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Research Article

Bifurcation Analysis in a Kind of Fourth-Order Delay Differential Equation

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A kind of fourth-order delay differential equation is considered. Firstly, the linear stability is investigated by analyzing the associated characteristic equation. It is found that there are stability switches for time delay and Hopf bifurcations when time delay cross through some critical values. Then the direction and stability of the Hopf bifurcation are determined, using the normal form method and the center manifold theorem. Finally, some numerical simulations are carried out to illustrate the analytic results.

1. Introduction

Sadek [1] has considered the following fourth-order delay differential equation:

$$x^{(4)}(t) + \alpha_1 \dddot{x}(t) + \alpha_2 \ddot{x}(t) + \phi(\dot{x}(t-\tau)) + f(x(t)) = 0. \tag{1.1}$$

By constructing Lyapunov functionals, it was given a group of conditions to ensure that the zero solution of (1.1) is globally asymptotically stable when the delay τ is suitable small, but if the sufficient conditions are not satisfied, what are the behaviors of the solutions? This is a interesting question in mathematics. The purpose of the present paper is to study the dynamics of (1.1) from bifurcation. We will give a detailed analysis on the above mentioned question. By regarding the delay τ as a bifurcation parameter, we analyze the distribution of the roots of the characteristic equation of (1.1) and obtain the existence of stability switches and Hopf bifurcation when the delay varies. Then by using the center manifold theory and normal form method, we derive an explicit algorithm for determining the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions.

We would like to mention that there are several articles on the stability of fourth-order delay differential equations, we refer the readers to [1–8] and the references cited therein.

The rest of this paper is organized as follows. In Section 2, we firstly focus mainly on the local stability of the zero solution. This analysis is performed through the study of a characteristic equation, which takes the form of a fourth-degree exponential polynomial. Using the approach of Ruan and Wei [9], we show that the stability of the zero solution can be destroyed through a Hopf bifurcation. In Section 3, we investigate the stability and direction of bifurcating periodic solutions by using the normal form theory and center manifold theorem presented in Hassard et al. [10]. In Section 4, we illustrate our results by numerical simulations. Section 5 with conclusion completes the paper.

2. Stability and Hopf Bifurcation

In this section, we will study the stability of the zero solution and the existence of Hopf bifurcation by analyzing the distribution of the eigenvalues. For convenience, we give the following assumptions:

$$\tau > 0, \qquad \alpha_1 > 0, \qquad \alpha_2 > 0, \qquad \phi(0) = 0, \qquad f(0) = 0, \tag{$\mathbf{H_1}$}$$

with ϕ and f are both continuous functions and those three-order differential quotients at origin are existent. We rewrite (1.1) as the following form:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= u, \\ \dot{u} &= v, \\ \dot{v} &= -\alpha_2 u - \alpha_1 v - f(x) - \phi(y(t-\tau)). \end{aligned} \tag{2.1}$$

It is easy to see that $(0,0,0,0)$ is the only trivial solution to the system (2.1) and the linearization around $(0,0,0,0)$ is given by

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= u, \\ \dot{u} &= v, \\ \dot{v} &= -f'(0)x - \alpha_2 u - \alpha_1 v - \phi'(0)y(t-\tau). \end{aligned} \tag{2.2}$$

Its characteristic equation is

$$\lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \phi'(0)\lambda e^{-\lambda\tau} + f'(0) = 0. \tag{2.3}$$

Lemma 2.1. *Suppose* $(\mathbf{H_1})$ *and*

$$\alpha_1\alpha_2 - \phi'(0) > 0, \quad f'(0) > 0, \quad \phi'(0)\left(\alpha_1\alpha_2 - \phi'(0)\right) - \alpha_1^2 f'(0) > 0 \tag{$\mathbf{H_2}$}$$

are satisfied. Then the trivial solution $(0,0,0,0)$ *is asymptotically stable when* $\tau = 0$.

Proof. When $\tau = 0$, (2.3) becomes

$$\lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \phi'(0)\lambda + f'(0) = 0. \tag{2.4}$$

By Routh-Hurwitz criterion, all roots of (2.4) have negative real parts if and only if

$$\alpha_1 > 0, \quad \alpha_1\alpha_2 - \phi'(0) > 0, \quad f'(0) > 0, \quad \phi'(0)\left(\alpha_1\alpha_2 - \phi'(0)\right) - \alpha_1^2 f'(0) > 0. \tag{2.5}$$

The conclusion follows from $(\mathbf{H_1})$ and $(\mathbf{H_2})$.

Let $i\omega$ $(\omega > 0)$ be a root of (2.3), then we have

$$\omega^4 - i\alpha_1\omega^3 - \alpha_2\omega^2 + \phi'(0)(i\omega)e^{-i\omega\tau} + f'(0) = 0. \tag{2.6}$$

Separating the real and imaginary parts gives

$$\begin{gathered} -\omega^4 + \alpha_2\omega^2 - f'(0) = \phi'(0)\omega\sin\omega\tau, \\ \alpha_1\omega^3 = \phi'(0)\omega\cos\omega\tau. \end{gathered} \tag{2.7}$$

Adding up the squares of both equations yields

$$\omega^8 + \left(\alpha_1^2 - 2\alpha_2\right)\omega^6 + \left(\alpha_2^2 + 2f'(0)\right)\omega^4 - \left(\phi'^2(0) + 2\alpha_2 f'(0)\right)\omega^2 + f'^2(0) = 0. \tag{2.8}$$

Let $V = \omega^2$, and denote

$$P = \alpha_1^2 - 2\alpha_2, \qquad Q = \alpha_2^2 + 2f'(0), \qquad K = -\phi'^2(0) - 2\alpha_2 f'(0). \tag{2.9}$$

Then (2.8) becomes

$$V^4 + PV^3 + QV^2 + KV + f'^2(0) = 0. \tag{2.10}$$

Set

$$h(V) = V^4 + PV^3 + QV^2 + KV + f'^2(0). \tag{2.11}$$

Then we have

$$h'(V) = 4V^3 + 3PV^2 + 2QV + K. \tag{2.12}$$

Consider

$$4V^3 + 3PV^2 + 2QV + K = 0. \tag{2.13}$$

Let $U = V + (3/4)P$. Then (2.13) becomes

$$U^3 + P_1 U + Q_1 = 0, \tag{2.14}$$

where

$$P_1 = \frac{Q}{2} - \frac{3}{16}P^2, \qquad Q_1 = \frac{1}{32}P^3 - \frac{1}{8}PQ + K. \tag{2.15}$$

Define

$$\begin{aligned}
M &= \left(\frac{Q_1}{2}\right)^2 + \left(\frac{P_1}{3}\right)^3, \\
\sigma &= \frac{-1 + \sqrt{3}i}{2}, \\
U_1 &= \sqrt[3]{-\frac{Q_1}{2} + \sqrt{M}} + \sqrt[3]{-\frac{Q_1}{2} - \sqrt{M}}, \\
U_2 &= \sqrt[3]{-\frac{Q_1}{2} + \sqrt{M}\sigma} + \sqrt[3]{-\frac{Q_1}{2} - \sqrt{M}\sigma^2}, \\
U_2 &= \sqrt[3]{-\frac{Q_1}{2} + \sqrt{M}\sigma^2} + \sqrt[3]{-\frac{Q_1}{2} - \sqrt{M}\sigma}, \\
V_i &= U_i - \frac{3}{4}P, \quad i = 1, 2, 3.
\end{aligned} \tag{2.16}$$

Then by Lemma 2.2 in Li and Wei [11], we have the following results on the distribution of the roots of (2.10). □

Lemma 2.2. (i) *If $M \geq 0$, then* (2.10) *has positive roots if and only if $V_1 > 0$ and $h(V_1) < 0$.*

(ii) *If $M < 0$, then* (2.10) *has positive roots if and only if there exists at least one $V^* \in \{V_1, V_2, V_3\}$, such that $V^* > 0$ and $h(V^*) \leq 0$.*

Without loss of generality, we assume that equation $h(V) = 0$ has four positive roots denoted by V_1, V_2, V_3, and V_4, respectively. Then (2.8) also has four positive roots, say $\omega_i = \sqrt{V_i}$, $i = 1, 2, 3, 4$.

From (2.7), and conditions ($\mathbf{H_1}$) and ($\mathbf{H_2}$), we have that

$$\cos\omega\tau = \frac{\alpha_1\omega^2}{\phi'(0)} > 0. \tag{2.17}$$

Hence, we define

$$\tau_k^j = \frac{1}{\omega_k}\left[\arccos\frac{\alpha_1\omega_k^2}{\phi'(0)} + 2j\pi\right], \quad k = 1,2,3,4, \ j = 0,1,\ldots, \tag{2.18}$$

when

$$\begin{gathered}\frac{-\omega_k^4 + \alpha_2\omega_k^2 - f'(0)}{\phi'(0)\omega_k} > 0, \\ \tau_k^j = \frac{1}{\omega_k}\left[-\arccos\frac{\alpha_1\omega_k^2}{\phi'(0)} + 2(j+1)\pi\right], \quad k = 1,2,3,4, \ j = 0,1,\ldots,\end{gathered} \tag{2.19}$$

when

$$\frac{-\omega_k^4 + \alpha_2\omega_k^2 - f'(0)}{\phi'(0)\omega_k} < 0. \tag{2.20}$$

Let

$$\lambda(\tau) = \alpha(\tau) + i\beta(\tau) \tag{2.21}$$

be the root of (2.3) satisfying $\alpha(\tau_k^j) = 0$, $\beta(\tau_k^j) = \omega_k$.

Lemma 2.3. *Suppose $h'(V_i) \neq 0$ ($i = 1,2,3,4$). If $\tau = \tau_k^j$, then $\pm i\omega_k$ is a pair of simple purely imaginary roots of* (2.3)*; and $\mathrm{Re}(d\lambda(\tau_k^j)/d\tau) > 0$ when $k = 2,4$; and $\mathrm{Re}(d\lambda(\tau_k^j)/d\tau) < 0$ when $k = 1,3$.*

Proof. Substituting $\lambda(\tau)$ into (2.3) and differentiating with respet to τ gives

$$\begin{aligned}\frac{d\lambda}{d\tau} &= \frac{\phi'(0)\lambda^2 e^{-\lambda\tau}}{4\lambda^3 + 3\alpha_1\lambda^2 + 2\alpha_2\lambda + \phi'(0)e^{-\lambda\tau} - \tau\phi'(0)\lambda e^{-\lambda\tau}} \\ &= \frac{-\lambda^2\left(\lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + f'(0)\right)}{\tau\lambda^5 + \left(3 + \tau\alpha_1\right)\lambda^4 + \left(3\alpha_1 + \tau\alpha_2\right)\lambda^3 + \alpha_2\lambda^2 + f'(0)\tau\lambda - f'(0)}.\end{aligned} \tag{2.22}$$

Then

$$\begin{aligned}
&\operatorname{Re}\frac{d\lambda(\tau^j)}{d\tau}\\
&=\frac{(\omega_k^6-\alpha_2\omega_k^4+f'(0)\omega_k^2)\left((3+\tau\alpha_1)\omega_k^4-\alpha_2\omega_k^2-f'(0)\right)-\alpha_1\omega_k^5\left(\tau\omega_k^5-(3\alpha_1+\tau\alpha_2)\omega_k^3+f'(0)\tau\omega_k\right)}{\left((3+\tau\alpha_1)\omega_k^4-\alpha_2\omega_k^2-f'(0)\right)^2+\left(\tau\omega_k^5-(3\alpha_1+\tau\alpha_2)\omega_k^3+f'(0)\tau\omega_k\right)^2}\\
&=\frac{\omega_k^2}{\Delta}h'(V_k),
\end{aligned} \tag{2.23}$$

where

$$\Delta=\left((3+\tau\alpha_1)\omega_k^4-\alpha_2\omega_k^2-f'(0)\right)^2+\left(\tau\omega_k^5-(3\alpha_1+\tau\alpha_2)\omega_k^3+f'(0)\tau\omega_k\right)^2; \tag{2.24}$$

and for $h'(V_k)\neq 0$ $(k=1,2,3,4)$, $h(0)=f'^2(0)>0$ and $\lim_{V\to\pm\infty}h(V)=\infty$, we can know that $\operatorname{Re}(d\lambda(\tau_k^j)/d\tau)>0$ when $k=2,4$; and $\operatorname{Re}(d\lambda(\tau_k^j)/d\tau)<0$ when $k=1,3$. This completes the proof. □

From $h(0)=f'^2(0)>0$ and $\lim_{V\to\pm\infty}h(V)=\infty$, it is easy to know that: if $h(V)$ satisfies $h'(V_i)\neq 0$ $(i=1,2,3,4)$, if the equation $h(V)=0$ has positive roots, then the number of the roots must be even; and from Lemma 2.3, we have that the sign of $\alpha'(\tau_k^j)$ changes as τ_k^j varies, and then the stability switches may happen.

From Lemmas 2.1–2.3 and the theory in [9], we have the following.

Lemma 2.4. *Suppose that* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $h'(V_i)\neq 0$ $(i=1,2,3,4)$ *are satisfied.*

(i) *If conditions (i) and (ii) in Lemma 2.2 are not satisfied, then all the roots of* (2.3) *have negative real parts for all* $\tau\geq 0$.

(ii) *If one of conditions (i) and (ii) in Lemma 2.2 is satisfied, let* $\tau^*=\min(\tau_2^0,\tau_4^0)$, *then all roots of* (2.3) *have negative real parts when* $\tau\in[0,\tau^*)$; *and there may exist an integer* $m\geq 0$ *such that* $0<\tau_1<\tau_2<\cdots<\tau_{m-1}<\tau_m<\tau_{m+1}<\cdots$, *and all the roots of* (2.3) *have negative real parts when* $\tau\in[0,\tau_1)\cup(\tau_2,\tau_3)\cup\cdots\cup(\tau_{m-1},\tau_m)$, *and* (2.3) *has at least a pair of roots with positive real parts when* $\tau\in(\tau_1,\tau_2)\cup(\tau_3,\tau_4)\cup\cdots\cup(\tau_m,\infty)$, *where* $\tau_m\in\{\tau_k^j\}$.

From Lemma 2.4 and applying the Hopf bifurcation theorem for functional differential equations [12, Chapter 11, Theorem 1.1], we have the following results.

Theorem 2.5. *Suppose* $(\mathbf{H_1})$, $(\mathbf{H_2})$, *and* $h'(V_i)\neq 0$ $(i=1,2,3,4)$ *are satisfied.*

(i) *If conditions (i) and (ii) in Lemma 2.2 are not satisfied, then the trivial solution* $(0,0,0,0)$ *of system* (2.1) *is asymptotically stable when* $\tau>0$.

(ii) *If one of conditions (i) and (ii) in Lemma 2.2 is satisfied, let* $\tau^*=\min\{\tau_2^0,\tau_4^0\}$, *then the trivial solution* $(0,0,0,0)$ *of system* (2.1) *is asymptotically stable when* $\tau\in[0,\tau^*)$; *and there may exist an integer* $m\geq 0$ *such that* $0<\tau_1<\tau_2<\cdots<\tau_{m-1}<\tau_m<\tau_{m+1}<\cdots$, *and the trivial solution* $(0,0,0,0)$ *of system* (2.1) *is* *asymptotically stable when* $\tau \in [0, \tau_1) \cup (\tau_2, \tau_3) \cup \cdots \cup (\tau_{m-1}, \tau_m)$, *and is unstable when* $\tau \in (\tau_1, \tau_2) \cup (\tau_3, \tau_4) \cup \cdots \cup (\tau_m, \infty)$, *where* $\tau_m \in \{\tau_k^j\}$.

(iii) *The system* (2.1) *undergoes a Hopf bifurcation at the origin when* $\tau = \tau_k^j$, *with* $k = 1, 2, 3, 4;\ j = 0, 1, 2, \ldots$.

3. Direction and Stability of the Hopf Bifurcation

In this section, we will study the direction, stability, and the period of the bifurcating periodic solution. The method we used is based on the normal form method and the center manifold theory presented by Hassard et al. [10].

We first rescale the time by $t \to t/\tau$ to normalize the delay so that system (2.1) can be written as the form

$$\begin{aligned} \dot{x} &= \tau y, \\ \dot{y} &= \tau u, \\ \dot{u} &= \tau v, \\ \dot{v} &= -\alpha_2 \tau u - \alpha_1 \tau v - \tau f(x) - \tau \phi(y(t-1)). \end{aligned} \tag{3.1}$$

The linearization around $(0, 0, 0, 0)$ is given by

$$\begin{aligned} \dot{x} &= \tau y, \\ \dot{y} &= \tau u, \\ \dot{u} &= \tau v, \\ \dot{v} &= -\tau f'(0) x - \alpha_2 \tau u - \alpha_1 \tau v - \tau \phi'(0) y(t-1); \end{aligned} \tag{3.2}$$

and the nonlinear term is

$$F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dfrac{-f''(0)}{2} x^2 - \dfrac{f'''(0) x^3}{6} - \dfrac{\tau \phi''(0) y^2(t-1)}{2} - \dfrac{\tau \phi'''(0) y^3(t-1)}{6} - \cdots \end{pmatrix}. \tag{3.3}$$

The characteristic equation associated with (3.2) is

$$\gamma^4 + \alpha_1 \tau \gamma^3 + \alpha_2 \tau^2 \gamma^2 + \phi'(0) \tau^3 \gamma e^{-\gamma} + \tau^4 f'(0) = 0. \tag{3.4}$$

Comparing (3.4) with (2.3), one can find out that $\gamma = \tau\lambda$, and hence, (3.4) has a pair of imaginary roots $\pm i \tau_k^j \omega_k$, when $\tau = \tau_k^j$ for $k = 1, 2, 3, 4$, $j = 0, 1, 2, \ldots$, and the transversal condition holds.

Let $\tau = \tau_0 + \mu$, $\mu \in R$ where $\tau_0 \in \{\tau_k^j\}$, $\omega_0 \in \{\omega_k\}$, $k = 1,2,3,4$, $j = 0,1,2,\ldots$. Then $\mu = 0$ is the Hopf bifurcation value for (3.1). Let $i\tau_0\omega_0$ be the root of (3.4).

For $\varphi \in C([-1,0], R^4)$, let

$$L_\mu\varphi = (\tau_0+\mu)\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -f'(0) & 0 & -\alpha_2 & -\alpha_1 \end{pmatrix}\varphi(0) - (\tau_0+\mu)\phi'(0)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\varphi(-1),$$

$$F(\mu,\varphi) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dfrac{-(\tau_0+\mu)f''(0)\varphi_1^2(0)}{2} - \dfrac{(\tau_0+\mu)f'''(0)\varphi_1^3(0)}{6} - \dfrac{(\tau_0+\mu)\phi''(0)\varphi_2^2(-1)}{2} - \dfrac{(\tau_0+\mu)\phi'''(0)\varphi_2^3(-1)}{6} - \cdots \end{pmatrix}. \tag{3.5}$$

By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions $\eta(\theta,\mu)$ in $\theta \in [-1,0]$ such that

$$L_\mu\varphi = \int_{-1}^{0} d\eta(\theta,\mu)\varphi(\theta), \quad \varphi \in C([-1,0], R^4). \tag{3.6}$$

In fact, we choose

$$\eta(\theta,\mu) = (\tau_0+\mu)\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -f'(0) & 0 & -\alpha_2 & -\alpha_1 \end{pmatrix}\delta(\theta) + (\tau_0+\mu)\phi'(0)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\delta(\theta+1), \tag{3.7}$$

where

$$\delta(\theta) = \begin{cases} 1, & \theta = 0, \\ 0, & \theta \neq 0. \end{cases} \tag{3.8}$$

For $\varphi \in C^1([-1,0], C^4)$, define

$$A(\mu)\varphi = \begin{cases} \dfrac{d\varphi(\theta)}{d\theta}, & \theta \in [-1,0), \\ \displaystyle\int_{-1}^{0} d\eta(t,\mu)\varphi(t), & \theta = 0, \end{cases}$$
$$R(\mu)\varphi = \begin{cases} 0, & \theta \in [-1,0), \\ F(\mu,\varphi), & \theta = 0. \end{cases} \tag{3.9}$$

Hence, we can rewrite (3.1) in the following form:

$$\dot{w}_t = A(\mu)w_t + R(\mu)w_t, \tag{3.10}$$

where $w = (x, y, u, v)^T$, $w_t = w(t+\theta)$ for $\theta \in [-1,0]$.

For $\psi \in C^1([0,1], C^4)$, define

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0,1], \\ \displaystyle\int_{-1}^{0} \psi(-t)d\eta(t,0), & s = 0. \end{cases} \tag{3.11}$$

For $\varphi \in C([-1,0], C^4)$ and $\psi \in C([0,1], C^4)$, define the bilinear form

$$\langle \psi, \varphi \rangle = \overline{\psi}(0)\varphi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{\psi}^T(\xi-\theta)d\eta(\theta)\varphi(\xi)d\xi, \tag{3.12}$$

where $\eta(\theta) = \eta(\theta, 0)$. Then A^* and $A(0)$ are adjoint operators, and $\pm i\tau_0\omega_0$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*.

By direct computation, we obtain that

$$q(\theta) = \begin{pmatrix} 1 \\ i\omega_0 \\ -\omega_0^2 \\ -i\omega_0^3 \end{pmatrix} e^{i\tau_0\omega_0\theta} \tag{3.13}$$

is the eigenvector of $A(0)$ corresponding to $i\tau_0\omega_0$, and

$$q^*(\theta) = D\begin{pmatrix} -i\omega_0^3 - \omega_0^2\alpha_1 + i\omega_0\alpha_2 + \phi'(0)e^{-i\tau_0\omega_0} \\ -\omega_0^2 + i\omega_0\alpha_1 + \alpha_2 \\ i\omega_0 + \alpha_1 \\ 1 \end{pmatrix}^T e^{i\tau_0\omega_0 s} \tag{3.14}$$

is the eigenvector of A^* corresponding to $-i\tau^0\omega_0$. Moreover,

$$\langle q^*, q\rangle = 1, \qquad \langle q^*, \overline{q}\rangle = 0, \tag{3.15}$$

where

$$D = \frac{1}{-\omega_0^2\alpha_1 + \phi'(0)e^{-i\tau_0\omega_0} - i\tau_0\omega_0\phi'(0)e^{i\tau_0\omega_0}}. \tag{3.16}$$

Using the same notation as in Hassard et al. [10], we first compute the coordinates to describe the center manifold $\mathcal{C}_0$ at $\mu = 0$. Let w_t be the solution of (3.1) when $\mu = 0$.

Define

$$z_t = \langle q^*, w_t\rangle, \qquad W(t,\theta) = w_t(\theta) - 2\,\mathrm{Re}\{z(t)q(\theta)\}. \tag{3.17}$$

On the center manifold $\mathcal{C}_0$, we have

$$W(t,\theta) = W(z(t), \overline{z}(t), \theta), \tag{3.18}$$

where

$$W(z,\overline{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \cdots, \tag{3.19}$$

z and $\overline{z}$ are local coordinates for center manifold $\mathcal{C}_0$ in the direction of q^* and $\overline{q}^*$. Note that W is real if w_t is real. We consider only real solutions.

For solution w_t in $\mathcal{C}_0$ of (3.1), since $\mu = 0$,

$$\begin{aligned} \dot{z}(t) &= i\tau_0\omega_0 z + \langle q^*(\theta), F(W(z,\overline{z},\theta) + 2\,\mathrm{Re}\{z(t)q(\theta)\})\rangle \\ &= i\tau_0\omega_0 z + \overline{q}^*(0)F(W(z,\overline{z},0) + 2\mathrm{Re}\{z(t)q(0)\}) \\ &\stackrel{\text{def}}{=} i\tau_0\omega_0 z + \overline{q}^*(0)F_0(z,\overline{z}). \end{aligned} \tag{3.20}$$

We rewrite this as

$$\dot{z}(t) = i\tau_0\omega_0 z(t) + g(z,\overline{z}), \tag{3.21}$$

where

$$F_0(z,\overline{z}) = F_{z^2}\frac{z^2}{2} + F_{\overline{z}^2}\frac{\overline{z}^2}{2} + F_{z\overline{z}}z\overline{z} + F_{z^2\overline{z}}\frac{z^2\overline{z}}{2} + \cdots, \tag{3.22}$$

$$\begin{aligned} g(z,\overline{z}) &= \overline{q}^*(0)F(W(z,\overline{z},0) + 2\,\mathrm{Re}\{z(t)q(0)\}) \\ &= g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots . \end{aligned} \tag{3.23}$$

Compare the coefficients of (3.20) and (3.21), noticing (3.23), we have

$$g_{20} = \overline{q}^*(0)F_{z^2}, \qquad g_{11} = \overline{q}^*(0)F_{z\overline{z}}, \qquad g_{02} = \overline{q}^*(0)F_{\overline{z}^2}, \qquad g_{21} = \overline{q}^*(0)F_{z^2\overline{z}}. \tag{3.24}$$

By (3.10) and (3.21), it follows that

$$\begin{aligned} \dot{W} &= \dot{w}_t - \dot{z}q - \dot{\overline{z}}\,\overline{q} \\ &= \begin{cases} AW - 2\,\mathrm{Re}\{\overline{q}^*(0)F_0 q(\theta)\} & -1 \le \theta < 0 \\ AW - 2\,\mathrm{Re}\{\overline{q}^*(0)F_0 q(0)\} + F_0 & \theta = 0 \end{cases} \\ &\overset{\mathrm{def}}{=} AW + H(z,\overline{z},\theta), \end{aligned} \tag{3.25}$$

where

$$H(z,\overline{z},\theta) = H_{20}\frac{z^2}{2} + H_{11}(\theta)z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots . \tag{3.26}$$

Expanding the above series and comparing the coefficients, we obtain

$$\begin{gathered} (A - 2i\tau_0\omega_0 I)W_{20}(\theta) = -H_{20}(\theta), \\ AW_{11}(\theta) = -H_{11}(\theta), \\ (A - 2i\tau_0\omega_0)W_{02}(\theta) = -H_{02}(\theta). \end{gathered} \tag{3.27}$$

Notice that

$$w_t(\theta) = W(z,\overline{z},\theta) + zq(\theta) + \overline{z}\overline{q}(\theta), \tag{3.28}$$

that is,

$$\begin{aligned}
x(t) &= z+\overline{z}+W^{(1)}(z,\overline{z},0)\\
&= z+\overline{z}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\overline{z}+W_{02}^{(1)}(0)\frac{\overline{z}^2}{2}+\cdots,\\
y(t-1) &= i\omega_0 e^{-i\tau_0\omega_0}z-i\omega_0 e^{i\tau_0\omega_0}\overline{z}+W^{(2)}(z,\overline{z},-1)\\
&= i\omega_0 e^{-i\tau_0\omega_0}z-i\omega_0 e^{i\tau_0\omega_0}\overline{z}+W_{20}^{(2)}(-1)\frac{z^2}{2}+W_{11}^{(2)}(-1)z\overline{z}+W_{02}^{(2)}(-1)\frac{\overline{z}^2}{2}+\cdots.
\end{aligned} \tag{3.29}$$

Thus

$$\begin{aligned}
x^2(x) &= 2\frac{z^2}{2}+2z\overline{z}+2\frac{\overline{z}^2}{2}+\left(4W_{11}^{1}(0)+2W_{20}^{1}(0)\right)\frac{z^2\overline{z}}{2}+\cdots,\\
x^3(x) &= 6\frac{z^2\overline{z}}{2}+\cdots,\\
y^2(t-1) &= -2\omega_0^2 e^{-2i\tau_0\omega_0}\frac{z^2}{2}+2\omega_0^2 z\overline{z}-2\omega_0^2 e^{2i\tau_0\omega_0}\frac{\overline{z}^2}{2}\\
&\quad+\left(4i\omega_0 W_{11}^{(2)}(-1)e^{-i\tau_0\omega_0}-i\omega_0 2W_{20}^{(2)}(-1)e^{i\tau_0\omega_0}\right)\frac{z^2\overline{z}}{2}+\cdots,\\
y^3(t-1) &= 6i\omega_0^3 e^{-i\tau_0\omega_0}\frac{z^2\overline{z}}{2}+\cdots;
\end{aligned} \tag{3.30}$$

and we have

$$\begin{gathered}
F(0,\omega_t)=\begin{pmatrix} 0\\ 0\\ 0\\ -\dfrac{\tau_0 f''(0)x^2(t)}{2}-\dfrac{\tau_0 f'''(0)x^3(t)}{6}-\dfrac{\tau_0\phi''(0)y^2(t-1)}{2}-\dfrac{\tau_0\phi''(0)y^3(t-1)}{6}-\cdots\end{pmatrix},\\
q^*(0)=D\begin{pmatrix} -i\omega_0^3-\omega_0^2\alpha_1+i\omega_0\alpha_2+\phi'(0)e^{-i\tau_0\omega_0}\\ -\omega_0^2+i\omega_0\alpha_1+\alpha_2\\ i\omega_0+\alpha_1\\ 1\end{pmatrix}^T,\\
g(z,\overline{z})=\overline{q}^*(0)F_0(z,\overline{z}).
\end{gathered} \tag{3.31}$$

Then we have

$$
\begin{aligned}
g_{20} &= -\tau_0\overline{D}f''(0) + \tau_0\overline{D}\phi''(0)\omega^2 e^{-2i\tau_0\omega_0},\\
g_{11} &= -\tau_0\overline{D}f''(0) - \tau_0\overline{D}\phi''(0)\omega^2,\\
g_{02} &= -\tau_0\overline{D}f''(0) + \tau_0\overline{D}\phi''(0)\omega^2 e^{2i\tau_0\omega_0},\\
g_{21} &= -\tau_0\overline{D}f''(0)\left[2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\right] - \tau_0\overline{D}f'''(0)\\
&\quad - \tau_0\overline{D}\phi''(0)\left[2i\omega_0 W_{11}^{(2)}(-1)e^{-i\tau_0\omega_0} - i\omega_0 W_{20}^{(2)}(-1)e^{i\tau_0\omega_0}\right] - \tau_0\overline{D}\phi'''(0).
\end{aligned}
\tag{3.32}
$$

So we only need to find out $W_{11}^{(1)}(0)$, $W_{20}^{(1)}(0)$, $W_{11}^{(2)}(-1)$, and $W_{20}^{(2)}(-1)$ to obtain g_{21}.

When $\theta \in [-1, 0)$, we have

$$
\begin{aligned}
H(z,\overline{z},\theta) &= -2\,\mathrm{Re}\{\overline{q}^*(0)F_0 q(\theta)\}\\
&= -\overline{q}^*(0)F_0 q(\theta) - q^*(0)F_0\overline{q}(\theta)\\
&= -g q(\theta) - \overline{g q}(\theta).
\end{aligned}
\tag{3.33}
$$

Comparing the coefficients with (3.26), we get

$$
\begin{aligned}
H_{20}(\theta) &= -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta),\\
H_{11}(\theta) &= -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta).
\end{aligned}
\tag{3.34}
$$

From (3.27), (3.32), (3.33), and (3.34), we derive

$$
\begin{aligned}
\dot{W}_{20}(\theta) &= 2i\tau_0\omega_0 W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta),\\
\dot{W}_{11}(\theta) &= g_{11}q(\theta) + \overline{g}_{11}\overline{q}(\theta).
\end{aligned}
\tag{3.35}
$$

Then we can get

$$
\begin{aligned}
W_{20}(\theta) &= \frac{ig_{20}}{\tau_0\omega_0}q(0)e^{i\tau_0\omega_0\theta} + \frac{i\,\overline{g}_{02}}{3\tau_0\omega_0}\overline{q}(0)e^{-i\tau_0\omega_0\theta} + E_1 e^{2i\tau_0\omega_0\theta},\\
W_{11}(\theta) &= \frac{g_{11}}{i\tau_0\omega_0}q(0)e^{i\tau_0\omega_0\theta} - \frac{\overline{g}_{11}}{i\tau_0\omega_0}\overline{q}(0)e^{-i\tau_0\omega_0\theta} + E_2.
\end{aligned}
\tag{3.36}
$$

Notice that

$$
\begin{aligned}
\left(2i\tau_0\omega_0 I - \int_{-1}^{0} e^{2i\tau_0\omega_0\theta}\,d\eta(\theta)\right)E_1 &= F_{z^2},\\
\left(\int_{-1}^{0} d\eta(\theta)\right)E_2 &= -F_{z\overline{z}}.
\end{aligned}
\tag{3.37}
$$

We obtain

$$E_1 = \begin{pmatrix} E^* \\ 2i\omega_0 E^* \\ -4\omega_0^2 E^* \\ -8i\omega_0^3 E^* \end{pmatrix}, \qquad E_2 = \begin{pmatrix} -\dfrac{f''(0) + \phi''(0)}{f'(0)} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.38}$$

where

$$E^* = \frac{-f''(0) + \phi''(0)\omega_0^2 e^{-2i\tau_0\omega_0}}{f'(0) + 2i\omega_0 e^{-2i\tau_0\omega_0} - 4\omega_0^2\alpha_2 - 8i\omega_0^3(2i\omega_0 + \alpha_1)}.$$

Hence

$$\begin{aligned}
W_{20}^{(1)}(0) &= \frac{ig_{20}}{\tau_0\omega_0} q_1(0) + \frac{i\overline{g}_{02}}{3\tau_0\omega_0}\overline{q}_1(0) + E^* e^{-2i\tau_0\omega_0} \\
&= \frac{i\left(-\overline{D}f''(0) + \overline{D}\phi''(0)\omega_0^2 e^{-2i\tau_0\omega_0}\right)}{\omega_0} + \frac{i\left(-Df''(0) + D\phi''(0)\omega_0^2 e^{-2i\tau_0\omega_0}\right)}{3\omega_0} \\
&\quad + E^* e^{-2i\tau_0\omega_0}, \\
W_{20}^{(2)}(-1) &= \frac{ig_{20}}{\tau_0\omega_0} q_2(0) e^{-i\tau_0\omega_0} + \frac{i\overline{g}_{02}}{3\tau_0\omega_0}\overline{q}_2(0) e^{i\tau_0\omega_0} + (2i\omega_0) E^* e^{-2i\tau_0\omega_0} \\
&= -\left(-\overline{D}f''(0) + \overline{D}\phi''(0)\omega_0^2 e^{-2i\tau_0\omega_0}\right) e^{-i\tau_0\omega_0} + \frac{-Df''(0) + D\phi''(0)\omega_0^2 e^{-2i\tau_0\omega_0}}{3} e^{i\tau_0\omega_0} \\
&\quad + (2i\omega_0) E^* e^{-2i\tau_0\omega_0}, \\
W_{11}^{(1)}(0) &= \frac{g_{11}}{i\tau_0\omega_0} q_1(0) - \frac{\overline{g}_{11}}{i\tau_0\omega_0}\overline{q}_1(0) - \frac{f''(0) + \omega_0^2\phi'(0)}{f'(0)} \\
&= \frac{-\overline{D}f''(0) - \overline{D}\phi''(0)\omega_0^2}{i\omega_0} + \frac{Df''(0) + D\phi''(0)\omega_0^2}{i\omega_0} - \frac{f''(0) + \omega_0^2\phi'(0)}{f'(0)}, \\
W_{11}^{(2)}(-1) &= \frac{g_{11}}{i\tau_0\omega_0} q_2(0) e^{-i\tau_0\omega_0} - \frac{\overline{g}_{11}}{i\tau_0\omega_0}\overline{q}_2(0) e^{i\tau_0\omega_0} \\
&= \left(-\overline{D}f''(0) - \overline{D}\phi''(0)\omega_0^2\right) e^{-i\tau_0\omega_0} + \left(Df''(0) + D\phi''(0)\omega_0^2\right) e^{i\tau_0\omega_0}.
\end{aligned} \tag{3.39}$$

Consequently, from (3.32),

$$
\begin{aligned}
g_{21} ={}& -\tau_0\overline{D}f''(0)\left[2W_{11}^{(1)}(0)+W_{20}^{(1)}(0)\right]-\tau_0\overline{D}f'''(0)\\
&-\tau_0\overline{D}\phi''(0)\left[2i\omega_0W_{11}^{(2)}(-1)e^{-i\tau_0\omega_0}-i\omega_0W_{20}^{(2)}(-1)e^{i\tau_0\omega_0}\right]-\tau_0\overline{D}\phi'''(0)\\
={}&-2\tau_0\overline{D}f''(0)\left[\frac{-\overline{D}f''(0)-\overline{D}\phi''(0)\omega_0^2}{i\omega_0}+\frac{Df''(0)+D\phi''(0)\omega_0^2}{i\omega_0}-\frac{f''(0)+\omega_0^2\phi''(0)}{f'(0)}\right]-\tau_0\overline{D}f''(0)\\
&\times\left[\frac{i\left(-\overline{D}f''(0)+\overline{D}\phi''(0)\omega_0^2e^{-2i\tau_0\omega_0}\right)}{\omega_0}+\frac{i\left(-Df''(0)+D\phi''(0)\omega_0^2e^{-2i\tau_0\omega_0}\right)}{3\omega_0}+E^*e^{-2i\tau_0\omega_0}\right]-\tau_0\overline{D}f'''(0)\\
&-2\tau_0\overline{D}\phi''(0)(i\omega_0)\left[\left(-\overline{D}f''(0)-\overline{D}\phi''(0)\omega_0^2\right)e^{-i\tau_0\omega_0}+\left(Df''(0)+D\phi''(0)\omega_0^2\right)e^{i\tau_0\omega_0}\right]e^{-i\tau_0\omega_0}\\
&+\tau_0\overline{D}\phi''(0)(i\omega_0)\left[-\left(-\overline{D}f''(0)+\overline{D}\phi''(0)\omega_0^2e^{-2i\tau_0\omega_0}\right)e^{-i\tau_0\omega_0}\right]e^{i\tau_0\omega_0}\\
&+\tau_0\overline{D}\phi''(0)(i\omega_0)\left[\frac{-Df''(0)+D\phi''(0)\omega_0^2e^{-2i\tau_0\omega_0}}{3}e^{i\tau_0\omega_0}+(2i\omega_0)E^*e^{-2i\tau_0\omega_0}\right]e^{i\tau_0\omega_0}-\tau_0\overline{D}\phi'''(0).
\end{aligned}
\tag{3.40}
$$

Substituting g_{20}, g_{11}, g_{02}, and g_{21} into

$$C_1(0)=\frac{i}{2\tau_0\omega_0}\left(g_{20}g_{11}-2|g_{11}|^2-\frac{1}{3}|g_{02}|^2\right)+\frac{g_{21}}{2}, \tag{3.41}$$

we can obtain $\operatorname{Re}C_1(0)$. Then we obtain the sign of

$$
\begin{aligned}
\beta_2 &= 2\operatorname{Re}C_1(0),\\
\mu_2 &= -\frac{\operatorname{Re}C_1(0)}{\alpha'(\tau_0)}.
\end{aligned}
\tag{3.42}
$$

By the general theory due to Hassard et al. [10], we know that the quantity of β_2 determines the stability of the bifurcating periodic solutions on the center manifold, and μ_2 determines the direction of the bifurcation; and we have the following.

Theorem 3.1. (i) *If $\mu_2 > 0(< 0)$, then the Hopf bifurcation at the origin of system* (1.1) *is supercritical (subcritical).*

(ii) *If $\beta_2 < 0(> 0)$, then the bifurcating periodic solutions of system* (1.1) *are asymptotically stable (unstable).*

4. An Example and Numerical Simulations

In this section, we give an example and present some numerical simulations to illustrate the analytic results.

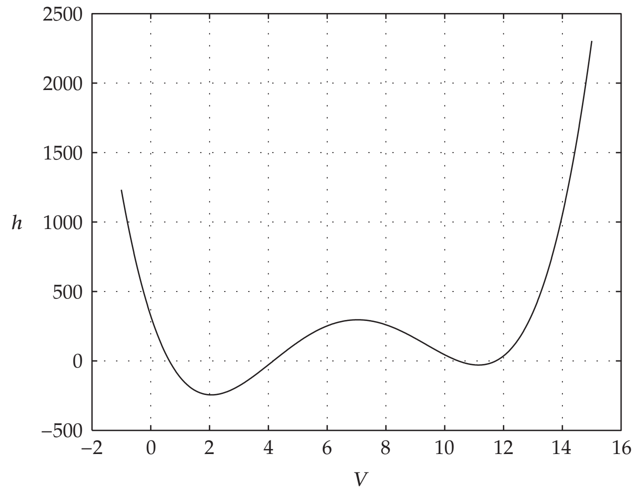


Figure 1: The curve of function $h(V) = V^4 - 27V^3 + 232V^2 - 648V + 324$.

Example 4.1. Consider the following equation:

$$x^{(4)}(t) + \dddot{x}(t) + 14\ddot{x}(t) + 12\sin\dot{x}(t-\tau) + x^3(t) + 18x(t) = 0. \tag{4.1}$$

Clearly,

$$\begin{gathered} \alpha_1 = 1, \qquad \alpha_2 = 14, \qquad \phi(\dot{x}(t-\tau)) = 12\sin\dot{x}(t-\tau), \qquad f(x) = x^3 + 18x, \\ \phi(0) = \phi''(0) = 0, \qquad \phi'(0) = 12, \qquad \phi'''(0) = -12, \\ f(0) = f''(0) = 0, \qquad f'(0) = 18, \qquad f'''(0) = 6. \end{gathered} \tag{4.2}$$

By direct computation, we know $(\mathbf{H_1})$ and $(\mathbf{H_2})$ are satisfied. That is, the data satisfy the conditions of Lemma 2.1. The characteristic equation is

$$\lambda^4 + \lambda^3 + 14\lambda^2 + 12\lambda e^{-\lambda\tau} + 18 = 0; \tag{4.3}$$

and we can obtain

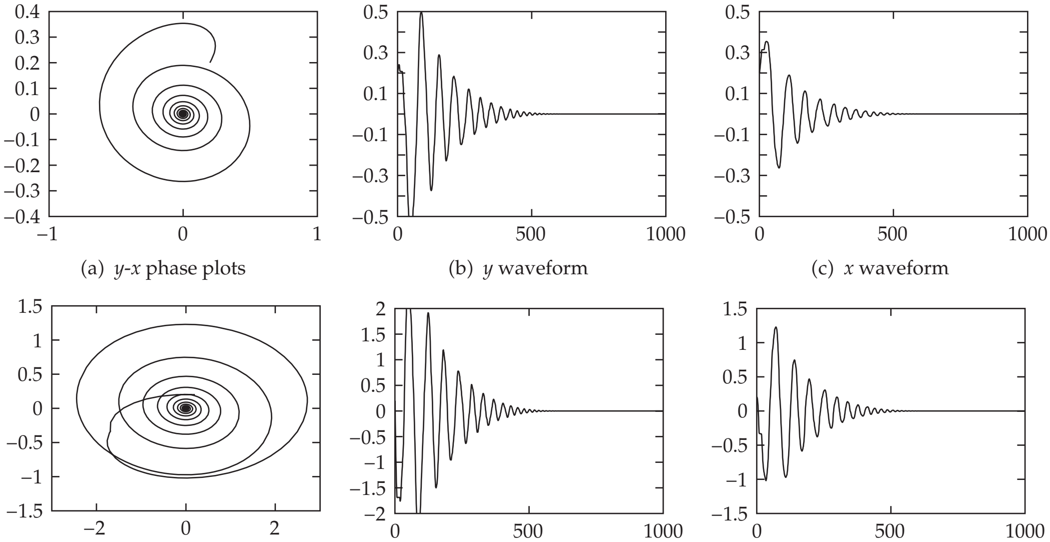
$$h(V) = V^4 - 27V^3 + 232V^2 - 648V + 324. \tag{4.4}$$

As shown in Figure 1, the equation $h(V) = 0$ has four roots as

$$V_1 = 0.633, \qquad V_2 = 4.166, \qquad V_3 = 10.464, \qquad V_4 = 11.737; \tag{4.5}$$

and

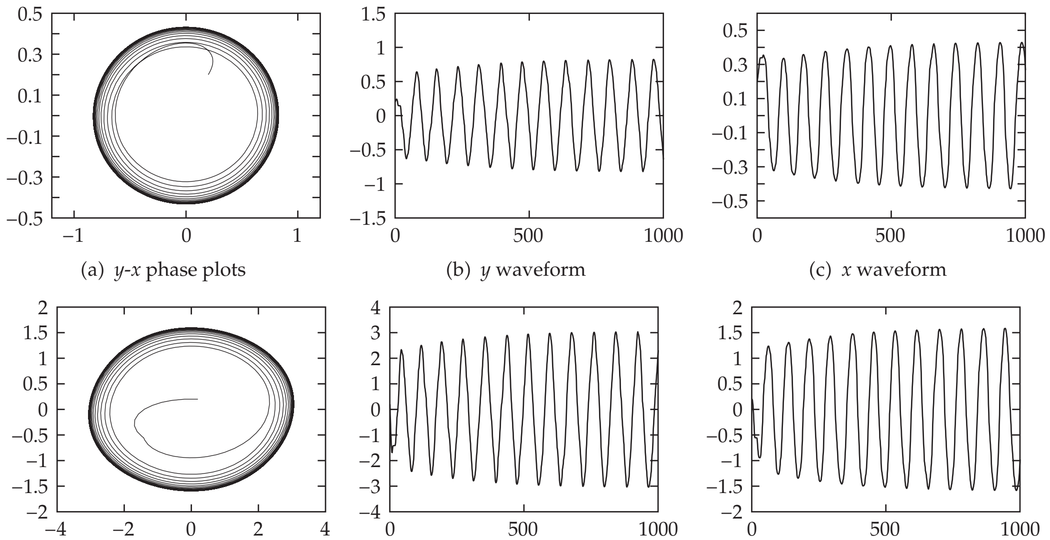
$$h'(V_1) < 0, \qquad h'(V_2) > 0, \qquad h'(V_3) < 0, \qquad h'(V_4) > 0. \tag{4.6}$$



(a) y-x phase plots (b) y waveform (c) x waveform

(d) v-u phase plots (e) v waveform (f) u waveform

Figure 2: The zero solution of system (4.1) is asymptotically stable when $\tau = 0.4$.



(a) y-x phase plots (b) y waveform (c) x waveform

(d) v-u phase plots (e) v waveform (f) u waveform

Figure 3: For (4.1) with $\tau = 0.64 > \tau_2^0$ and sufficiently near $\tau_2^0 = 0.596$, the bifurcating periodic solution from zero solution occurs and is asymptotically stable.

Hence

$$\begin{aligned} &\omega_1 = 0.796, \qquad \omega_2 = 2.041, \qquad \omega_3 = 3.235, \qquad \omega_4 = 3.426, \\ &\tau_1^0 = 5.988, \qquad \tau_2^0 = 0.596, \qquad \tau_3^0 = 0.158, \qquad \tau_4^0 = 0.061. \end{aligned} \tag{4.7}$$

For $\tau_4^0 < \tau_3^0 < \tau_2^0 < \tau_1^0$, we obtain that the zero solution of system (4.1) is asymptotically stable when $\tau \in [0, 0.061) \cup (0.158, 0.596)$.

According to the formula given in Section 3, we can obtain that

$$D_4 = -1.898 + 0.133i, \qquad D_3 = -0.323 + 0.056i, \qquad D_2 = -0.061 + 0.027i,$$
$$g_{21}^{(4)} = -0.697 + 0.049i, \qquad g_{21}^{(3)} = -0.306 + 0.053i, \qquad g_{21}^{(2)} = -0.217 + 0.098i, \tag{4.8}$$
$$g_{02} = g_{11} = g_{02} = E^* = 0.$$

Then we have

$$C_1(0) = \frac{g_{21}}{2}. \tag{4.9}$$

Hence, when $\tau \in \{\tau_1^0, \tau_2^0, \tau_3^0\}$, we have

$$\beta_2 = 2\mathrm{Re}\, C_1(0) < 0,$$
$$\mu_2 = -\frac{\mathrm{Re}\, C_1(0)}{\alpha'(\tau_0)} > 0. \tag{4.10}$$

Conclusion of (4.1)

The zero solution of system (4.1) is asymptotically stable when $\tau \in [0, 0.061) \cup (0.158, 0.596)$. The Hopf bifurcation at the origin when $\tau_0 = \tau_k^0$ is supercritical, and the bifurcating periodic solutions are asymptotically stable.

The following is the results of numerical simulations to system (4.1).

(i) We choose $\tau = 0.4 \in (0.158, 0.596)$, then the zero solution of system (4.1) is asymptotically stable, as shown in Figure 2.

(ii) We choose $\tau = 0.64$ being near to $\tau_2^0 = 0.596$, a periodic solution bifurcates from the origin and is asymptotically stable, as shown in Figure 3.

5. Conclusion

In this paper, we consider a certain fourth-order delay differential equation. The linear stability is investigated by analyzing the associated characteristic equation. It is found that there may exist the stability switches when delay varies, and the Hopf bifurcation occurs when the delay passes through a sequence of critical values. Then the direction and the stability of the Hopf bifurcation are determined using the normal form method and the center manifold theorem. Finally, an example is given and numerical simulations are carried out to illustrate the results. By using Lyapunov's second method, Sadek [1] investigated the stability of system (1.1). The main result is as the following.

Theorem 5.1. *Suppose that the following hold.*

(i) *There are constants* $\alpha_1 > 0,\ \alpha_2 > 0,\ \alpha_3 > 0,\ \alpha_4 > 0$, *and* $\Delta > 0$ *such that*

$$(\alpha_1\alpha_2 - \phi'(y))\alpha_3 - \alpha_1^2\alpha_4 \geq \Delta \tag{5.1}$$

for all y.

(ii) $f(0) = 0,\ xf(x) > 0\ (x \neq 0),\ F(x) = \int_0^x f(\xi)d\xi \rightarrow \infty$ *as* $|x| \rightarrow \infty$, *and*

$$0 \leq \alpha_4 - f'(x) \leq \varepsilon d_0 \alpha_1^0 \tag{5.2}$$

for all x, *where* ε *is a positive constant such that*

$$\varepsilon = \min\left(\frac{1}{\alpha_1}, \frac{\Delta}{4\alpha_1\alpha_3 d_0}, \frac{\alpha_3}{4\alpha_4 d_0}\left(\frac{2\Delta\alpha_4}{\alpha_1\alpha_3^2} - \delta_1\right)\right) \tag{5.3}$$

with $d_0 = \alpha_1\alpha_2 + \alpha_2\alpha_3\alpha_4^{-1}$.

(iii) $\phi(0) = 0$ *and* $\phi'(y) \geq \alpha_3 > 0$ *for all* y, *and* $0 \leq \phi'(y) - \phi(y)/y \leq \delta_1 < 2\Delta\alpha_4/\alpha_1\alpha_3^2$ *for all* $y \neq 0$.

Then the zero solution of (1.1) *is asymptotically stable, provided that*

$$\tau < \min\left\{\frac{\varepsilon}{d_2\alpha_2}, \frac{\Delta}{2\alpha_1\alpha_3(\alpha_2\alpha_3 + 2\mu)}, \frac{\alpha_1\varepsilon}{d_1\alpha_2\alpha_3}\right\}, \tag{5.4}$$

with $\mu = (\alpha_2\alpha_3/2)(d_1 + d_2 + 1) > 0$.

Comparing Theorem 5.1 with Theorem 2.5 obtained in Section 2, one can find out that if the sufficient conditions to ensure the globally asymptotical stability of system (1.1) given in [10] are not satisfied, we can also get the stability of system (1.1), but here the stability means local stability, and the system undergoes a Hopf bifurcation at the origin. Otherwise, here we just need to give the condition on the origin of $f(x)$ and $\phi(x)$, the condition is relatively weak.

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References

[1] A. I. Sadek, "On the stability of solutions of certain fourth order delay differential equations," *Applied Mathematics and Computation*, vol. 148, no. 2, pp. 587–597, 2004.

[2] H. Bereketoglu, "Asymptotic stability in a fourth order delay differential equation," *Dynamic Systems and Applications*, vol. 7, no. 1, pp. 105–115, 1998.

[3] E. O. Okoronkwo, "On stability and boundedness of solutions of a certain fourth-order delay differential equation," *International Journal of Mathematics and Mathematical Sciences*, vol. 12, no. 3, pp. 589–602, 1989.
[4] A. S. C. Sinha, "On stability of solutions of some third and fourth order delay-differential equations," *Information and Computation*, vol. 23, no. 2, pp. 165–172, 1973.
[5] C. Tunç, "Some stability results for the solutions of certain fourth order delay differential equations," *Differential Equations and Applications*, vol. 4, pp. 133–140, 2005.
[6] C. Tunç, "Some stability and boundedness results for the solutions of certain fourth order differential equations," *Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica*, vol. 44, pp. 161–171, 2005.
[7] C. Tunç, "On stability of solutions of certain fourth-order delay differential equations," *Applied Mathematics and Mechanics*, vol. 27, no. 8, pp. 1141–1148, 2006.
[8] C. Tunç, " On the stability of solutions to a certain fourth-order delay differential equation," *Nonlinear Dynamics*, vol. 51, no. 1-2, pp. 71–81, 2008.
[9] S. Ruan and J. Wei, "On the zeros of transcendental functions with applications to stability of delay differential equations with two delays," *Dynamics of Continuous, Discrete and Impulsive Systems. Series A*, vol. 10, no. 6, pp. 863–874, 2003.
[10] B. D. Hassard, N. D. Kazarinoff, and Y. H. Wan, *Theory and Applications of Hopf Bifurcation*, vol. 41 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, UK, 1981.
[11] X. Li and J. Wei, "On the zeros of a fourth degree exponential polynomial with applications to a neural network model with delays," *Chaos, Solitons & Fractals*, vol. 26, no. 2, pp. 519–526, 2005.
[12] J. Hale, *Theory of Functional Differential Equations*, vol. 3 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 2nd edition, 1977.