}(\ln 3)(2+\cos t) y_1(t) + \frac{1}{4\pi} e^{(\sin t)-1} g_1(t, y_1(t-\tau_2), y_2(t-\tau_2)), \\ y_2'(t) &= -\frac{1}{4\pi}(\ln 2)(2+\sin t) y_2(t) + \frac{1}{4\pi} e^{(\cos t)-1} g_2(t, y_1(t-\tau_2), y_2(t-\tau_2)), \end{aligned} \tag{4.26}$$

where $\tau_2 > 0$ is a fixed constant and

$$g_1(t, u_1, u_2) := \begin{cases} \dfrac{2}{5}, & t \in R,\ u_1 + u_2 \in [0,1], \\ \dfrac{(3 - u_1 - u_2)}{5} + (36e^2 + 1)(u_1 + u_2 - 1), & t \in R,\ u_1 + u_2 \in [1,3], \\ 72e^2 + 2, & t \in R,\ u_1 + u_2 \in [3,12], \\ \dfrac{5}{12}(u_1 + u_2) - 3 + 72e^2, & t \in R,\ u_1 + u_2 \in [12,\infty), \end{cases} \tag{4.27}$$

$$g_2(t, u_1, u_2) := \text{constant} < \frac{7}{15}, \quad (t, u_1, u_2) \in R \times [0,\infty) \times [0,\infty).$$

We set

$$\begin{gathered}
a_1(t) := \frac{1}{4\pi}(\ln 3)(2+\cos t), \quad a_2(t) := \frac{1}{4\pi}(\ln 2)(2+\sin t), \quad t \in R, \\
f_1(t, y_1(t-\tau), y_2(t-\tau)) := \frac{1}{4\pi}e^{(\sin t)-1}g_1(t, y_1(t-\tau), y_2(t-\tau)), \quad t \in R, \\
f_2(t, y_1(t-\tau), y_2(t-\tau)) := \frac{1}{4\pi}e^{(\cos t)-1}g_2(t, y_1(t-\tau), y_2(t-\tau)), \quad t \in R, \\
C_1 := 1, \qquad C_2 := 3.
\end{gathered} \tag{4.28}$$

Then

$$T = 2\pi, \qquad \delta = \frac{1}{3}, \qquad M_1 = \frac{1}{3}, \qquad M_2 = 2. \tag{4.29}$$

We may verify also that conditions (H_4), (H_5), and (H_6) hold. Hence, Theorem 4.1 tells us that system (4.26) has at least three 2π-positive periodic solutions:

$$y^1 = \left(y_1^1, y_2^1\right), \qquad y^2 = \left(y_1^2, y_2^2\right), \qquad y^3 = \left(y_1^3, y_2^3\right) \tag{4.30}$$

such that

$$\begin{gathered}
\left\|y^1\right\| < 1, \qquad 3 < \min_{t\in[0,2\pi]} y_1^2 + \min_{t\in[0,2\pi]} y_2^2, \\
\left\|y^3\right\| > 1, \qquad \min_{t\in[0,2\pi]} y_1^3 + \min_{t\in[0,2\pi]} y_2^3 < 3.
\end{gathered} \tag{4.31}$$

The second existence result of three positive solutions of system (1.4) is as follows.

Theorem 4.3. *Assume that conditions* (H_4), (H_5), *and* (H_7) *hold. Then system* (1.4) *has at least three T-positive periodic solutions.*

Proof. If we can get (4.8) with C_5 replaced by C_4 in this case, then the proof is complete. In fact, for any $y = (y_1, y_2, \ldots, y_m) \in \overline{P_{C_4}}$, condition (H_7) implies

$$\begin{aligned}
\|Ay\| &= \sum_{i=1}^{m} \max_{t\in[0,T]} \int_t^{t+T} G_i(t,s) f_i(s, Y_1(s), Y_2(s), \ldots, Y_m(s))ds \\
&\le \sum_{i=1}^{m} \max_{t\in[0,T]} \int_t^{t+T} G_i(t,t+T) f_i(s, Y_1(s), Y_2(s), \ldots, Y_m(s))ds \\
&\le m \max_{t\in[0,T]} \int_t^{t+T} M_2 \frac{C_4}{mTM_2} ds \\
&\le C_4.
\end{aligned} \tag{4.32}$$

Then

$$A : \overline{P_{C_4}} \longrightarrow \overline{P_{C_4}}, \tag{4.33}$$

as desired. This ends the proof. □

The following example is an application of Theorem 4.3.

Example 4.4. Consider system

$$\begin{aligned} y_1'(t) &= -\frac{1}{4\pi}(\ln 2)(2+\cos t)y_1(t) + \frac{1}{4\pi}e^{(\cos t)-1}g_1\big(t, y_1(t-\tau_3), y_2(t-\tau_3)\big), \\ y_2'(t) &= -\frac{1}{4\pi}(\ln 2)(2+\sin t)y_2(t) + \frac{1}{4\pi}e^{(\cos t)-1}g_2\big(t, y_1(t-\tau_3), y_2(t-\tau_3)\big), \end{aligned} \tag{4.34}$$

where $\tau_3 > 0$ is a fixed constant and

$$\begin{aligned} g_1(t,u_1,u_2) &:= \begin{cases} \dfrac{2(1+u_2)}{3(u_1+u_2+1)}, & t \in R,\ u_1+u_2 \in [0,2], \\[2ex] \dfrac{(1+u_2)(4-u_1-u_2)}{3(u_1+u_2+1)} + \dfrac{1}{2}\big(24e^2+1\big)(u_1+u_2-2), & t \in R,\ u_1+u_2 \in [2,4], \\[2ex] 24e^2+1, & t \in R,\ u_1+u_2 \in [4,\infty), \end{cases} \\ g_2(t,u_1,u_2) &:= \text{constant} < 1, \quad (t,u_1,u_2) \in R \times [0,\infty) \times [0,\infty). \end{aligned} \tag{4.35}$$

If we set

$$\begin{aligned} a_1(t) &:= \frac{1}{4\pi}(\ln 2)(2+\cos t), \quad a_2(t) := \frac{1}{4\pi}(\ln 2)(2+\sin t), \quad t \in R, \\ f_i\big(t, y_1(t-\tau), y_2(t-\tau)\big) &= \frac{1}{4\pi}e^{(\cos t)-1}g_i\big(t, y_1(t-\tau), y_2(t-\tau)\big), \quad t \in R,\ i = 1,2, \end{aligned} \tag{4.36}$$

then

$$T = 2\pi, \qquad \delta = \frac{1}{2}, \qquad M_1 = 1, \qquad M_2 = 2. \tag{4.37}$$

We choose

$$C_1 := 2, \qquad C_2 := 4, \qquad C_4 := 50e^2. \tag{4.38}$$

Then assumptions (H_4), (H_5), and (H_7) hold. Hence we know from Theorem 4.3 that system (4.34) has at least three 2π-positive periodic solutions:

$$y^1 = \left(y_1^1, y_2^1\right), \qquad y^2 = \left(y_1^2, y_2^2\right), \qquad y^3 = \left(y_1^3, y_2^3\right) \tag{4.39}$$

such that

$$\begin{aligned} &\left\|y^1\right\| < 2, \qquad 4 < \min_{t\in[0,2\pi]} y_1^2 + \min_{t\in[0,2\pi]} y_2^2, \\ &\left\|y^3\right\| > 2, \qquad \min_{t\in[0,2\pi]} y_1^3 + \min_{t\in[0,2\pi]} y_2^3 < 4. \end{aligned} \tag{4.40}$$

We end this paper by the following remark.

Remark 4.5. For a general positive integer $m \geq 2$, main results of this paper do not appear in former literatures as we know. Comparing with the existing results, our Theorems 3.1, 4.1, and 4.3 are new also when $m = 1$.

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