

*Research Article*

## Existence of Positive Solutions for a Discrete Three-Point Boundary Value Problem

Huting Yuan, Guang Zhang, and Hongliang Zhao

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A discrete three-point boundary value problem  $\Delta^2 x_{k-1} + \lambda f_k(x_k) = 0$ ,  $k = 1, 2, \dots, n$ ,  $x_0 = 0$ ,  $\alpha x_l = x_{n+1}$ , is considered, where  $1 \leq l \leq n$  is a fixed integer,  $a$  is a real constant number, and  $\lambda$  is a positive parameter. A characterization of the values of  $\lambda$  is carried out so that the boundary value problem has the positive solutions. Particularly, in this paper the constant  $a$  can be negative numbers. The similar results are not valid for the three-point boundary value problem of differential equations.

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### 1. Introduction

It is of interest to note here that the three-point or multipoint boundary value problems in the continuous case have been studied in great detail in the recent papers [1–11] since the early 1980s. In numerical integration of differential equations, it is taken by granted that their difference approximations retain the same existence and uniqueness property of solutions. However, in the case of boundary value problems, a number of examples can be cited where this assumption fails. This has led a number of recent investigations providing necessary and/or sufficient conditions for the existence and uniqueness of the solutions of discrete boundary value problems (see [12–18]). Then, how do we consider the three-point or multipoint boundary value problems of difference equations?

Recently, in [19] we have considered the existence of positive solutions for the nonlinear discrete three-point boundary value problem

$$\begin{aligned} \Delta^2 x_{k-1} + f(x_k) &= 0, & k &= 1, 2, \dots, n, \\ x_0 &= 0, & \alpha x_l &= x_{n+1}, \end{aligned} \tag{1.1}$$

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where  $n \in \{2, 3, \dots\}$ ,  $l \in [1, n] = \{1, 2, \dots, n\}$ ,  $\alpha$  is a positive number, and  $f \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}_+)$ . For (1.1), the existence of one or two positive solutions was established when  $f$  is superlinear or sublinear.

In fact, system (1.1) can be regarded as a discrete reality model. A horizontal string of negligible mass is stretched between the points  $x = 0$  and  $x = n + 1$ , and concentrated forces with magnitudes  $f(x_1), f(x_2), \dots, f(x_n)$  and downward directions are applied at the points  $x = 1, 2, \dots, n$ , respectively. Suppose an end of the string is fixed, and another end of the string has some relation with the points  $1, 2, \dots, n - 1$ , or  $n$ . For example, we can suppose that  $x_{n+1} = ax_l$  for some  $l \in [1, n]$ . By the Hooker law, we can obtain the discrete three-point boundary value problem (1.1).

In this paper, we will consider a more general nonlinear discrete three-point boundary problem of the form

$$\Delta^2 x_{k-1} + \lambda f_k(x_k) = 0, \quad k = 1, 2, \dots, n, \quad (1.2)$$

$$x_0 = 0, \quad ax_l = x_{n+1}, \quad (1.3)$$

where  $\lambda$  is a positive parameter,  $a$  is a real constant number,  $n$  is a positive integer,  $f_k \in \mathbb{C}(\mathbb{R}^+, \mathbb{R}^+)$  for  $k \in [1, n]$ , and  $\Delta$  denotes the forward difference operator defined by  $\Delta x_k = x_{k+1} - x_k$  and  $\Delta^2 x_{k-1} = x_{k+1} - 2x_k + x_{k-1}$ .

By a solution  $x$  of (1.2)-(1.3), we mean a nontrivial  $x: [0, n+1] \rightarrow \mathbb{R}$  satisfying (1.2) with the boundary value condition (1.3). A solution  $\{x_k\}_{k=0}^{n+1}$  of (1.2)-(1.3) is called to be positive if  $x_k > 0$  for  $k \in [1, n]$ . If, for a particular  $\lambda$  the boundary value problem (1.2)-(1.3) has a positive solution  $x$ , then  $\lambda$  is called an eigenvalue and  $x$  a corresponding eigenfunction of (1.2)-(1.3). We let

$$\Gamma = \{\lambda > 0 \mid (1.2)-(1.3) \text{ has a positive solution}\} \quad (1.4)$$

be the set of eigenvalues of (1.2)-(1.3). Further, we introduce the notations

$$f_{k0} = \lim_{u \rightarrow 0^+} \frac{f_k(u)}{u}, \quad f_{k\infty} = \lim_{u \rightarrow \infty} \frac{f_k(u)}{u} \quad \text{for } k \in [1, n]. \quad (1.5)$$

The following is the plan of this paper. It is well known that Green's functions are important for the boundary value problems. Thus, in the next section we will give the Green function of (1.2)-(1.3) by considering the inversive matrix of corresponding vector matrix equation. The properties of the Green function are also considered in this section. In this paper, we will be concerned with several eigenvalue characterizations of (1.2)-(1.3). Such problems have been extensively studied for discrete or continuous two-point boundary value problems. In Section 3, we will consider the eigenvalue characterizations of existence of one positive solution. The existence of triple solutions will be established in Section 4. In Section 5, we will give some remarks which explain some difference between differential equation and difference equation for the corresponding problem.

## 2. Inverse matrix and Green's function

For the boundary value problems, Green's functions and its properties are important. To this end, we let  $x = \text{col}(x_1, x_2, \dots, x_n)$  and

$$F(x) = \text{col}(f_1(x_1), f_2(x_2), \dots, f_n(x_n)), \quad (2.1)$$

then system (1.2)-(1.3) can be rewritten by the matrix form  $Ax = \lambda F(x)$ , where  $A = (a_{ij})_{n \times n}$ ,  $a_{ii} = 2$  ( $i = 1, 2, \dots, n$ ),  $a_{j,j+1} = a_{j+1,j} = -1$  ( $j = 1, 2, \dots, n-1$ ),  $a_{nl} = -a$ , and other entries are zero.

When  $|a| < (n+1)/l$ , we can get that  $A^{-1}$  exists. In fact, let  $B = (b_{ij})_{n \times n}$  (where  $b_{ii} = 2$  ( $i = 1, 2, \dots, n$ ),  $b_{j,j+1} = b_{j+1,j} = -1$  ( $j = 1, 2, \dots, n-1$ ), and other entries are zero) and  $H = (h_{ij})_{n \times n}$  (where  $h_{nl} = -a$  and other entries are zero), we have  $A = B + H$  and

$$A^{-1} = (B + H)^{-1} = (B(E + B^{-1}H))^{-1} = (E + B^{-1}H)^{-1}B^{-1}. \quad (2.2)$$

Note that  $B^{-1} = (g_{ij})_{n \times n}$ , where

$$g_{ij} = \begin{cases} \frac{j(n+1-i)}{n+1}, & 1 \leq j \leq i \leq n, \\ \frac{i(n+1-j)}{n+1}, & 1 \leq i \leq j \leq n. \end{cases} \quad (2.3)$$

Thus, we have

$$B^{-1}H = \frac{-a}{n+1}D, \quad (2.4)$$

where  $D = (d_{ij})_{n \times n}$ ,  $d_{il} = i$  for  $i = 1, 2, \dots, n$  and other entries are zero. It is well known that

$$(E + B^{-1}H)^{-1} = E + \sum_{k=1}^{\infty} (-1)^k (B^{-1}H)^k, \quad (2.5)$$

it is easy to get

$$(B^{-1}H)^k = l^{k-1} \left( \frac{-a}{n+1} \right)^k D, \quad (2.6)$$

and furthermore,

$$\sum_{k=1}^{\infty} (-1)^k (B^{-1}H)^k = \sum_{k=1}^{\infty} l^{k-1} \left( \frac{a}{n+1} \right)^k D = \frac{a}{n+1-la} D. \quad (2.7)$$

Thus, we have

$$\begin{aligned} (E + B^{-1}H)^{-1} &= E + \frac{a}{n+1-la} D, \\ A^{-1} &= (E + B^{-1}H)^{-1}B^{-1} = \left( g_{ij} + \frac{ia}{n+1-la} g_{lj} \right)_{n \times n}. \end{aligned} \quad (2.8)$$

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LEMMA 2.1. When  $|a| < (n+1)/l$ , the matrix  $A$  is invertible and its inversion is

$$A^{-1} = \left( g_{ij} + \frac{ia}{n+1-la} g_{lj} \right)_{n \times n}. \quad (2.9)$$

In view of Lemma 2.1, system (1.2)-(1.3) can be rewritten by  $x = \lambda A^{-1}F(x)$  or

$$x_i = \lambda \sum_{j=1}^n \left( g_{ij} + \frac{ia}{n+1-la} g_{lj} \right) f_j(x_j), \quad i = 1, 2, \dots, n. \quad (2.10)$$

Naturally, we can call that

$$G(i, j) = g_{ij} + \frac{ia}{n+1-la} g_{lj}, \quad 1 \leq i, j \leq n, \quad (2.11)$$

is the Green function of problem (1.2)-(1.3), where  $1 \leq l \leq n$  is a fixed integer,  $a$  is a real constant, and they satisfy the condition  $|a| < (n+1)/l$ . For  $1 \leq i, j \leq n$ ,  $g_{ij}$  is defined by (2.3).

In the following, we will discuss the properties of  $G(i, j)$ . In this paper, we will be concerned with the existence of positive solutions for (1.2)-(1.3). Thus, we ask  $G(i, j) > 0$  for  $1 \leq i, j \leq n$ . When  $0 \leq a < (n+1)/l$ , it easily follows that  $G(i, j) > 0$  for  $1 \leq i, j \leq n$ . In the following, we assume that  $a < 0$ . Note that

$$\min_{1 \leq i, j \leq n} g_{ij} = g_{1n} = g_{n1} = \frac{1}{n+1}. \quad (2.12)$$

Thus, we only need to consider the sign of

$$\frac{1}{n+1} + \frac{na}{n+1-la} g_{lj}. \quad (2.13)$$

By the definition of  $g_{lj}$ , we have

$$\begin{aligned} \frac{1}{n+1} + \frac{na}{n+1-la} g_{lj} &= \begin{cases} \frac{1}{n+1} \left( 1 + \frac{naj(n+1-l)}{n+1-la} \right), & 1 \leq j \leq l \leq n, \\ \frac{1}{n+1} \left( 1 + \frac{nal(n+1-j)}{n+1-la} \right), & 1 \leq l \leq j \leq n, \end{cases} \\ &\geq \frac{1}{n+1} \left( 1 + \frac{nal(n+1-l)}{n+1-la} \right) > 0 \end{aligned} \quad (2.14)$$

which implies that

$$a > -\frac{n+1}{l[n(n+1-l)-1]}. \quad (2.15)$$

LEMMA 2.2. Assume that

$$-\frac{n+1}{l[n(n+1-l)-1]} < a < \frac{n+1}{l} \quad (2.16)$$

holds, then the Green function  $G(i, j)$  is positive for  $1 \leq i, j \leq n$ .

In the following, we will give the estimation of  $G(i, j)$ .

When  $0 \leq a < (n+1)/l$ , we have

$$\begin{aligned} g_{ij} + \frac{ia}{n+1-la} g_{lj} &\leq g_{\lceil (n+1)/2 \rceil, \lceil (n+1)/2 \rceil} + \frac{na}{n+1-la} g_{ll} \\ &= \frac{1}{n+1} \left( \left\lceil \frac{n+1}{2} \right\rceil \left( n+1 - \left\lceil \frac{n+1}{2} \right\rceil \right) + \frac{nal(n+1-l)}{n+1-la} \right) = M, \\ g_{ij} + \frac{ia}{n+1-la} g_{lj} &\geq \frac{1}{n+1} \left( 1 + \frac{a(n+1-l)}{n+1-la} \right) = m, \end{aligned} \quad (2.17)$$

where  $\lceil (n+1)/2 \rceil$  is the largest integer part of  $(n+1)/2$ .

When  $a$  is a negative number and the condition (2.15) holds, we have known that

$$g_{ij} + \frac{ia}{n+1-la} g_{lj} \geq \frac{1}{n+1} \left( 1 + \frac{nal(n+1-l)}{n+1-la} \right) = m', \quad (2.18)$$

and we have also

$$g_{ij} + \frac{ia}{n+1-la} g_{lj} \leq \frac{1}{n+1} \left( \left\lceil \frac{n+1}{2} \right\rceil \left( n+1 - \left\lceil \frac{n+1}{2} \right\rceil \right) + \frac{a(n+1-l)}{n+1-la} \right) = M'. \quad (2.19)$$

### 3. Existence of one positive solution

In the following, we will ask that the condition (2.16) hold. Let  $E$  be the Banach space defined by  $E = \{x \mid x : [1, n] \rightarrow \mathbb{R}\}$  with norm  $\|x\| = \max_{k \in [1, n]} |x_k|$ , and let

$$\begin{aligned} P &= \{x \in E \mid x_k \geq 0, k \in [1, n]\}, \\ C &= \{x \in E \mid x_k \geq 0, k \in [1, n]; x_k \geq \delta \|x\|\}, \end{aligned} \quad (3.1)$$

where  $\delta = m/M$  when  $0 \leq al < n+1$  and  $\delta = m'/M'$  when  $a$  is a negative number and (2.15) holds. Define an operator  $T : P \rightarrow E$  by

$$Tx_i = \lambda \sum_{j=1}^n G(i, j) f_j(x_j), \quad i = 1, 2, \dots, n. \quad (3.2)$$

Then for  $x \in P$ , we have

$$\begin{aligned} \|Tx\| = \max_{i \in [1, n]} |Tx_i| &\leq \begin{cases} \lambda M \sum_{j=1}^n f_j(x_j), & 0 \leq a < \frac{n+1}{l}, \\ \lambda M' \sum_{j=1}^n f_j(x_j), & -\frac{n+1}{l[n(n+1-l)-1]} < a < 0, \end{cases} \\ Tx_i &\geq \begin{cases} \lambda m \sum_{j=1}^n f_j(x_j), & 0 \leq a < \frac{n+1}{l}, \\ \lambda m' \sum_{j=1}^n f_j(x_j), & -\frac{n+1}{l[n(n+1-l)-1]} < a < 0, \end{cases} \end{aligned} \quad (3.3)$$

which imply that  $Tx_i \geq \delta \|Tx\|$ , that is,  $TP \subset C$ .

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**THEOREM 3.1.** *Assume that  $f_k(u)$  is nondecreasing for all  $k \in [1, n]$  and there is  $k_0 \in [1, n]$  such that  $f_{k_0}(u) > 0$  for  $u > 0$ . Then there exists  $c > 0$  such that the interval  $(0, c] \subset \Gamma$ .*

*Proof.* Let  $L > 0$  be given and denote

$$C(L) = \{x \in \mathbb{C} \mid \|x\| \leq L\}. \quad (3.4)$$

Define

$$c = \frac{L}{\max_{i \in [1, n]} \sum_{j=1}^n G(i, j) f_j(L)}. \quad (3.5)$$

Then we have

$$Tx_i \leq \lambda \sum_{j=1}^n G(i, j) f_j(L) \leq c \max_{i \in [1, n]} \sum_{j=1}^n G(i, j) f_j(L) = L \quad (3.6)$$

for  $\lambda \in (0, c]$  and  $x \in C(L)$ . By Schauder fixed point theorem,  $T$  has a fixed point in  $C(L)$ . The proof is complete.  $\square$

The following theorem is immediately obtained by using Theorem 3.1.

**THEOREM 3.2.** *Assume that all conditions of Theorem 3.1 hold. Then  $\lambda_0 \in \Gamma$  implies that  $(0, \lambda_0] \subset \Gamma$ .*

**THEOREM 3.3.** *Assume that all conditions of Theorem 3.1 hold and let  $\lambda$  be an eigenvalue of (1.2)-(1.3) and let  $x \in \mathbb{C}$  be a corresponding eigenfunction. Further, let  $\|x\| = d$ . Then,*

$$\frac{d}{\max_{i \in [1, n]} \sum_{j=1}^n G(i, j) f_j(d)} \leq \lambda \leq \frac{d}{\max_{i \in [1, n]} \sum_{j=1}^n G(i, j) f_j(\delta d)}. \quad (3.7)$$

*Proof.* In fact, from

$$d = \|Tx\| = \lambda \max_{i \in [1, n]} \sum_{j=1}^n G(i, j) f_j(x_j), \quad (3.8)$$

we can obtain that

$$\lambda = \frac{d}{\max_{i \in [1, n]} \sum_{j=1}^n G(i, j) f_j(x_j)}, \quad (3.9)$$

which implies that

$$\frac{d}{\max_{i \in [1, n]} \sum_{j=1}^n G(i, j) f_j(d)} \leq \lambda \leq \frac{d}{\max_{i \in [1, n]} \sum_{j=1}^n G(i, j) f_j(\delta d)}. \quad (3.10)$$

The proof is complete.  $\square$

By Theorems 3.1–3.3 and the definition of  $G(i, j)$ , we can also obtain the following theorem.

**THEOREM 3.4.** *Assume that all conditions of Theorem 3.1 hold. Then the following hold:*

(a) if

$$\frac{u}{\sum_{j=1}^n f_j(u)} \quad (3.11)$$

is bounded for  $u \in (0, \infty)$ , then there exists  $c \in (0, \infty)$  such that  $\Gamma = (0, c)$  or  $(0, c]$ ;

(b) if

$$\lim_{u \rightarrow \infty} \frac{u}{\sum_{j=1}^n f_j(u)} = 0, \quad (3.12)$$

then there exists  $c \in (0, \infty)$  such that  $\Gamma = (0, c]$ ;

(c) if

$$\lim_{u \rightarrow \infty} \frac{u}{\sum_{j=1}^n f_j(u)} = \infty, \quad (3.13)$$

then  $\Gamma = (0, \infty)$ .

*Proof.* (a) and (c) are clear. In the following, we will prove that (b) is a fact. By Theorem 3.1, we know that there exists  $c' > 0$  such that  $(0, c'] \subset \Gamma$ . Note that from the condition

$$\lim_{u \rightarrow \infty} \frac{u}{\sum_{j=1}^n f_j(u)} = 0 \quad (3.14)$$

and Theorem 3.3, we can get that  $\Gamma$  is bounded. Let  $c = \sup \Gamma$  and suppose that the eigenvalues sequence  $\{\lambda_k\}_{k=1}^{\infty}$  of (1.2)–(1.3) is strict increasing and satisfies  $\lim_{k \rightarrow \infty} \lambda_k = c$ , the sequence  $\{x^{(k)}\}_{k=1}^{\infty}$  is the corresponding solutions sequence of  $\{\lambda_k\}_{k=1}^{\infty}$ . Then, we have

$$\begin{aligned} \|x^{(k)}\| = \|Tx^{(k)}\| &= \lambda_k \max_{i \in [1, n]} \sum_{j=1}^n G(i, j) f_j(x_j^{(k)}) \geq \lambda_k \min\{m, m'\} \sum_{j=1}^n f_j(\delta \|x^{(k)}\|), \\ \lambda_k \min\{m, m'\} &\leq \frac{\|x^{(k)}\|}{\sum_{j=1}^n f_j(\delta \|x^{(k)}\|)} \end{aligned} \quad (3.15)$$

which implies that

$$\frac{\|x^{(k)}\|}{\sum_{j=1}^n f_j(\delta \|x^{(k)}\|)} \quad (3.16)$$

is bounded below. By this and the condition

$$\lim_{u \rightarrow \infty} \frac{u}{\sum_{j=1}^n f_j(u)} = 0, \quad (3.17)$$

we get that  $\{\|x^{(k)}\|\}$  is bounded. For every  $i \in [1, n]$ , choosing the subsequence  $\{x_i^{(k_i)}\}$  such that  $\lim_{k_i \rightarrow \infty} x_i^{(k_i)} = \limsup_{k \rightarrow \infty} x_i^{(k)} = x_i^{(0)}$ , then  $x^{(0)}$  is a positive solution of the equation

$$Tx_i = c \sum_{j=1}^n G(i, j) f_j(x_j), \quad i = 1, 2, \dots, n. \quad (3.18)$$

The proof is complete.  $\square$

In the following, we do not require the monotonicity of  $f_k$  for  $k \in [1, n]$ , but a fixed point theorem will be used. It can be seen in [20, 21].

LEMMA 3.5. *Let  $E$  be a Banach space, and let  $C \subset E$  be a cone. Assume  $\Omega_1, \Omega_2$  are open subsets of  $E$  with  $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$ , and let  $T : C \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow C$  be a completely continuous operator such that either (a)  $\|Ty\| \leq \|y\|$ ,  $y \in C \cap \partial\Omega_1$ , and  $\|Ty\| \geq \|y\|$ ,  $y \in C \cap \partial\Omega_2$ , or (b)  $\|Ty\| \geq \|y\|$ ,  $y \in C \cap \partial\Omega_1$ , and  $\|Ty\| \leq \|y\|$ ,  $y \in C \cap \partial\Omega_2$ . Then,  $T$  has a fixed point in  $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$ .*

For the sake of convenience, we set

$$A_0 = \max_{i \in [1, n]} \sum_{j=1}^n G(i, j), \quad B_0 = \min_{i \in [1, n]} \sum_{j=1}^n G(i, j). \quad (3.19)$$

THEOREM 3.6. *Suppose that there exist two positive numbers  $a$  and  $b$  such that  $a \neq b$  and*

$$\frac{b}{B_0 \min_{k \in [1, n], u \in [\delta x, x] \cap [0, b]} f_k(u)} < \frac{a}{A_0 \max_{k \in [1, n], u \in [0, a]} f_k(u)}. \quad (3.20)$$

Then for every  $\lambda$  satisfying

$$\frac{b}{B_0 \min_{k \in [1, n], u \in [\delta x, x] \cap [0, b]} f_k(u)} \leq \lambda \leq \frac{a}{A_0 \max_{k \in [1, n], x \in [0, a]} f_k(u)}, \quad (3.21)$$

the boundary value problem (1.2)-(1.3) has a positive solution.

*Proof.* Assume that  $a < b$  and choose  $x \in \partial C_a$ . Then we have

$$Tx_i \leq \lambda \max_{k \in [1, n], x \in [0, a]} f_k(x) \sum_{j=1}^n G(i, j) \leq \lambda A_0 \max_{k \in [1, n], x \in [0, a]} f_k(x) \leq a \quad (3.22)$$

which implies that  $\|Tx\| \leq \|x\|$  for  $x \in \partial C_a$ . Let  $x \in \partial C_b$ , then we get that

$$Tx_i \geq \lambda B_0 \min_{k \in [1, n], u \in [\delta y, y] \cap [0, b]} f_k(u) \geq b \quad \text{for } x \in \partial C_b, \quad (3.23)$$

which implies that  $\|Tx\| \leq \|x\|$  for  $x \in \partial C_b$ . The proof is complete by Lemma 3.5. When  $a > b$ , the proof is similar.  $\square$



The following theorem is immediately obtained by using Theorem 3.6.

**THEOREM 3.7.** *Assume that there exists  $k_0 \in [1, n]$  such that  $f_{k_0}(u) > 0$  for  $u > 0$  and that  $f_{k_\infty}$  and  $f_{0k}$  are finite for any  $k \in [1, n]$ . Then for each  $\lambda$  satisfying*

$$\frac{1}{\delta B_0 \min_{k \in [1, n]} f_{k_\infty}} < \lambda < \frac{1}{A_0 \max_{k \in [1, n]} f_{k_0}} \quad (3.24)$$

or

$$\frac{1}{\delta B_0 \min_{k \in [1, n]} f_{k_0}} < \lambda < \frac{1}{A_0 \max_{k \in [1, n]} f_{k_\infty}}, \quad (3.25)$$

the boundary value problem (1.2)-(1.3) has a positive solution.

*Proof.* If the condition (3.24) is a fact, we can choose  $\varepsilon > 0$  such that

$$\frac{1}{\delta B_0 (\min_{k \in [1, n]} f_{k_\infty} - \varepsilon)} < \lambda < \frac{1}{A_0 (\max_{k \in [1, n]} f_{k_0} + \varepsilon)}. \quad (3.26)$$

Let  $H_1 > 0$  such that  $f_k(u) \leq (\max_{k \in [1, n]} f_{k_0} + \varepsilon)u$  for  $0 < u < H_1$  and  $k \in [1, n]$ . For  $x \in \mathbb{C}$  which satisfies  $\|x\| = H_1$ , we have

$$Tx_i \leq \lambda \left( \max_{k \in [1, n]} f_{k_0} + \varepsilon \right) \|x\| \sum_{j=1}^n G(i, j) \leq \lambda A_0 H_1 \left( \max_{k \in [1, n]} f_{k_0} + \varepsilon \right) < H_1, \quad (3.27)$$

which implies that  $\|Tx\| \leq \|x\|$  for  $x \in \partial C_{H_1}$ .

Choosing  $\bar{H}_2 > 0$  such that  $f(u) \geq (\min_{k \in [1, n]} f_{k_\infty} - \varepsilon)u$  for  $x \geq \delta \bar{H}_2$  and let  $H_2 = \max\{2H_1, \bar{H}_2\}$ , for  $x \in \partial C_{H_2}$ , we have

$$Tx_i \geq \lambda \left( \min_{k \in [1, n]} f_{k_\infty} - \varepsilon \right) \|x\| \sum_{j=1}^n G(i, j) \geq \lambda \left( \min_{k \in [1, n]} f_{k_\infty} - \varepsilon \right) \delta H_2 B_0 > H_2, \quad (3.28)$$

which implies that  $\|Tx\| \geq \|x\|$  for  $x \in \partial C_{H_2}$ .

When the condition (3.25) holds, the proof is similar. The proof is complete.  $\square$

By Theorem 3.6, we can similarly obtain the following results. Their proofs will be omitted.

**THEOREM 3.8.** *For every  $k \in [1, n]$ ,  $f_{k_0} = 0$ , and  $f_{k_\infty} = \infty$  or  $f_{k_0} = \infty$  and  $f_{k_\infty} = 0$ , then  $\Gamma = (0, \infty)$ .*

**THEOREM 3.9.** *For every  $k \in [1, n]$ ,  $f_{k_0} = \infty$  or  $f_{k_\infty} = \infty$ , then there exists  $\lambda^* > 0$  such that  $(0, \lambda^*) \subset \Gamma$ .*

**THEOREM 3.10.** *For every  $k \in [1, n]$ ,  $f_{k_0} = 0$ , or  $f_{k_\infty} = 0$ , then there exists  $\lambda^{**} > 0$  such that  $(\lambda^{**}, \infty) \subset \Gamma$ .*

**THEOREM 3.11.** *For every  $k \in [1, n]$ ,  $f_{k_0} = \infty$ , or  $f_{k_\infty} = L$ , then  $(0, 1/(A_0 L)) \subset \Gamma$ . For every  $k \in [1, n]$ ,  $f_{k_0} = l$ , or  $f_{k_\infty} = \infty$ , then  $(0, 1/(A_0 l)) \subset \Gamma$ .*

#### 4. Existence of triple positive solutions

In this section, we will consider the existence of triple positive solutions for the system (1.2)-(1.3). To this end, we firstly give the definition of a concave nonnegative continuous functional and the Leggett-Williams fixed point theorem.

Let  $E$  be a Banach space, and let  $P \subset E$  be a cone. By a concave nonnegative continuous functional  $\psi$  on  $P$ , we mean a continuous mapping  $\psi : P \rightarrow [0, +\infty)$  with

$$\psi(\mu x + (1 - \mu)y) \geq \mu\psi(x) + (1 - \mu)\psi(y), \quad x, y \in P, \mu \in [0, 1]. \quad (4.1)$$

Let  $\xi, \alpha, \beta$  be positive constants, we will employ the following notations:

$$\begin{aligned} P_\xi &= \{y \in P : \|y\| < \xi\}, & \bar{P}_\xi &= \{y \in P : \|y\| \leq \xi\}, \\ P(\psi, \alpha, \beta) &= \{y \in \bar{P}_\beta : \psi(y) \geq \alpha\}. \end{aligned} \quad (4.2)$$

Our existence criteria will be based on the Leggett-Williams fixed point theorem (see [22]).

**LEMMA 4.1.** *Let  $E$  be a Banach space,  $P \subset E$  a cone of  $E$ , and  $R > 0$  a constant. Suppose there exists a concave nonnegative continuous functional  $\psi$  on  $P$  with  $\psi(y) \leq \|y\|$  for  $y \in \bar{P}_R$ . Let  $T : \bar{P}_R \rightarrow \bar{P}_R$  be a completely continuous operator. Assume there are numbers  $r, L$ , and  $K$  with  $0 < r < L < K \leq R$  such that*

- (H1) *the set  $\{y \in C(\psi, L, K) : \psi(y) > L\}$  is nonempty and  $\psi(Ty) > L$  for all  $y \in P(\psi, L, K)$ ;*
- (H2)  *$\|Ty\| < r$  for  $y \in \bar{P}_r$ ;*
- (H3)  *$\psi(Ty) > L$  for all  $y \in P(\psi, L, R)$  with  $\|Ty\| > K$ .*

*Then  $T$  has at least three fixed points  $y_1, y_2$ , and  $y_3 \in \bar{P}_R$ . Furthermore,  $y_1 \in P_r, y_2 \in \{y \in P(\psi, L, R) : \psi(y) > L\}$ , and  $y_3 \in \bar{P}_R \setminus (P(\psi, L, R) \cup \bar{P}_r)$ .*

We use Lemma 4.1 to establish the existence of three positive solutions to (1.2)-(1.3).  $E$  and  $P$  are defined by the above section.

**THEOREM 4.2.** *Assume that  $f_k(u)$  is nondecreasing for  $k \in [1, n]$  and that  $f_{k\infty} = f_{k0} = 0$  for  $k \in [1, n]$ . Suppose further that there is a number  $L > 0$  such that  $f_k(L) > 0$  for  $k \in [1, n]$ . Let  $R, K, L$ , and  $r$  be four numbers such that*

$$R \geq K > \frac{L}{\delta} \geq L > r > 0, \quad (4.3)$$

$$\frac{\max_{k \in [1, n]} f_k(r)}{r} < \frac{\max_{k \in [1, n]} f_k(R)}{R} < \frac{B_0 \min_{k \in [1, n]} f_k(L)}{A_0 L}. \quad (4.4)$$

*Then for each  $\lambda \in (L/(B_0 \min_{k \in [1, n]} f_k(L)), R/(A_0 \max_{k \in [1, n]} f_k(R))$ , there exist three non-negative solutions  $x^{(1)}, x^{(2)}$ , and  $x^{(3)}$  of (1.2)-(1.3) associated with  $\lambda$  such that  $x_k^{(1)} < r < x_k^{(2)} < L < x_k^{(3)} \leq R$  for  $k \in [1, n]$ .*

*Proof.* Note that if  $f_k(L) > 0$ , then by the monotonicity of  $f_k, f_k(R) > 0$  for any  $R$  greater than  $L$ . In view of  $f_{k\infty} = 0$ , we may choose  $R \geq K > L$  such that the second inequality in (4.4) holds, and in view of  $f_{k0} = 0$ , we may choose  $r \in (0, L)$  such that the first inequality in (4.4) holds. Let us set  $\lambda_1 = L/(B_0 \min_{k \in [1, n]} f_k(L))$  and  $\lambda_2 = R/(A_0 \max_{k \in [1, n]} f_k(R))$ .

Then  $\lambda_1, \lambda_2 > 0$ . Furthermore,  $\lambda_1 < \lambda_2$  in view of (4.4). We now define for each  $\lambda \in (\lambda_1, \lambda_2]$  a continuous mapping  $T : P \rightarrow P$  by

$$Tx_i = \lambda \sum_{j=1}^n G(i, j) f_j(x_j), \quad i = 1, 2, \dots, n, \quad (4.5)$$

and a functional  $\psi : P \rightarrow [0, \infty)$  by

$$\psi(x) = \min_{k \in [1, n]} x_k. \quad (4.6)$$

In view of  $f_{k\infty} = f_{k0} = 0$  and (4.4), we have

$$Tx_i = \lambda \sum_{j=1}^n G(i, j) f_j(x_j) \leq \lambda \max_{k \in [1, n]} f_k(R) \sum_{j=1}^n G(i, j) \leq \lambda_2 A_0 \max_{k \in [1, n]} f_k(R) = R, \quad (4.7)$$

for  $k \in [1, n]$  and all  $x \in \bar{P}_R$ . Therefore,  $T(\bar{P}_R) \subset \bar{P}_R$ . We assert that  $P$  is completely continuous on  $\bar{P}_R$  because  $E$  is a finite-dimensional space.

We now assert that (H2) of Lemma 4.1 holds. Indeed,

$$Tx_i = \lambda \sum_{j=1}^n G(i, j) f_j(x_j) \leq \lambda A_0 \max_{k \in [1, n]} f_k(r) \leq \lambda_2 A_0 \max_{k \in [1, n]} f_k(r) < r \quad (4.8)$$

for all  $y \in \bar{P}_r$ , where the last inequality follows from (4.4).

In addition, we can show that the condition (H1) of Lemma 4.1 holds. Obviously  $\psi(x)$  is a concave continuous function on  $P$  with  $\psi(x) \leq \|x\|$  for  $y \in \bar{P}_R$ . We notice that if  $u_k = (1/2)(L + K)$  for  $k \in [1, n]$ , then  $u \in \{x \in P(\psi, L, K) : \psi(x) > L\}$ , which implies that  $\{x \in P(\psi, L, K) : \psi(x) > L\}$  is nonempty. For  $x \in P(\psi, L, K)$ , we have  $\psi(x) = \min_{k \in [1, n]} x_k \geq L$  and  $\|x\| \leq K$ . In view of the conditions of Theorem 3.7, we have

$$\psi(Tx) = \lambda \min_{k \in [1, n]} \sum_{j=1}^n G(i, j) f_j(x_j) \geq \lambda B_0 \min_{k \in [1, n]} f_k(L) > \lambda_1 B_0 \min_{k \in [1, n]} f_k(L) = L, \quad (4.9)$$

for all  $x \in C(\psi, L, K)$ .

Finally, we prove condition (H3) in Lemma 4.1. Let  $x \in P(\psi, L, R)$  with  $\|Tx\| > K$ . We notice that (4.5) implies

$$\|Tx\| \leq \lambda \max\{M, M'\} \sum_{j=1}^n f_j(x_j). \quad (4.10)$$

Thus,

$$\psi(Tx) = \lambda \min_{i \in [1, n]} \sum_{j=1}^n G(i, j) f_j(x_j) \geq \lambda \min\{m, m'\} \sum_{j=1}^n f_j(x_j) \geq \delta \|Tx\| > \delta K > L. \quad (4.11)$$

An application of Lemma 4.1 stated above now yields our proof.  $\square$

**THEOREM 4.3.** *Assume that  $f_k(u)$  is nondecreasing for  $k \in [1, n]$  and that  $f_{k_\infty} > f_{k_0} > 0$  for  $k \in [1, n]$ . Suppose there is a number  $L > 0$  such that  $f_k(L) > 0$  and  $0 < f_{k_0} < f_{k_\infty} < B_0 \min_{k \in [1, n]} f_k(L)/(A_0 L)$ . Denote that*

$$\begin{aligned} \lim_{u \rightarrow 0} \frac{\max_{k \in [1, n]} f_k(u)}{u} &= l_1, \\ \lim_{u \rightarrow \infty} \frac{\max_{k \in [1, n]} f_k(u)}{u} &= l_2. \end{aligned} \tag{4.12}$$

Let  $R, K, L$ , and  $r$  be four numbers such that

$$R \geq K > \frac{L}{\delta} \geq L > r > 0, \tag{4.13}$$

$$\frac{\max_{k \in [1, n]} f_k(R)}{R} < l_2 + \varepsilon, \tag{4.14}$$

$$\frac{\max_{k \in [1, n]} f_k(r)}{r} < l_1 + \varepsilon, \tag{4.15}$$

where  $\varepsilon$  is a positive number such that

$$l_2 + \varepsilon < \frac{B_0 \min_{k \in [1, n]} f_k(L)}{A_0 L}. \tag{4.16}$$

Then for each  $\lambda \in (L/(B_0 \min_{k \in [1, n]} f_k(L)), 1/(A_0 l_2))$ , system (1.2)-(1.3) has at least three nonnegative solutions  $x^{(1)}$ ,  $x^{(2)}$ , and  $x^{(3)}$  associated with  $\lambda$  such that  $x_k^{(1)} < r < x_k^{(2)} < L < x_k^{(3)} \leq R$  for  $k \in [1, n]$ .

*Proof.* Note that if  $f_k(L) > 0$ , then  $f_k(R) > 0$  for any  $R$  greater than  $L$ . Let  $\lambda_1 = L/(B_0 \min_{k \in [1, n]} f_k(L))$  and  $\lambda_2 = 1/(A_0 l_2)$ . Then  $\lambda_1, \lambda_2 > 0$ . Furthermore,  $0 < \lambda_1 < \lambda_2$  in view of the condition  $0 < l_1 < l_2 < B_0 \min_{k \in [1, n]} f_k(L)/(A_0 L)$ . For the positive  $\varepsilon$  that satisfies (4.16) and any  $\lambda \in (\lambda_1, \lambda_2)$ , there is  $R \geq K > L$  such that (4.14) holds, and there is  $r \in (0, L)$  such that (4.15) holds.

We now define for each  $\lambda \in (\lambda_1, \lambda_2)$  a continuous mapping  $T : P \rightarrow P$  by (4.5) and a functional  $\psi : C \rightarrow [0, +\infty)$  by (4.6). For all  $x \in \bar{P}_R$ , we have

$$\begin{aligned} Tx_i &= \lambda \sum_{j=1}^n G(i, j) f_j(x_j) \leq \lambda \max_{k \in [1, n]} f_k(R) \sum_{j=1}^n G(i, j) \\ &\leq \lambda A_0 \max_{k \in [1, n]} f_k(R) \leq \lambda_2 A_0 (l_2 + \varepsilon) R = R, \end{aligned} \tag{4.17}$$

Furthermore, condition (H2) of Lemma 4.1 holds. Indeed, for  $x \in \bar{P}_r$ , we have

$$\begin{aligned} (Ay)(t) &= \lambda \sum_{j=1}^n G(i, j) f_j(x_j) \leq \lambda \max_{k \in [1, n]} f_k(r) \sum_{j=1}^n G(i, j) \\ &\leq \lambda A_0 \max_{k \in [1, n]} f_k(r) \leq \lambda A_0 (l_1 + \varepsilon) r < r. \end{aligned} \tag{4.18}$$

Similarly, we can prove that the conditions (H1) and (H3) of Lemma 4.1 hold. An application of Lemma 4.1 now yields our proof.  $\square$

**THEOREM 4.4.** Assume that  $f_k(u)$  is nondecreasing for  $k \in [1, n]$  and there is  $k_0 \in [1, n]$  such that  $f_{k_0}(0) > 0$ . Suppose there exist four numbers  $L, R, K$ , and  $r$  such that (4.3) and (4.4) hold. Then for each  $\lambda \in (L/(B_0 \min_{k \in [1, n]} f_k(L)), R/(A_0 \max_{k \in [1, n]} f_k(R))]$ , system (1.2)-(1.3) has at least three positive solutions  $x^{(1)}, x^{(2)}$ , and  $x^{(3)}$  associated with  $\lambda$  such that  $0 < x^{(1)} < r < x^{(2)} < L < x^{(3)} \leq R$  for  $k \in [1, n]$ .

The proof is similar to Theorem 4.2, and hence is omitted.

**5. Some remarks**

The second-order difference equation

$$\Delta^2 x_{k-1} + \lambda f(k, x_k) = 0, \quad k = 1, 2, \dots, n, \tag{5.1}$$

is the discrete analog of equation

$$x''(t) + \lambda f(t, x) = 0, \quad t \in [0, 1]. \tag{5.2}$$

The three-point or multipoint boundary value problems in the continuous case have been studied in great detail in the recent papers [1–11] since the early 1980s. For the continuous case, the boundary value condition (1.3) is of the form

$$x(0) = 0, \quad x(1) = ax(\eta), \quad \eta \in (0, 1). \tag{5.3}$$

When we consider the existence of positive solutions of problem (5.2)-(5.3),  $a > 0$  is clearly a necessary condition.

*Remark 5.1.* When

$$-\frac{n+1}{l[n(n+1-l)-1]} < a < 0, \tag{5.4}$$

our theorems cannot be similarly established for the continuous case.

*Remark 5.2.* In Section 2, we obtain the Green function by using matrix’s method. Such method is different from the continuous case.

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Huting Yuan: Department of Mathematics, Yanbei Normal University, Datong, Shanxi 037000, China  
*Email address:* sxdtyht@163.com

Guang Zhang: School of Science, Tianjin University of Commerce, Tianjin 300134, China  
*Email address:* qd\_gzhang@126.com

Hongliang Zhao: Department of Mathematics, Qingdao Technological University, 11 Fushun Road, Qingdao 266033, China  
*Email address:* shzhl@qtech.edu.cn