A DYNAMICAL MODEL OF TERRORISM

FIRDAUS UDWADIA, GEORGE LEITMANN, AND LUCA LAMBERTINI

This paper develops a dynamical model of terrorism. We consider the population in a given region as being made up of three primary components: terrorists, those susceptible to both terrorist and pacifist propaganda, and nonsusceptibles, or pacifists. The dynamical behavior of these three populations is studied using a model that incorporates the effects of both direct military/police intervention to reduce the terrorist population, and nonviolent, persuasive intervention to influence the susceptibles to become pacifists. The paper proposes a new paradigm for studying terrorism, and looks at the long-term dynamical evolution in time of these three population components when such interventions are carried out. Many important features—some intuitive, others not nearly so—of the nature of terrorism emerge from the dynamical model proposed, and they lead to several important policy implications for the management of terrorism. The different circumstances in which nonviolent intervention and/or military/police intervention may be beneficial, and the specific conditions under which each mode of intervention, or a combination of both, may be useful, are obtained. The novelty of the model presented herein is that it deals with the *time evolution* of terrorist activity. It appears to be one of the few models that can be tested, evaluated, and improved upon, through the use of actual field data.

1. Introduction

The 21st century has been marked by a new kind of warfare—terrorism. The roots of terrorism can be identified as lying in economic, religious, psychological, philosophical, and political aspects of society. Incidents of terrorism can often be sparked by the deterioration of some local conditions, in a spatial sense, as perceived by a small segment of society. Several qualitative models of terrorism are related to different ways of prioritization

of terrorist targets, security measures available, and the psychological impact that such activity may have on the local population in a given area, and in the world at large. Other qualitative models argue in terms of group behavior that is differentiated as being ideological, grievance-driven, and understandable (Pitchford [13]). Several attempts to study the phenomenon of terrorism using the case-study method have been carried out, attempting to look at the similarities and differences between various acts of terrorism while placing them in a historical context. Some models that are more quantitative utilize the economic approach using utility functions to model terrorist behavior, and to assess the losses created by terrorist actions (see Blomberg et al. [4]; Chen and Siems [5]; Frey et al. [10]), while others use optimal control methods to model governmental actions aimed at maximizing security under the constraints posed by the optimal trajectories selected by terrorist (see Faria [8]). While it is somewhat questionable whether such rational behavior can be imputed to fundamentalist extremist groups, the main difficulties posed by such models appear to be the ad hoc nature of the utility functions and "costs" assigned to terrorist groups and their pursuers (see Blomberg et al. [3]; Anderson and Carter [1]). Also whether extreme events such as suicide bombings fall within the framework of utility theory appears questionable. Still others try to develop static "rational-actor" models for negotiating and bargaining with the demands of transnational terrorists (see Sandler et al. [17]; Atkinson et al. [2]; Sandler and Enders [16]). Strategic response to terrorist activities using game theoretic approaches has been looked at by Sandler and Arce [15], Frey and Luechinger [9], d'Artigues and Vignolo [6], and Sandler [14]. Probabilistic assessment of terrorist activities through the analysis of historical data is another approach that has been used in modeling terrorist activity. Work on rudimentary models of terrorist activities using game theory by developing attack-defense strategies when multiple targets are to be defended under resource constraints has also been initiated (Guy Carpenter [11]). One of the main aims for developing such models of terrorism is the determination of suitable measures to counteract and control it, and to be able to get a prognosis of the environment in terms of its level of security and safety. Yet there appear to be very few models that are truly dynamical in nature and which therefore attempt to look at the time evolution of terrorist activity in a manner that can be usefully employed to yield actionable information. (For a thorough overview of the growing literature in this field, see Enders and Sandler [7].)

In this paper we present a simple dynamical model of terrorism in terms of the dynamics of the population of individuals who engage in terrorist activities. We imagine the population of a certain area (say the Gaza Strip, or the West Bank) as divided into two categories: terrorists (T) and nonterrorists (NT). The nonterrorists are further divided into those that are susceptible to terrorist propaganda—this segment of the population (perhaps, the Wahabis in certain areas of the world, or those educated in madrassas) we call the susceptibles (S)—and those that are not susceptible to such propaganda, who we refer to as nonsusceptibles (NS). We stipulate a reasonable model for the dynamics which includes the effect of military/police action against terrorists and the effect of nonviolent means to wean away the susceptible population from turning to terrorism. (Another stream of literature concerns the presence of any causal connections between factors like

education and poverty and the arising and growth of terrorism. See Krueger and Maleck-ove [12], *inter alia*.) One of the main motivations for the development of this dynamical model is the insights it provides to help understand the dynamical evolution of these different populations, to understand the different regimes of dynamical behavior that arise, and to point us in the right direction for asking the proper questions in order to predict and interdict terrorist activity.

2. The dynamical model

Let us say that the number of terrorists (T) in a certain geographical region (say, a city) at time τ is $x(\tau)$. As mentioned before, we will think of the nonterrorist population in the area as being made up of the population of susceptibles (S), $y(\tau)$, and of nonsusceptibles (NS), $z(\tau)$.

The number of terrorists in a given period of time can change because of several reasons: (1) direct recruitment by the terrorists of individuals from the susceptible population; the effectiveness of this is taken to be proportional to the product of the number of terrorists and the number of susceptibles; (2) effect of antiterrorist measures that are directed directly at reducing the terrorist population, such as military and police action/intervention, which we assume increases rapidly with, and as the square of, the number of terrorists in the region under concern; (3) number of terrorists that die from natural causes, or are killed in action, and/or self-destruct (as in the case of suicide bombers), which we assume to be proportional to the terrorist population itself; and (4) increase in the terrorist population primarily through the appeals by terrorists (in the region under concern) to other terrorist groups, through global propaganda using news media, and/or through the organized or voluntary recruitment/movement of terrorists from other regions into the region of concern, and also through population growth in this section of the population; this brings about an increase in the terrorist population that we assume is proportional to the number of terrorists. We capture these four effects then through the following differential equation that we posit for the evolution of the terrorist population in the geographical region of concern:

$$\frac{dx}{d\tau} = \hat{a}xy - \hat{b}x^2 + (\hat{c}_1 - \hat{c}_2)x, \tag{2.1}$$

where we assume for convenience that the parameters $\hat{a}$, $\hat{b}$, $\hat{c}_1$, and $\hat{c}_2$ are constant over the time-horizon of interest, and nonnegative. We denote time by the parameter τ. The term containing $\hat{c}_2$ refers to the death/destruction of members of the terrorist population either through natural causes or through suicide bombings, and the term containing $\hat{c}_1$ refers to their increase either through recruitment from among their own or the importation of terrorists from other geographical areas. The effectiveness of terrorists to attract susceptibles to their cause is described by the parameter $\hat{a}$, and the effectiveness of military/police action in reducing the numbers of terrorists is characterized by the parameter $\hat{b}$.

The change in the number of the susceptibles (S) in a given interval of time is likewise caused by several factors. (1) Depletion in their population caused by their direct contact with terrorists whose point of view they adopt. This is just the number of susceptibles that entered the ranks of terrorists, given before by $\hat{a}xy$. (2) Depletion in the population

of susceptibles caused by nonviolent propaganda done by governmental and nongovern-mental authorities that convince members of this population to use peaceful methods of engagement; and/or concessions made to disgruntled groups of susceptibles—these "carrots" offered may be of an economic, political, or other nature—to convince them to enter the ranks of the nonsusceptible (NS) population. We model this effect by assuming that the propaganda and/or concessions are targeted to the susceptible population, and that this propaganda intensifies rapidly as the number of terrorists in the geographical area under concern increases. We assume that the change this causes is proportional to the product $x^2 y$. (3) Increase in the population of susceptibles caused by the propaganda that is created through the notoriety and publicity of terrorist acts that are broadcast on global information channels, like television and printed media, that cause some members of the NS population to become susceptibles. (4) Increase in the susceptible population when individuals from outside the geographical area of concern are incited to move into the area, first as susceptibles (S), perhaps later going on to become terrorists. We assume that the changes in the S population attributable to this cause and the previous one are proportional to the number of terrorists in the region under concern. (5) The increase in the susceptible population proportional to its own size (e.g., children of individuals edu-cated in madrassas being educated, likewise, in madrassas). The evolution of the suscep-tible population adduced from these effects can be expressed by the differential equation

$$\frac{dy}{d\tau} = -\hat{a}xy - \hat{e}x^2 y + (\hat{f}_1 + \hat{f}_2)x + \hat{g}y, \qquad (2.2)$$

where we again assume that the parameters $\hat{e}$, $\hat{f}_1$, $\hat{f}_2$, and $\hat{g}$ are each a constant and non-negative over the time-horizons of interest. The parameter $\hat{e}$ signifies the effectiveness of nonviolent means in weaning away susceptibles into the NS (pacifist) population. The effect of individuals from the NS population moving to the S population is given by the term $\hat{f}_1 x$; the effect of individuals from outside of the region of concern being attracted to the region and becoming susceptibles is indicated by the term $\hat{f}_2 x$. The growth rate of the susceptible population is given by $\hat{g}$.

Lastly, the change in the number of nonsusceptibles (NS) in a given interval of time is described by (1) those members of the susceptible population that become NS by virtue of having altered their persuasions because of the nonviolent actions/propaganda/induce-ments of governmental and nongovernmental authorities, (2) those who become suscep-tibles due to the effects of global propaganda done by terrorists through news media, and the like, and (3) the increase in the NS population, which is proportional to their population numbers. This then may be described by the equation

$$\frac{dz}{d\tau} = \hat{e}x^2 y - \hat{f}_1 x + \hat{h}z, \qquad (2.3)$$

where we assume, for simplicity again, that the parameter $\hat{h}$, which is the growth rate of the NS population, is constant and nonnegative. The dynamical system is schematically illustrated in Figure 2.1. For the purposes of our analysis we will assume that $z \gg x, y$.
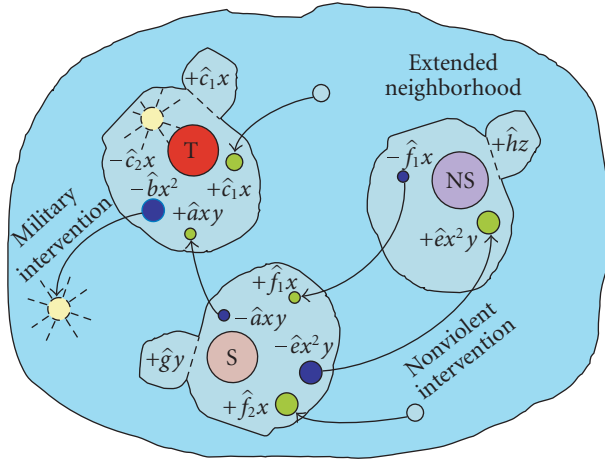
Figure 2.1. Schematic showing the dynamical system described by (2.1), (2.2), and (2.3).

We begin by dividing (2.1)–(2.3) by $\hat{c}_2$ and using the dimensionless time $t = \hat{c}_2\tau$. This yields the equations

$$\frac{dx}{dt} = axy - bx^2 + (c - 1)x, \tag{2.4}$$

$$\frac{dy}{dt} = -axy - ex^2y + fx + gy, \tag{2.5}$$

$$\frac{dz}{dt} = ex^2y - f_1x + hz, \tag{2.6}$$

where, all the constants are normalized so that $a = \hat{a}/\hat{c}_2$, $b = \hat{b}/\hat{c}_2$, $c = \hat{c}_1/\hat{c}_2$, $e = \hat{e}/\hat{c}_2$, $f_1 = \hat{f}_1/\hat{c}_2$, $f_2 = \hat{f}_2/\hat{c}_2$, $f = (f_1 + f_2)$, $g = \hat{g}/\hat{c}_2$, and $h = \hat{h}/\hat{c}_2$. We thus have a nonlinear system of three differential equations containing a total of 8 constant parameters all of which we will assume, for the purposes of this analysis, to be nonnegative.

3. Model dynamics

In this section our aim is to understand the unfolding of the dynamics of the system, to study its various regimes of behavior in the phase space (x, y, z), and investigate the manner in which the behavior changes as the values of the 8 parameters change.

We begin by noting that (2.6) is uncoupled from (2.4) and (2.5), and hence the entire dynamics of the evolution of the various populations is dependent only on the latter two equations. Once $x(t)$ and $y(t)$ are known, the dynamics of the NS population $z(t)$ is determined from (2.6). Since the system dynamics is then only dependent on the two coupled equations (2.4) and (2.5), we can rule out the possibility of having populations $x(t)$ and $y(t)$ that, in the strict technical sense, chaotically change with time. Because $x(t)$ and $y(t)$ represent the number of terrorists and the number of susceptibles, the region in

phase space that is of interest to us is limited to $x(t), y(t) \geq 0$. We will now show that in this quadrant of the phase space the orbits of the nonlinear dynamical system described by (2.4) and (2.5) cannot be closed for $b, f > 0$, and hence the nonlinear system is devoid of any limit cycles.

3.1. On the orbits of the dynamical system

Result 3.1. The nonlinear dynamical system described by (2.4) and (2.5) above does not have any limit cycles (closed orbits) in the first quadrant of the phase plane for $b, f > 0$.

Proof. Consider the function

$$
\begin{aligned}
q(x, y) &= \frac{\partial}{\partial x}[x^m y^n \{axy - bx^2 + (c-1)x\}] + \frac{\partial}{\partial y}[x^m y^n \{-axy - ex^2 y + fx + gy\}] \\
&= a(m+1)x^m y^{n+1} - b(m+2)x^{m+1} y^n + (c-1)(m+1)x^m y^n \\
&\quad - a(n+1)x^{m+1} y^n - e(n+1)x^{m+2} y^n + fn x^{m+1} y^{n-1} + g(n+1)x^m y^n.
\end{aligned}
\tag{3.1}
$$

Setting $m = n = -1$, we then obtain

$$
q(x, y) = -\frac{b}{y} - \frac{f}{y^2}.
\tag{3.2}
$$

Since $q(x, y)$ is negative in the first quadrant for $b, f > 0$, by Dulac's criterion there can be no closed orbits. $\square$

Result 3.2. All orbits that start at $t = 0$ in the first quadrant $x, y \geq 0$ remain in that quadrant for all time $t > 0$.

Proof. We begin by noting that the origin of the phase plane is always a fixed point; we therefore only need to concentrate on the flow on the x- and y-axes of the positive quadrant. Consider the flow on the x-axis. By (2.5) we see that the flow velocity in the y-direction is given by $dy/dt = fx \geq 0$. Since the flow does not have a negative component of velocity at any point along the positive x-axis, it cannot cross it. Similarly, along the y-axis, the x-component of the flow velocity is zero, so the flow cannot leave the first quadrant. $\square$

Furthermore, the flow at any point of the positive x-axis is pointed in either the positive or negative x-direction. When, in addition, $b = f = 0$, the flow is pointed in the positive x-direction for $c > 1$ and along the negative x-direction for $c < 1$; when $c = 1$, the x-axis becomes a line of fixed points. When $f = 0$, $b > 0$, and $c > 1$, the flow field at any point on the positive x-axis is pointed in the positive x-direction for $x < (c-1)/b$ and in the negative x-direction for $x > (c-1)/b$. When $c = 1$, the fixed point at $x = (c-1)/b$ moves to the origin, and the flow at any point on the entire positive x-axis is along the negative x-direction. For $c < 1$, $b \geq 0$, the x-component of the flow velocity at any point of the x-axis is always directed along negative x-direction.

3.2. Fixed points of the dynamical system. We begin by understanding the long-term evolutionary dynamics of the population of terrorists (T) and susceptibles (S) by identifying the fixed points of this nonlinear dynamical system described by (2.4) and (2.5). We observe that the point $x_0 = y_0 = 0$ is always a fixed point of the dynamical system. We now look at the other fixed points of the nonlinear equations (2.4) and (2.5) that lie in the positive quadrant of the phase space, and to begin with, we differentiate between four different possible situations. These cases are provided to introduce the different scenarios of interest in observing the dynamics of terrorism, and later on we will take up each of them in greater detail.

Case 1. When no intervention (military or nonviolent) is undertaken, that is, when the parameters $b = e = 0$. The fixed point of (2.4) and (2.5) is then given by

$$x_0 = \frac{g(1-c)}{a(1-c-f)}, \qquad y_0 = \frac{1-c}{a}. \tag{3.3}$$

We note that for the fixed point to lie in the first quadrant, we require $c \leq 1$ and $(f+c) \leq 1$. We will treat this case where there is no intervention as the "baseline situation," and in what follows in this section we will assume $c < 1$ and $f + c < 1$. What happen when $c \geq 1$ and/or $f + c \geq 1$ will be taken up later on.

Case 2. When nonviolent intervention is carried out against terrorists while abstaining from military/police intervention, the parameter $b = 0$. The fixed point of the dynamical system is now

$$x_0 = \frac{a(f+c-1) + \sqrt{a^2(f+c-1)^2 + 4eg(1-c)^2}}{2e(1-c)}, \qquad y_0 = \frac{1-c}{a}. \tag{3.4}$$

We note that the restriction $(f+c) < 1$ is no longer required for x_0 to be positive. However, as in the previous case, for y_0 to be nonnegative, we require $c \leq 1$. We observe that the effect of nonviolent intervention does not affect the steady state value (y_0) of the population of susceptibles when compared to that with no intervention at all.

Case 3. When military/police intervention against terrorism is carried out in the absence of nonviolent actions, $e = 0$, and the fixed point moves to

$$x_0 = \frac{gb - a(1-f-c) + \sqrt{[gb - a(1-f-c)]^2 + 4gab(1-c)}}{2ab}, \qquad y_0 = \frac{1-c+bx_0}{a}. \tag{3.5}$$

Military/police intervention appears to increase the steady state value of the susceptible population when compared with that for no intervention at all.

However, this case is a bit more complicated, as we will see later on, and it could lead to two fixed points when $g > 0$, $f = 0$, and $c > 1$, one of which will be shown to be unstable.

Case 4. When both military and nonviolent interventions against terrorism are implemented, that is, when $b, e > 0$, the fixed point is located at

$$y_0 = \frac{1 - c + bx_0}{a}, \tag{3.6}$$

where x_0 is the real positive root of the cubic equation

$$ebx^3 + [e(1 - c) + ab]x^2 + [a(1 - c - f) - gb]x - g(1 - c) = 0. \tag{3.7}$$

We observe that the addition of nonviolent intervention to military action does not affect the steady state population of susceptibles. Also, when $c < 1$, the first two coefficients of (3.7) (corresponding to the cubic term and the square term) are always positive, the last coefficient is always negative, and the coefficient of x is sign indefinite. Hence, from Descartes' rule of signs there can be only *one* positive root of this cubic equation. We will take up the case when $c > 1$ later on.

3.3. Stability of fixed points. We next inquire into the stability of the fixed points related to each of the above-mentioned cases.

Linearization of the nonlinear equations around the fixed point (x_0, y_0) so that $x(t) = x_0 + u(t)$, $y(t) = y_0 + v(t)$ leads to the linearized equations

$$\begin{bmatrix} \dfrac{du}{dt} \\ \dfrac{dv}{dt} \end{bmatrix} = \begin{bmatrix} ay - 2bx - (1 - c) & ax \\ f - ay - 2exy & g - ax - ex^2 \end{bmatrix}_{(x_0, y_0)} \begin{bmatrix} u \\ v \end{bmatrix} = J_{(x_0, y_0)} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{3.8}$$

In relation (3.8) we have denoted the Jacobian evaluated at the fixed point (x_0, y_0) by $J_{(x_0, y_0)}$. Since

$$J_{(0,0)} = \begin{bmatrix} -(1 - c) & 0 \\ f & g \end{bmatrix}, \tag{3.9}$$

its eigenvalues are $\lambda = -(1 - c)$ and $\lambda = g$, so that the fixed point $(0,0)$ is an *unstable saddle point* as long as $g > 0$ (later on, we will briefly consider the case $g = 0$ which is nonhyperbolic). The eigenvectors corresponding to the stable and unstable manifolds are then $[(1 - c + g)/f \quad -1]^T$ and $[0 \quad 1]^T$, respectively. The y-axis is thus part of the unstable manifold of the fixed point $(0,0)$.

Also, for the fixed point (x_0, y_0), we have, using the equations that govern the fixed point of the system (2.4) and (2.5),

$$J_{(x_0, y_0)} = \begin{bmatrix} -bx_0 & ax_0 \\ -\left(\dfrac{gy_0}{x_0} + ex_0 y_0\right) & -\dfrac{fx_0}{y_0} \end{bmatrix}, \quad x_0, y_0 > 0. \tag{3.10}$$

The eigenvalues, λ, of the matrix $J_{(x_0, y_0)}$ are the roots of the characteristic equation

$$\lambda^2 - \left(-\frac{fx_0}{y_0} - bx_o\right)\lambda + ax_0 \left(\frac{gy_0}{x_0} + ex_0 y_0\right) + \frac{bfx_0^2}{y_0} = 0. \tag{3.11}$$

Noting that for $b, f > 0$, the trace of the matrix in (3.10) is negative and its determinant is positive, we find that the fixed point (x_0, y_0) is either a *stable spiral* or a *stable node*.

Thus the fixed point at (x_0, y_0) is

(1) a stable spiral when

$$\left(\frac{f x_0}{y_0} - b x_0\right)^2 < 4 a x_0 \left(\frac{g y_0}{x_0} + e x_0 y_0\right), \tag{3.12}$$

(2) a stable node when

$$\left(\frac{f x_0}{y_0} - b x_0\right)^2 > 4 a x_0 \left(\frac{g y_0}{x_0} + e x_0 y_0\right). \tag{3.13}$$

For each of the four cases considered earlier, the location of the fixed points is provided by the relation (3.3)–(3.7); they are functions of the six parameters that specify the particular dynamical system being considered. As an example, for our baseline situation with $b = e = 0$, the fixed point given by relation (3.3) is a stable node when $f^2 > 4(1 - c)(1 - c - f)^2/g$, and, a stable spiral when $f^2 < 4(1 - c)(1 - c - f)^2/g$.

It should be observed that the system dynamics looks very different when the parameter $g = 0$. For then, the entire line $x = 0$ constitutes a line of fixed points. The Jacobian matrix given in relation (3.8) then becomes

$$J_{(0,y;g=0)} = \begin{bmatrix} ay - (1 - c) & 0 \\ f - ay & 0 \end{bmatrix} \tag{3.14}$$

making the stability of these nonisolated fixed points difficult to ascertain. The eigenvalues of the matrix in (3.14) are $\lambda = 0$ and $\lambda = ay - (1 - c)$. And so we may only conjecture, since the fixed points are nonhyperbolic, that for $y < (1 - c)/a$ the fixed points along the y-axis in the phase plane are stable (so that $\lambda < 0$), and that for $y > (1 - c)/a$ they become unstable. Numerical simulation shows that this conjecture is correct.

We observe in the above analysis (see (3.3)) that for the baseline situation, for which $b = e = 0$, we assumed in addition that $c < 1$ and $(f + c) < 1$. In order to obtain a better understanding of the "baseline situation" in which there is no military/police intervention as well as no nonviolent intervention, so that we can later on compare the behavior of the dynamics when we introduce interventions, we look at this situation here in some greater detail.

3.4. Baseline case ($b = e = 0$). We begin by looking at the feasible range of parameters that might be of interest in a more-or-less realistic situation. We begin with setting the time scale (recall, we are using $t = \hat{c}_2 \tau$ to denote our dimensionless time). Imagine a population of 400,000 inhabitants in a certain region (say, the city of Ar Ramadi in Iraq). Let us say that in a month, 5% of the terrorists in this city resort to successful suicide bombings. Thus $\hat{c}_2 = 0.05$/month, and each unit of dimensionless time, t, then corresponds to an actual duration, τ, of 20 months ($\tau = 1/0.05$). In this period of 20 months, let us assume that, on average, each terrorist persuades 1 member of the S population of this city from every 10,000 of its members to join the terrorist's cause. Thus

the parameter $a = 1/10{,}000 = 10^{-4}$. Let us say that during these 20 months, the susceptible population grows by 20 individuals for every 1,000 members of the S population, so that $g = 0.02$. Furthermore, we assume that 5 individuals, on average, from the nonsusceptible (NS) population are "softened up" and become susceptibles by the global notoriety/propaganda (through the news media, say) created by the collective actions/destruction wreaked by every 100 terrorists in this period of time (20 months), so that $f_1 = 0.05$. Also, since the region under consideration has "porous boundaries" and attracts terrorists from its neighboring regions (cities), we assume that the destructive activities of a 100 terrorists within the region cause 10 individuals to move from a neighboring region into the area and join the susceptibles camp, so that $f_2 = 0.1$. Lastly, we assume that during the 20-month unit of time under consideration, 120 trained terrorists (on average) move from outside the area into the area under consideration either voluntarily or through an organized network of world-wide terrorists as a consequence of the propaganda and/or requests made by every 100 terrorists who reside within the region; this makes $c = 1.2$. We then see that the parameters $a = 10^{-4}$, $c = 1.2$, $f = f_1 + f_2 = 0.15$, and $g = 0.02$ may well be within the realm of possibilities. Were the boundaries of the region to be assumed "secure," then we might expect only a trickle of individuals coming in from neighboring regions/cities into our region of concern (say, the city of Ar Ramadi) so that both the parameters f_2 and c would be much less than unity. However, in reality, at present it appears unlikely that they would be zero.

Consider the dynamics for $b = e = 0$ when $c = \hat{c}_1/\hat{c}_2 > 1$, so that the rate of increase of terrorists exceeds their destruction/death as seen from the linear term on the right-hand side of (2.1). Then the dynamical system, as seen from (3.3), points out that there is now no fixed point in the first quadrant of the phase space. The phase portrait of the nonlinear system can now best be understood geometrically by noting that (2.4) indicates that the x-component of the phase velocity is always positive in the first quadrant. The nullcline, which describes the curve for which the y-component of the phase velocity goes to zero, is given by the relation $axy - fx - gy = 0$, whose slope goes to zero when $y = f/a$. Above this nullcline the y-component of the velocity is negative, while below the nullcline it is positive causing the phase trajectories to asymptote to the line $y = f/a$. The positive x-component of the velocity of the phase particle causes x to be unbounded as $t \to \infty$. We illustrate this dynamical behavior in Figure 3.1.

We observe that when $c > 1$, there appears a rapid terrorist expansion; note the increase in the S population before its sudden drop in Figure 3.1(b). When $f = 0$, the S population asymptotically tends to zero.

When $c > 1$, but $(f + c) > 1$, (3.3) again points out that there is no fixed point in the positive first quadrant. This situation might arise when the borders of the region are maintained relatively secure, so that the influx of terrorists entering our region of concern is limited, which nonetheless may have a high number of people from the NS population (say, disgruntled pacifists who may no longer want to "sit on the sidelines") who are persuaded to become susceptibles so that the value of f_1 may be large. Another instance of such dynamical behavior might arise when the boundaries are relatively secure from known terrorists to prevent their free movement into a region, but there are a substantial number of "motivated" susceptibles (so that the value of f_2 is now large), who are harder

Phase portrait: $a = 0.0001$, $b = 0$,
$c = 1.2$, $e = 0$, $f = 0.15$, $g = 0.02$
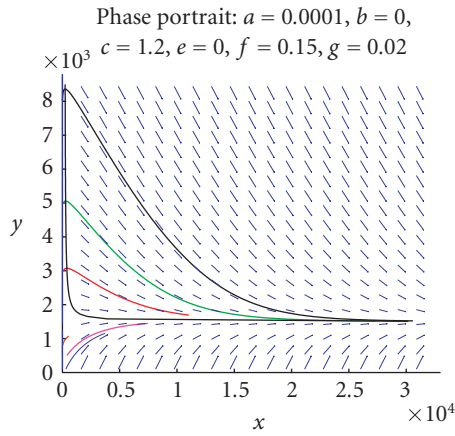
(a)
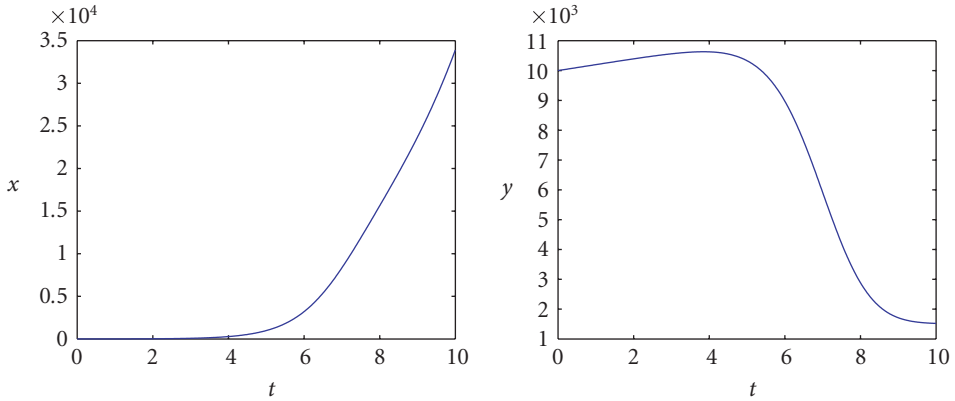
(b)

Figure 3.1. (a) Phase plot showing the flow field and the asymptotic behavior of the dynamical system, $c > 1$. Trajectories starting from different initial conditions are shown in color. The solid black line is the nullcline $axy - fx - gy = 0$. Notice that the terrorist population increases constantly while the S population reaches steady state given by $y = f/a = 1500$. (b) The time evolution of the dynamical system starting from $x(0) = 2$, $y(0) = 10,000$ shows the run-away terrorist population; the susceptible population reduces to its asymptotic value of $1,500$.

to keep track of, and who enter the geographical region, relatively speaking, unchecked. Figure 3.2 shows the dynamical behavior of the system. We notice that the main geometrical difference between the phase portraits in Figures 3.1 and 3.2 is the presence of the nullcline shown in green, which now moves to the positive quadrant. The asymptotic value of x is unbounded, while that of y again approaches f/a, as illustrated.

Figures 3.1 and 3.2 make the bifurcation from unstable to stable behavior clear. When $c > 1$, the green nullcline $y = (1 - c)/a$, which is horizontal in the phase plane, intersects the y-axis at a negative ordinate. Hence there is no fixed point since the other nullcline
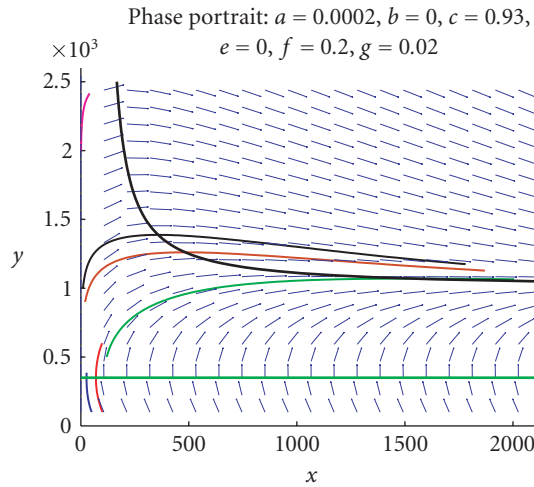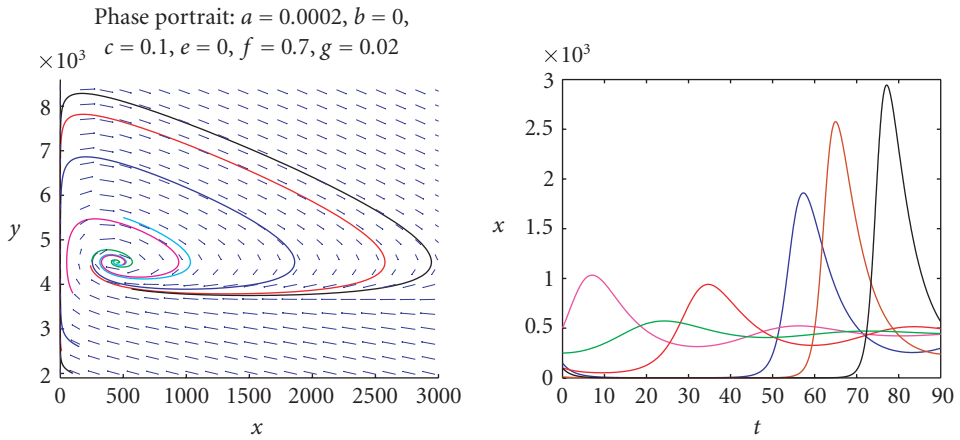
Figure 3.2. Phase plot showing the flow field and the asymptotic behavior of the dynamical system, when $c < 1$, $f + c > 1$. Trajectories starting from different initial conditions are shown in color. The solid black line is the nullcline $axy - fx - gy = 0$; the solid green line shows the nullcline $ay = 1 - c$. The phase particle has a negative x-component of velocity below this green line, indicating a drop in the terrorist population, and a positive component above it. Notice that the terrorist population asymptotically increases constantly while the S population again reaches a steady state given by $y = f/a = 1000$.

$axy - fx - gy = 0$ cannot intersect it in the first quadrant. When $c = 1$, the horizontal nullcline coincides with the x-axis. As c further decreases, this nullcline moves into the positive quadrant. However, since the asymptote to the nullcline $axy - fx - gy = 0$ is at $y = f/a$, until $f + c < 1$, the two nullclines cannot cross, and hence there is no fixed point. The phase flow indicates that the system is unstable when $f + c > 1$, with the population of terrorists continually increasing. When $f + c = 1$, the nullclines cross at $x \to \infty$, and the system remains unstable.

When $c < 1$ and $(f + c) < 1$, (3.3) points out that we have a fixed point in the first quadrant that may be either a stable node or a stable spiral. The two nullclines now intersect each other in the first quadrant. We illustrate this in Figure 3.3(a), where we show the stable spiral correctly predicted, with $x_0 = 450$, $y_0 = 4500$. Here we take $c = 0.1$ and $f = 0.7$. Condition (3.12) is satisfied by the parameters and we obtain a stable spiral. Figure 3.3(b) shows the dynamics when $c = 0.7$ and $f = 0.1$, so that for both these simulations $f + g = 0.8$. We note that an increase in the value of c paradoxically reduces the asymptotic populations of terrorists and susceptibles to $x_0 = 150$, $y_0 = 1500$.

Increasing the parameter a by a factor of 10 yields the time histories shown in Figure 3.4. Comparing Figures 3.3 and 3.4, we notice that the time to reach an effective steady state has reduced dramatically, somewhat paradoxically.

Lastly, we point out that when $b = e = f = 0$, the conditions in Result 3.1 of Section 3.1 are no longer satisfied and therefore we can no longer guarantee that there are no limit

(a)
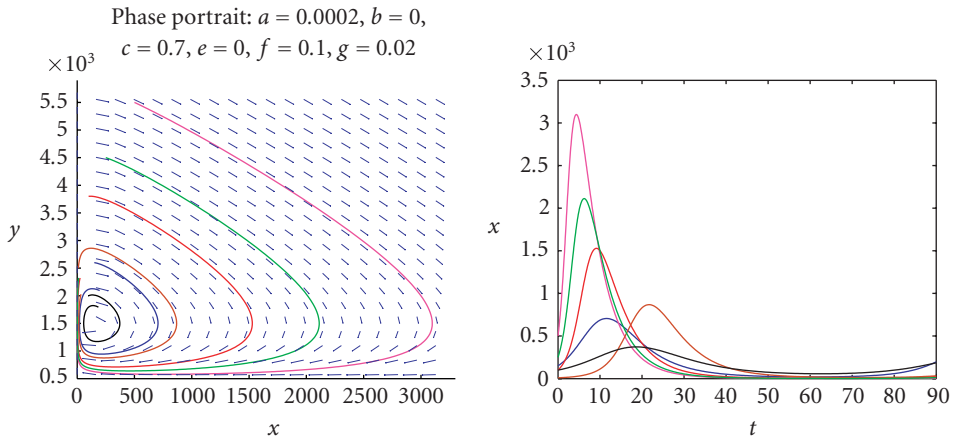
(b)

Figure 3.3. (a) Phase plot showing the flow field (with $b = e = 0$) and the asymptotic behavior of the dynamical system, when $c < 1$, $f + c < 1$, with $f + c = 0.8$ and $c = 0.1$. The fixed point is at $x_0 = 450$, $y_0 = 4500$. Trajectories starting from different initial conditions are shown in color. (b) Phase plot showing the flow field (with $b = e = 0$) and the asymptotic behavior of the dynamical system, when $c < 1$, $f + c < 1$, with $f + c = 0.8$ and $c = 0.7$. The fixed point is at $x_0 = 150$, $y_0 = 1500$. Trajectories starting from different initial conditions are shown in color. The same initial conditions as in Figure 3.3(a) are used.

cycles in the first quadrant of the phase space (see relation (3.2)). We then have the simplified dynamical system given by

$$\frac{dx}{dt} = axy - (1 - c)x, \tag{3.15}$$
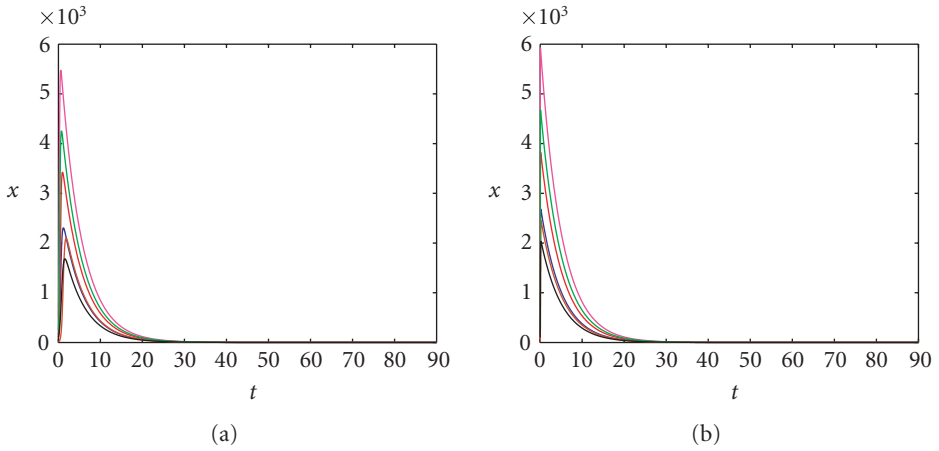
$$\frac{dy}{dt} = -axy + gy, \tag{3.16}$$

Figure 3.4. (a) Time history of the dynamical system shown in Figure 3.3(b) with $a = 0.002$. (b) Time history of the dynamical system shown in Figure 3.3(b) with $a = 0.02$.

which has the invariant

$$e^{a(x+y)} = c_0 x^g y^{(1-c)}, \tag{3.17}$$

where c_0 is a constant, determined from the initial conditions, $x(0)$ and $y(0)$. Relation (3.17) results in closed orbits for $g > 0$, $c < 1$. The fixed point given by (3.3) is now no longer hyperbolic, and because of the closed orbits around it, it is a nonlinear center. Figure 3.5 shows the periodic orbits for this situation, the periods being functions of c_0.

However, when $c > 1$, the orbits are no longer closed as seen from the vector field, since the x-component (see (2.4)) of the phase particle's velocity is now always positive. The y-component of the field is negative for $x > g/a$ and positive for $x < g/a$, pointing to the fact that $y \to 0$ as $t \to \infty$. We note from (3.16) that y may not vary monotonically with time if $x(0) < g/a$. As the limiting case of the situation shown in Figure 3.1(a), when $f \to 0$, $y(t \to \infty) = 0$.

The phase portrait undergoes a dramatic change when c moves from less than unity to greater than unity, and we have a bifurcation when $c = 1$. Thus when the rate of death/self-destruction of terrorists equals the rate at which new terrorists are imported into the area from its neighborhood, the behavior of the differential equations shows that the fixed point y_0 moves from $1/a$ when $c = 0$, to $y_0 = 0$ when $c = 1$. When $c = 1$, this fixed point thus moves to the x-axis. Also, the fixed points now lie along the x-axis, $x \geq 0$. The nullcline is given by $x = g/a$, and for initial values of the terrorist population, $x(0) < g/a$, the population of susceptibles (S) initially increases before going to zero (see Figure 3.6). The phase trajectories all end on the x-axis, the final (steady state) population of terrorists, x_f, being given by the implicit relation (see (3.17))

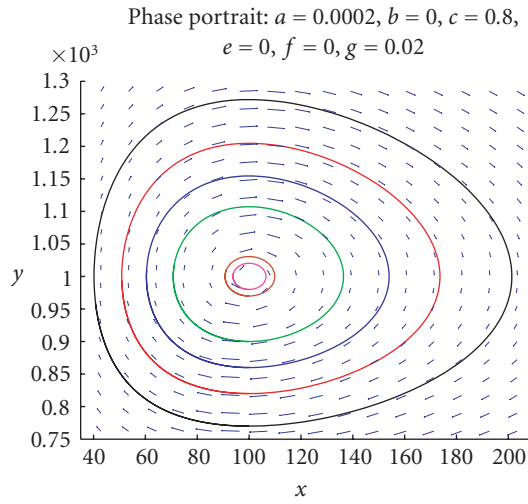$$a(x_f - x(0)) = g \ln \left(\frac{x_f}{x(0)} \right) + a y(0). \tag{3.18}$$

Phase portrait: $a = 0.0002, b = 0, c = 0.8,$
$e = 0, f = 0, g = 0.02$

Figure 3.5. Phase portrait showing the vector field for $b = e = f = 0$, $c < 1$ and the periodic orbits around the nonlinear center at $x_0 = 100$, $y_0 = 1000$, as given by (3.3). The orbits are described by the closed form relation (3.17).

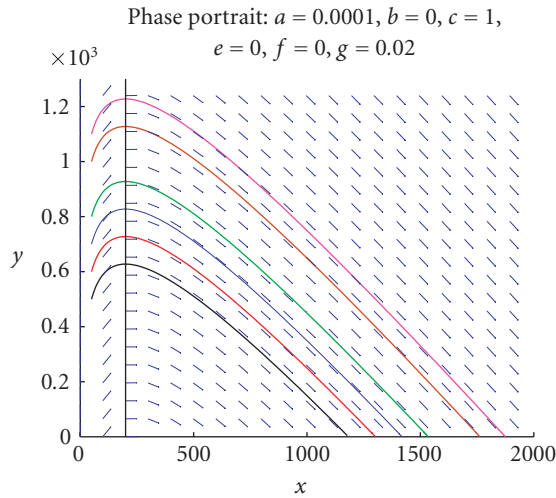Phase portrait: $a = 0.0001, b = 0, c = 1,$
$e = 0, f = 0, g = 0.02$

Figure 3.6. The x-axis becomes a line of fixed points when $c = 1$, and all trajectories end up on the x-axis. The black solid line shows the nullcline at $x_0 = g/a = 200$. Initial conditions that fall to the left of the nullcline show an increase in the S population before it eventually fades away to zero.

To appreciate the difference caused by the parameter c in the dynamics, we show the time trajectories below for the two regimes of behavior when $0 \leq c < 1$, when $c > 1$, as well as at the bifurcation point $c = 1$ (see Figure 3.7).
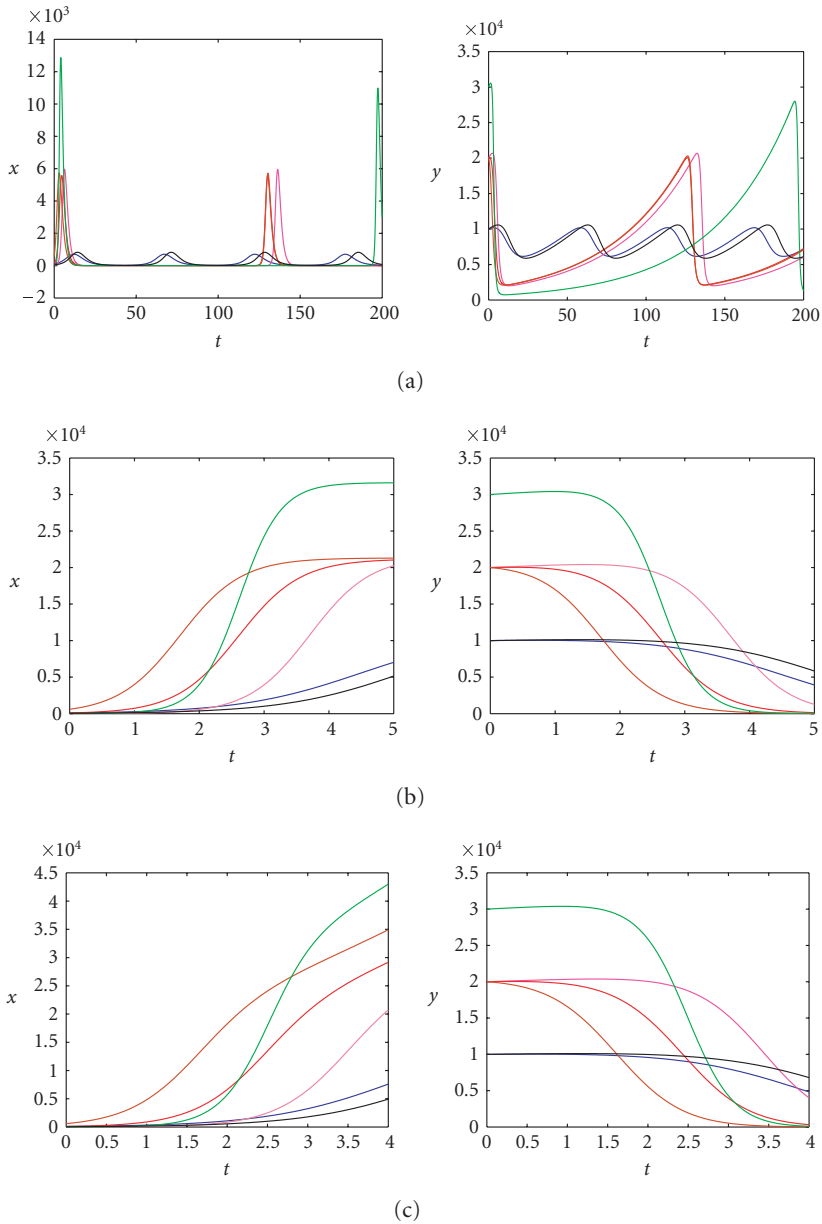
Figure 3.7. (a) Stable limit cycle behavior when $c = 0.2 < 1$. Note the soliton-like behavior in the dynamics of the terrorist population (T) when the initial population of susceptibles (S) far exceeds that of T. The parameter values (except for the value of c) are the same as those shown on the phase plot in Figure 3.6. (b) Time histories of the dynamics when $c = 1$ starting from different initial conditions. The parameters (except for the value of c) are the same as those for Figure 3.7(a). The initial conditions (same as in Figure 3.7(a)) can be more clearly seen here. (c) Unstable behavior showing an explosion in the terrorist population for $c = 1.2 > 1$; the population of susceptibles declines to zero eventually.

Phase portrait: $a = 0.0002$, $b = 0$,
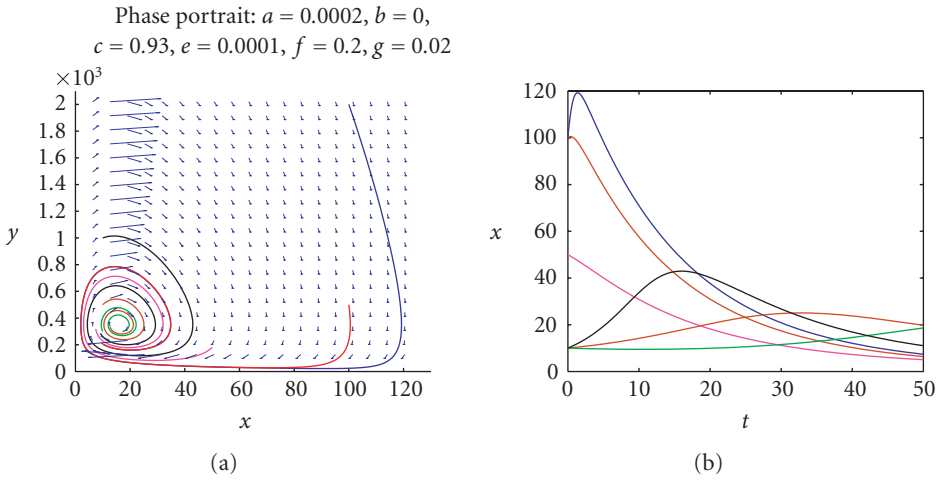$c = 0.93$, $e = 0.0001$, $f = 0.2$, $g = 0.02$

(a)

(b)

Figure 3.8. The presence of nonviolent intervention causes the dynamical system to exhibit a fixed point. Unlike the explosion in the terrorist population that occurs when $b = e = 0$ and $c + f > 1$, with $c < 1$, the terrorist population in the presence of nonviolent intervention is bounded and the dynamical trajectories here are spirals in phase space. The fixed point is at $x_0 \approx 16$, $y_0 = 350$. The parameters are the same as those used in Figure 3.2 except that $e > 0$.

Having now understood the dynamics of the baseline situation we are ready to study the effect caused by nonzero values of the parameters b and e, which represent intervention by different means.

3.5. Effects of nonviolent intervention ($b = 0$, $e > 0$). We begin by considering the locations of the fixed points when $b = 0$ and $e > 0$, and comparing it with the baseline situation. We note that the existence of fixed points in the baseline situation requires (see (3.3)) that $c < 1$ and $c + f < 1$. Relations (3.3) and (3.4) point out that the presence of nonviolent intervention will have no effect on the steady state value of the S population, which will remain $(1 - c)/a$. Denoting the steady state terrorist populations $\tilde{x}_0 = x_0|_{b=0, e>0}$ and $x_0 = x_0|_{b=0, e=0}$, we can show, after some algebra, that

$$\frac{x_0 - \tilde{x}_0}{x_0} = \frac{e(1-c)x_0}{e(1-c)(x_0 + \tilde{x}_0) + a(1-c-f)}. \tag{3.19}$$

From (3.19) we see that if $g > 0$, $c < 1$, and $c + f < 1$, we find that $x_0 > \tilde{x}_0$. Thus nonviolent intervention causes the steady state value of the terrorist population to decrease when compared to the situation with no intervention of any kind.

Furthermore, unlike what happens when $e = 0$, in the presence of nonviolent intervention, when $c < 1$ and $c + f > 1$, the dynamical behavior generates an attracting fixed point. Figure 3.8 when contrasted with Figure 3.2 shows that the presence of intervention causes the terrorist population (and the S population) to be bounded.

The reason for this is that the asymptote to the nullcline $axy + ex^2y - fx - gy = 0$ as $x \to \infty$ is now the x-axis, and hence the horizontal nullcline $y = (1 - c)/a$ will intersect it
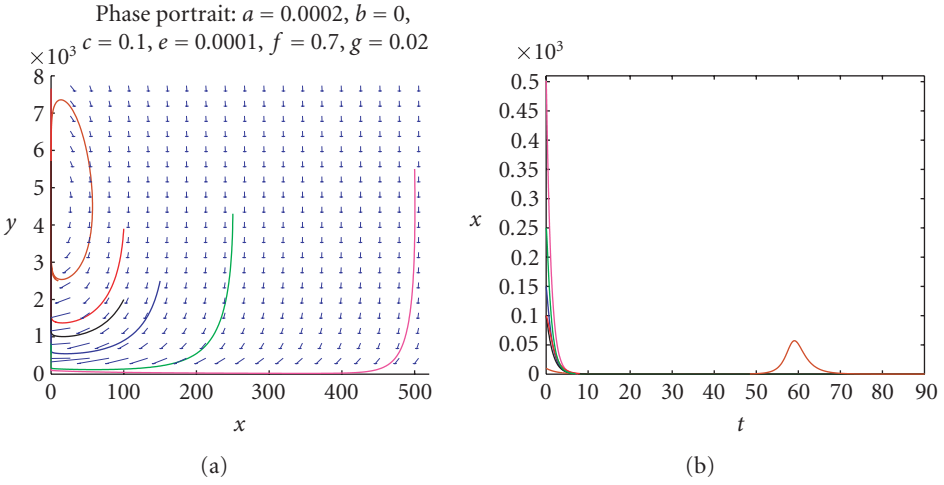
Figure 3.9. Using the same parameters as for Figure 3.3(a) except that $e = 0.0001$, we see a dramatic drop in the equilibrium population of terrorists through nonviolent intervention (compare with Figure 3.3(a)). The fixed point is now at $x_0 \approx 14$, $y_0 = 4500$.

as long as $c < 1$, irrespective of the value of $f + c$, and a stable fixed point results in the first quadrant $x, y > 0$. When $c = 1$, the horizontal nullcline lies along the x-axis, and the two nullclines intersect at $x \rightarrow \infty$, and the system is unstable.

For $c < 1$ and $c + f < 1$, we see a drastically lower steady state population of terrorists when nonviolent intervention is provided, as shown in Figure 3.9. For comparison between the situation with and without nonviolent intervention we show the dynamics using the same parameters as in Figure 3.3(a) except that we now have $e > 0$.

Lastly, we observe from relation (3.4) that for large values of the parameter e, and $c < 1$, $x_0 \propto 1/\sqrt{e}$, and so the T population can be, theoretically speaking, driven down to zero; note, however, that the S population is unaffected by the presence of nonviolent intervention.

When $c > 1$, the effect of nonviolent intervention cannot stop the explosion in the terrorist population (see Figure 3.10), though it asymptotically brings the population of susceptibles under control. The reason is the same as for the situation with no intervention—the x-component of the velocity of the phase flow as seen in (2.4) is always positive in the positive quadrant and hence there can be no fixed point with $x_0 > 0$.

3.6. Effects of military/police intervention ($b > 0$, $e = 0$).
The equations that determine the fixed points of the dynamical system are

$$y_0 = \frac{1 - c + bx_0}{a}, \tag{3.20}$$

$$r(x) = abx^2 + [a(1 - c - f) - gb]x - g(1 - c) = 0. \tag{3.21}$$

Phase portrait: $a = 0.0001$, $b = 0$,
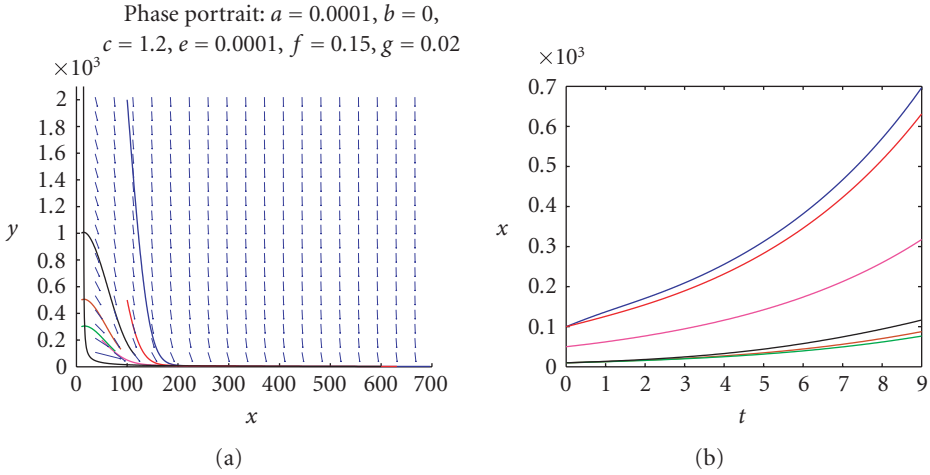$c = 1.2$, $e = 0.0001$, $f = 0.15$, $g = 0.02$

(a)

(b)

Figure 3.10. Phase plot showing the ineffectiveness of nonviolent intervention in curbing the terrorist population. Parameter values are same as those for **Figure 3.1** except that $e = 0.0001$. The vector field has a positive x-component of velocity and hence the terrorist population continually increases.

The roots of (3.21) are given by

$$x_0 = \frac{A \pm \sqrt{B}}{2ab},$$
(3.22)

where

$$A = gb - a(1-c) + af, \qquad B = [gb + a(1-f-c)]^2 + 4abfg.$$
(3.23)

We begin by noting that for $b, f, g > 0$, $c < 1$, $f + c < 1$, we have, after some algebra, $x_0|_{b=0, e=0} > x_0|_{b>0, e=0}$. We next consider 2 cases.

(A) *When $f, g > 0$.* We begin by noting that in the presence of military intervention ($b > 0$) a fixed point exists when $f, g > 0$. This is because B is always positive. Three regimes of behavior now surface, as far as the fixed points are concerned. (i) When $c < 1$ and $A < 0$, then $\sqrt{B} > A$, and we get one positive root $x_0 > 0$. (ii) $c < 1$ and $A \geq 0$, we again get one positive root $x_0 > 0$. (iii) When $c > 1$, we find that $A > 0$ and $0 < \sqrt{B} < A$, and we get two real positive roots of the equation $r(x) = 0$ with the two roots $x_0^{(2)} > x_0^{(1)} > 0$. When $c < 1$ and $x_0 > 0$, we always have, by relation (3.20), $y_0 > 0$. However, when $c > 1$, for $y_0 > 0$, we require by (3.20) that $x_0 > (c-1)/b$. Furthermore, when $x_0 = (c-1)/b$, we find that $r(x_0) < 0$ (see the appendix, part (a)), so only the larger of the two roots, $x_0^{(2)}$, of the equation $r(x) = 0$ will yield a corresponding value of $y_0 > 0$. Thus military intervention causes the dynamics to always have a fixed point in the first quadrant, which is stable.

We observe from (3.5) that when $c < 1$, the steady state value of the S population is always larger than when nonviolent intervention is solely used and also when no intervention of any kind is used. For values of b that are large, the fixed point $x_0 \propto g/a$, and
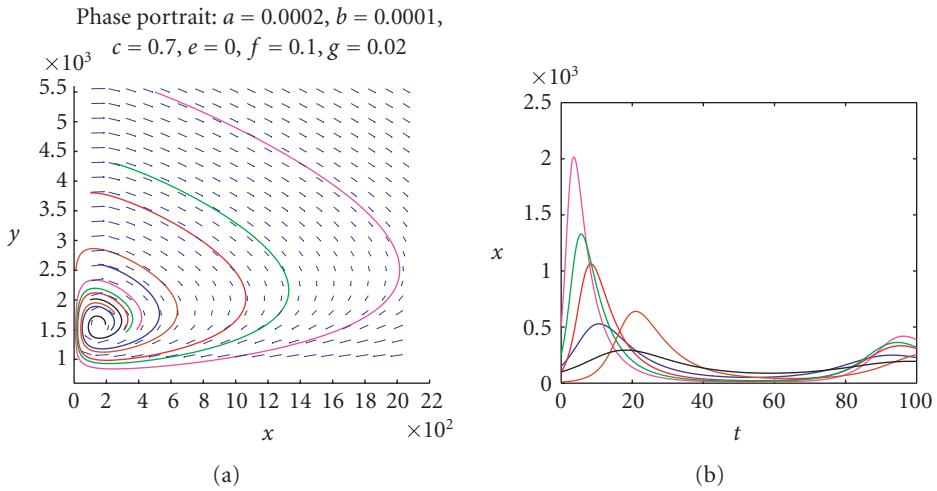
Phase portrait: $a = 0.0002$, $b = 0.0001$,
$c = 0.7$, $e = 0$, $f = 0.1$, $g = 0.02$

(a)

(b)

Figure 3.11. A stable spiral is seen when $c < 1$, $A < 0$. Note the long time that the trajectory stays in the vicinity of the y-axis and the same parameters used as in **Figure 3.3**(b) (except for the value of b). The fixed point occurs at $x_0 = 146$, $y_0 = 1573$.

Phase portrait: $a = 0.0002$, $b = 0.0025$,
$c = 0.7$, $e = 0$, $f = 0.1$, $g = 0.02$
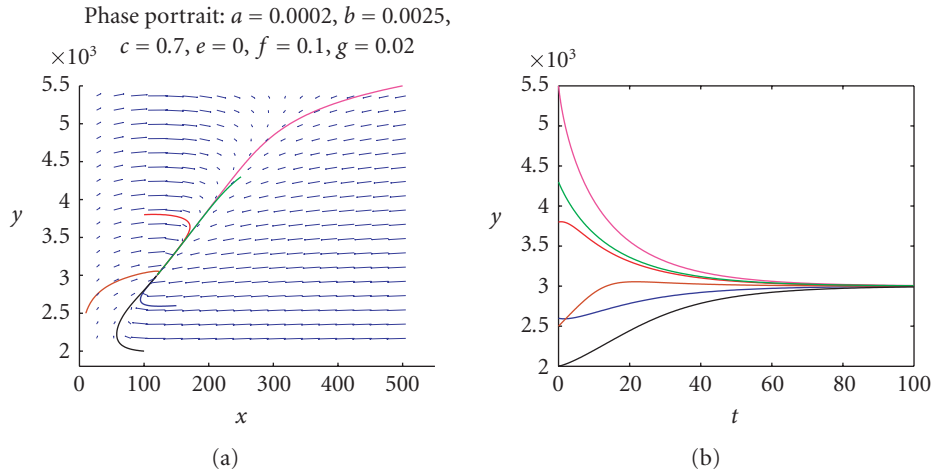

(a)

(b)

Figure 3.12. A stable node is seen when $c < 1$, $A > 0$. Compare with **Figure 3.3**(b). The fixed point occurs at $x_0 = 120$, $y_0 = 3000$.

as b increases the steady state value of the S population, is larger by $b/(ga^2)$ when compared to that obtained in the presence of no military intervention (or with nonviolent intervention).

Figures 3.11 and 3.12 show the dynamical behavior when $c < 1$. When $A < 0$, we see a stable spiral. As the value of b increases, A becomes positive and the dynamical behavior

Phase portrait: $a = 0.0001$, $b = 0.0001$,
$c = 0.7$, $e = 0$, $f = 0.15$, $g = 0.02$

Phase portrait: $a = 0.0001$, $b = 0.0001$,
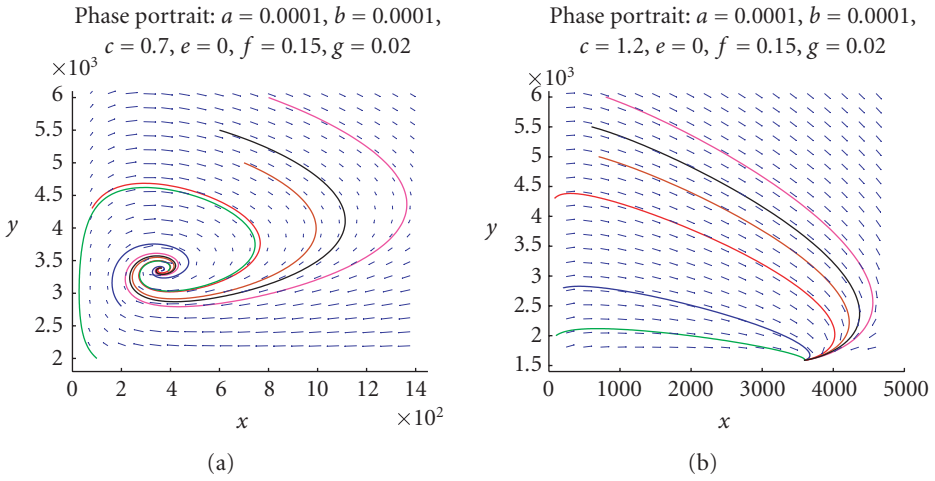$c = 1.2$, $e = 0$, $f = 0.15$, $g = 0.02$

(a)

(b)

Figure 3.13. The effect of the parameter c in its influence on the steady state values and the dynamical behavior of the system. (a) $c = 0.7$. The fixed point is at $x_0 \approx 361$, $y_0 \approx 3361$. (b) $c = 1.2$. The fixed point is at $x_0 \approx 3588$, $y_0 \approx 1588$.

switches to a stable node. For comparison, we have used the same parameter values we used in Figure 3.3(b), except for the value of b.

The influence of the parameter c can be substantial in determining the steady state values of the T and S populations. For example, with $a = b = 0.0001$, $f = 0.15$, and $g = 0.02$, the steady state T populations for $c = 0.7$ and for $c = 1.2$ are about 361 and 3588, respectively; the steady state S populations are about 3361 and 1588, respectively. Figure 3.13 shows the phase portraits for the two cases. Both cases lead to stable spirals. However, the time taken to reach these steady states also differs substantially, the system with the larger c value reaching a steady state much more rapidly (see Figures 3.14 and 3.15 that show the time histories). We have used the same parameters in Figure 3.13(b) as were used in Figure 3.10 except for the values of b and e. This comparison illustrates that the presence of military action will cause a stable fixed point to appear even when $c > 1$, something that does not happen when only nonviolent intervention is undertaken.

(B) *When $g > 0$, $f = 0$.* The situation when $f = 0$ is interesting in that we can have two fixed points in the first quadrant. We find that

$$x_0^{(1)} = \frac{g}{a}, \quad y_0^{(1)} = \frac{1-c}{a} + \frac{bg}{a^2}, \quad x_0^{(2)} = \frac{c-1}{b}, \quad y_0^{(2)} = 0 \quad \text{when } \frac{gb}{a} + 1 > c, \quad (3.24)$$

$$\tilde{x}_0^{(1)} = \frac{c-1}{b}, \qquad \tilde{y}_0^{(1)} = 0 \quad \text{when } \frac{gb}{a} + 1 < c. \quad (3.25)$$

It can be shown (see the appendix, part (b)) that the fixed point $(x_0^{(1)}, y_0^{(1)})$ is either a stable node or a spiral. For $\tilde{x}_0^{(1)}, x_0^{(2)} > 0$, we require $c > 1$. Thus when $c > 1$ and $gb/a + 1 > c$, we have two fixed points, as shown in relations (3.24). The fixed point $(x_0^{(2)}, y_0^{(2)})$ is, however,
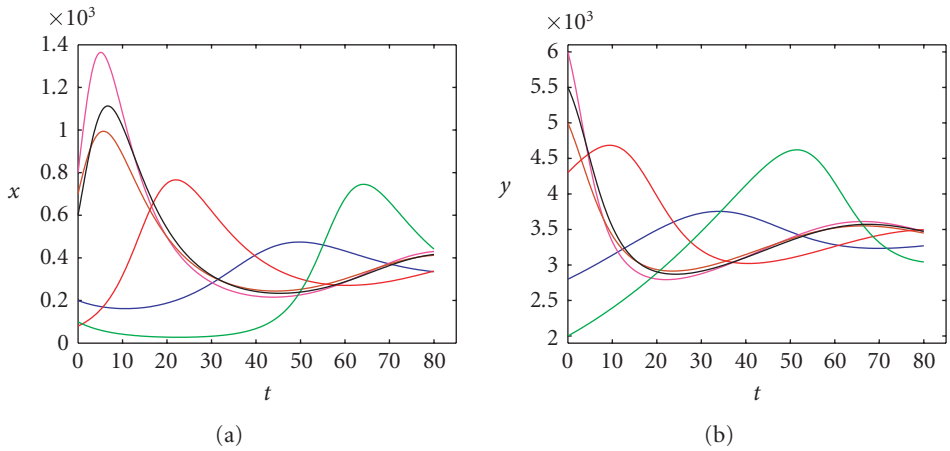
Figure 3.14. Time histories of response of the dynamical system with $c = 0.7$, and the parameters shown in Figure 3.13(a). (a) Terrorist population versus time. (b) Susceptible populations versus time.
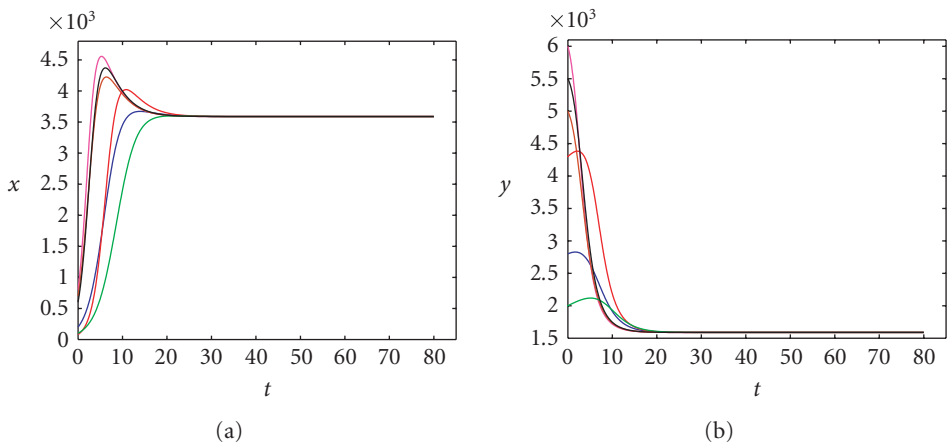

Figure 3.15. Time histories of response of the dynamical system with $c = 1.2$, and the parameters shown in Figure 3.13(b). (a) Terrorist population versus time. (b) Susceptible populations versus time. Note the much larger steady state terrorist population as compared to Figure 3.14.

unstable. The fixed point $(\tilde{x}_0^{(1)}, \tilde{y}_0^{(1)})$ which occurs when $1 < gb/a + 1 < c$ is a stable node, as shown in the appendix, part (b).

We next address the question of comparing the steady state population of terrorists when we only have nonviolent intervention as compared to when we only have military intervention, keeping the parameters c, f, and g the same. We note that when $b = 0$, $e > 0$, we have a stable fixed point $x_0, y_0 > 0$ as long as $c < 1$. On the other hand, when $b > 0$, $e = 0$, we always have a fixed point in the first quadrant with $\tilde{x}_0, \tilde{y}_0 > 0$ for all values of $c > 0$. In order to compare the steady state values of the T and S populations in the two

situations, we therefore restrict our attention to the situation when $c < 1$. We then have the following result.

Result 3.3. When $c < 1$, and $e(1 - c) - ab > 0$, then $\tilde{x}_0 > x_0$ when the parameters c, f, and g are kept unchanged. This means that under these circumstances, the steady state population of terrorists is larger when only military/police intervention is utilized, as compared to when only nonviolent intervention is used.

Proof. From (3.7) we see that $x_0 > 0$ satisfies the relation

$$e(1 - c)x_0^2 + a(1 - c - f)x_0 = g(1 - c) \tag{3.26}$$

and $\tilde{x}_0 > 0$ satisfies the relation

$$ab\tilde{x}_0^2 + [a(1 - c - f) - gb]\tilde{x}_0 = g(1 - c). \tag{3.27}$$

Substituting for $g(1 - c)$ from (3.26) in (3.27) gives

$$ab(\tilde{x}_0 - x_0)(\tilde{x}_0 + x_0) + [ab - e(1 - c)]x_0^2 + a(1 - c - f)(\tilde{x}_0 - x_0) - gb\tilde{x}_0 = 0, \tag{3.28}$$

where we have added and subtracted the quantity abx_0^2. This results in

$$(\tilde{x}_0 - x_0) = \frac{gb\tilde{x}_0 + [e(1 - c) - ab]x_0^2}{ab(\tilde{x}_0 + x_0) + a(1 - c - f)}. \tag{3.29}$$

Since from (3.26) we have $ab\tilde{x}_0^2 + [a(1 - c - f)]\tilde{x}_0 = g(1 - c) + gb\tilde{x}_0$, (3.29) becomes upon multiplying the numerator and denominator on both sides by $\tilde{x}_0$,

$$(\tilde{x}_0 - x_0) = \frac{gb\tilde{x}_0^2 + [e(1 - c) - ab]\tilde{x}_0 x_0^2}{ab(\tilde{x}_0 x_0) + g(1 - c) + gb\tilde{x}_0}. \tag{3.30}$$

The denominator is always positive, and to ensure that $\tilde{x}_0 > x_0$, the result now follows. From the proof we note that the condition stated in Result 3.3 when $c < 1$ is a sufficient condition for $\tilde{x}_0 > x_0$, but not a necessary condition. $\square$

Figure 3.16 illustrates Result 3.3, where we compare the effect of military intervention with that produced by nonviolent intervention. We use the parameter values as in Figure 3.13(a), and here $e(1 - c) - ab > 0$. We find that the fixed point has moved from $\tilde{x}_0 \approx 361$, $\tilde{y}_0 \approx 3361$ when using military intervention to $x_0 \approx 14$, $y_0 = 3000$ when using nonviolent intervention, though the time needed to achieve the steady state value is much greater in the latter case. While the change in the population of susceptibles is not appreciable, that in the terrorist population is substantial, as seen from the two figures.

3.7. Effects of combined military/police and nonviolent intervention $(e, b > 0)$**.** When both military/police intervention and nonviolent intervention are present, the fixed point x_0 satisfies relation (3.7). When $c < 1$, as pointed out in Section 3.2, we have a fixed point (x_0, y_0), where x_0 is positive real root of

$$r(x) = ebx^3 + [e(1 - c) + ab]x^2 + [a(1 - c - f) - gb]x - g(1 - c) = 0. \tag{3.31}$$
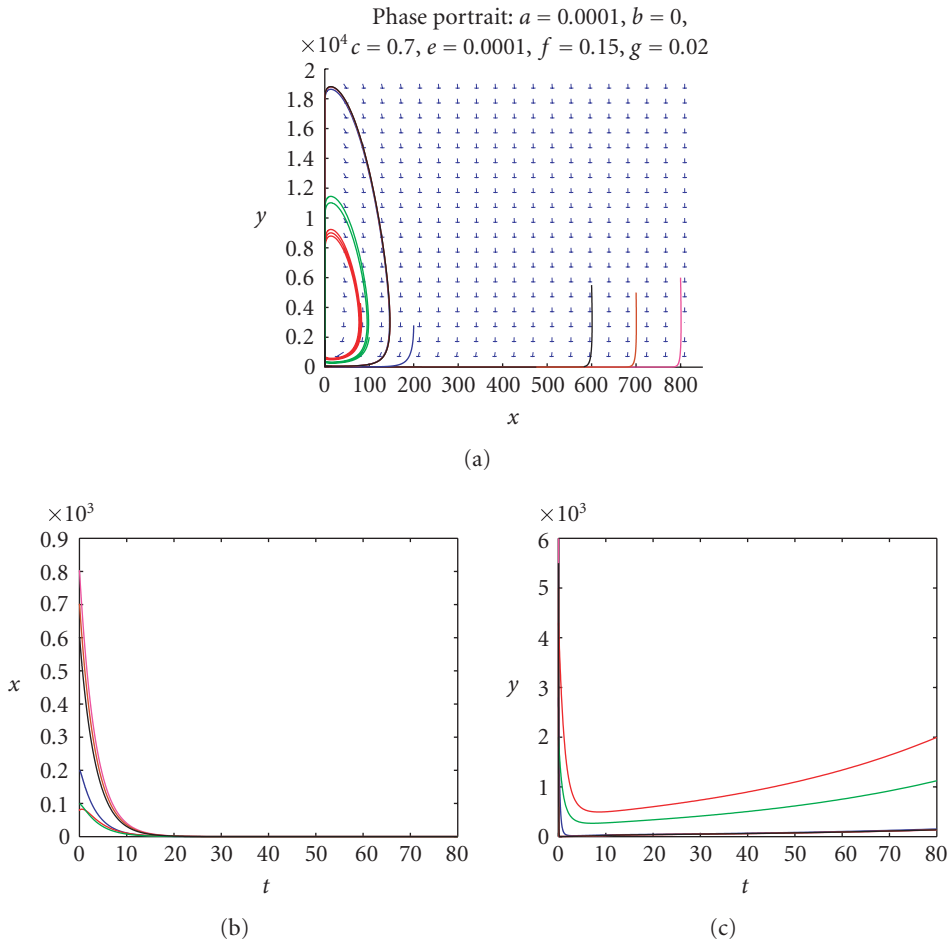
Figure 3.16. A comparison with Figures 3.13(a) and 3.14 showing that nonviolent intervention brings about lower steady state terrorist populations than those brought about by military/police intervention, when $e(1-c) - ab > 0$. The fixed point occurs at $x_0 \approx 14$, $y_0 = 3000$.

This fixed point could be a stable spiral or a stable node depending on whether relation (3.12) or (3.13) is satisfied.

However, when $c > 1$, the coefficient of the x^2 term in (3.31) is sign indefinite while the coefficient of the x term is negative. Also,

$$r\left(\frac{c-1}{b}\right) = -af\frac{(c-1)}{b} \tag{3.32}$$

is negative when $f > 0$ and $c > 1$. Hence Descarte's rule points out that we would have 2 positive roots, $x_0^{(2)} > x_0^{(1)}$, of the equation $r(x) = 0$, and one negative root. Since we require $x_0 > (c-1)/b$, only the root $x_0^{(2)}$ will then yield a corresponding y_0 that is positive.

Phase portrait: $a = 0.0001$, $b = 0.0001$,
$\times 10^3$ $c = 0.7$, $e = 0.0001$, $f = 0.15$, $g = 0.02$
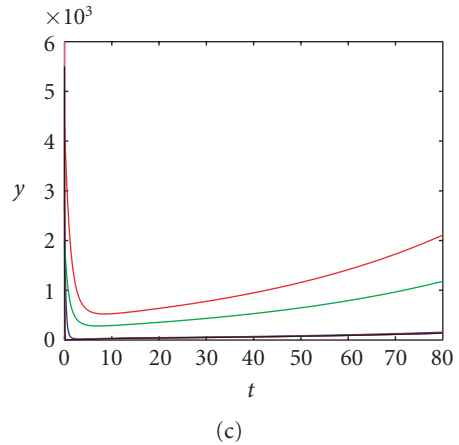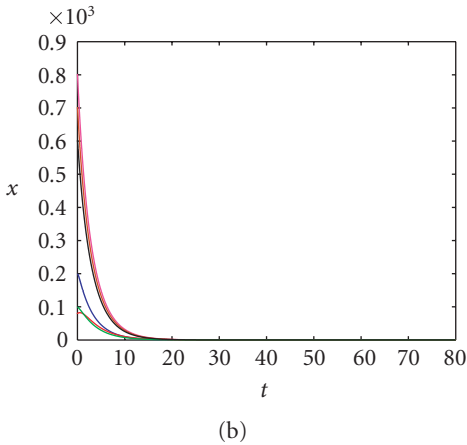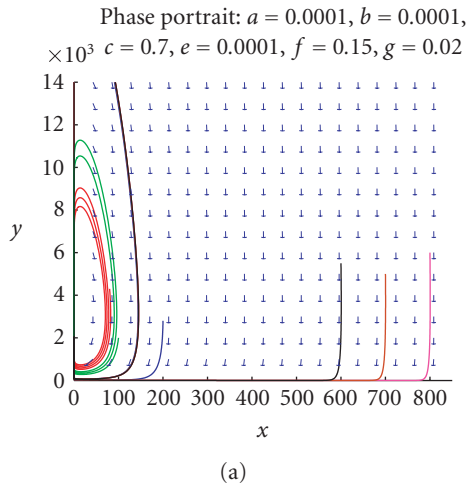

(a)

(b)

(c)

Figure 3.17. The effects of both military and nonviolent actions are illustrated here for $c < 1$. The fixed point is at $x_0 \approx 14$, $y_0 \approx 3013$, and the trajectories appear to reach the equilibrium point marginally faster than with only nonviolent intervention. All the parameters are the same for Figures 3.13(a), 3.14, 3.16, and 3.17, except for the values of b and e. A soliton-like behavior is observed from the phase plot.

Hence we will have *only one* fixed point (x_0, y_0) for which $x_0, y_0 > 0$. As pointed out in Section 3.3, the fixed point will again be either a stable spiral or a stable node depending on whether relation (3.12) or (3.13) is satisfied.

Figure 3.17 shows the effect of combined military/police and nonviolent interventions when $c < 1$. For $c > 1$, we see that as in the case of only military intervention, we have a stable node, that is reached relatively rapidly when compared with $c < 1$, as seen in Figure 3.18.

A comparison of Figures 3.13(b) and 3.15 with Figure 3.18 indicates the difference that nonviolent intervention makes when added to military/police intervention. We note
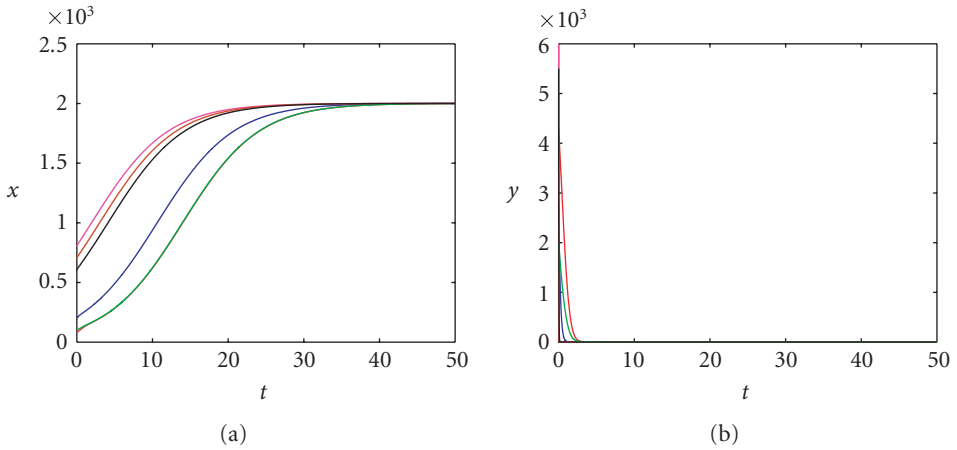
Figure 3.18. The parameters used here are the same as those for **Figure 3.17**, except that $c = 1.2$. The fixed point is a stable node with $x_0 \approx 2000$, $y_0 \approx 1$.

that the steady state value of the terrorist population is reduced substantially, as is the population of susceptibles, while the time required to reach steady state is increased considerably.

From our previous understanding of the various cases, we can heuristically say that the presence of military intervention causes the terrorist population to be asymptotically bounded, something that cannot occur with only nonviolent intervention; it also reduced the steady state value of the susceptible population. The addition of nonviolent intervention reduces the steady state value of the terrorist population as compared with what it might have been only with military/police intervention, but also increases the time required to reach the steady state.

Result 3.4. The addition of military/police intervention to nonviolent intervention always reduces the steady state population of terrorists from what it would be with only nonviolent intervention. Hence,

$$x_0\big|_{\substack{e>0\\b>0}} < x_0\big|_{\substack{e>0\\b=0}}. \tag{3.33}$$

Proof. We note that a steady state is reached with only nonviolent intervention when $c < 1$. Let $x_0 > 0$ be the steady state terrorist population when only nonviolent intervention is used, and let $\tilde{x}_0 > 0$ be the steady state terrorist population when both military/police and nonviolent interventions are employed. Thus we have, with $c < 1$,

$$e(1-c)x_0^2 + a(1-c-f)x_0 - g(1-c) = 0, \tag{3.34}$$

$$eb\tilde{x}_0^3 + [e(1-c) + ab]\tilde{x}_0^2 + [a(1-c-f) - gb]\tilde{x}_0 - g(1-c) = 0. \tag{3.35}$$

Substituting for $g(1-c)$ from (3.34) into (3.35) we get

$$\widetilde{x}_0 - x_0 = -\frac{abfx_0^2\widetilde{x}_0/(1-c)}{eb\widetilde{x}_0(x_0\widetilde{x}_0 + x_0^2) + [e(1-c) + ab]x_0\widetilde{x}_0 + g(1-c)} \tag{3.36}$$

from which the result follows. □

Result 3.5. The addition of nonviolent intervention to military/police intervention always reduces the steady state population of terrorists from what it would be with only nonviolent intervention. Hence, with $f, g > 0$, we have

$$x_0 \big|_{\substack{e>0\\b>0}} < x_0 \big|_{\substack{e=0\\b>0}}. \tag{3.37}$$

Proof. We note that a steady state is reached with only military/police intervention for any $c > 0$. Let $x_0 > 0$ be the steady state terrorist population when only military/police intervention is used, and let $\widetilde{x}_0 > 0$ be the steady state terrorist population when both military/police and nonviolent interventions are employed. Thus we have

$$r(x_0) \equiv abx_0^2 + [a(1-c-f) - gb]x_0 + g(c-1) = 0, \tag{3.38}$$

$$eb\widetilde{x}_0^3 + [e(1-c) + ab]\widetilde{x}_0^2 + [a(1-c-f) - gb]\widetilde{x}_0 + g(c-1) = 0. \tag{3.39}$$

Substituting for $g(1-c)$ from (3.39) into (3.38) we get

$$x_0 - \widetilde{x}_0 = \frac{ex_0\widetilde{x}_0^2[b\widetilde{x}_0 + (1-c)]}{abx_0\widetilde{x}_0 + g(1-c)}, \tag{3.40}$$

which is clearly positive when $0 \le c \le 1$, since both x_0 and $\widetilde{x}_0$ are positive.

As pointed out in Section 3.6, when $c > 1$ and $f, g > 0$, (3.38) has two positive roots of which the larger yields $\widetilde{y}_0 > 0$, and for this to happen, from relation (3.6) we see that $\widetilde{x}_0 > (c-1)/b$, so that the numerator in relation (3.40) is positive. To prove that the denominator in (3.40) is positive when $c > 1$, we note that since $r(g/a) = -gf < 0$, $x_0 > g/a$, and hence

$$abx_0\widetilde{x}_0 > ab\left(\frac{g}{a}\right)\left(\frac{c-1}{b}\right) = g(c-1). \tag{3.41}$$

Thus we have shown that $x_0 - \widetilde{x}_0 > 0$ when $c \ge 0$. □

We have thus shown that a combination of military/police action and nonviolent intervention yields a fixed point with a lower terrorist population that with any one of these interventions excluded.

4. Conclusions

In this paper we have developed a preliminary dynamical systems approach to understanding the dynamics of terrorism. We have conceptualized the population in a given region in terms of terrorists, susceptibles, and nonsusceptibles. We consider terrorists to be individuals who are firmly committed to terrorism, who will not yield to nonviolent

actions such as persuasion through economic, political, social, or other means. They are what might be called, for example, the "insurgent" population in areas like Mosul and Baghdad. The segment we term as susceptibles constitutes those that can be influenced, either into becoming terrorists through terrorist propaganda or into becoming pacifists that constitute what we call the nonsusceptible population. The population segment of susceptibles would, for example, include individuals who have been educated in madrassas, and/or those who might be influenced from a young age by Wahabi thinking, and/or those who may have grown up in conditions of depravity brought about by geopolitical forces outside their control (e.g., some people in the West Bank and Gaza Strip). We assume that military/police intervention is used for reducing the rate of expansion of the terrorist population, while nonviolent action is used to target susceptibles into becoming nonterrorists. The flow between these three populations and the effect of military/police action as well as nonviolent action are illustrated in Figure 2.1. We assume that military/police action increases as the square of the terrorist population, and that nonviolent action is proportional to the product of the size of the susceptible population and the square of the terrorist population.

The model we have used here is normative, and the usefulness of the results adduced herein would naturally depend on the extent to which our model approximates reality. In this respect, our model is somewhat different from other socio-politico-economic models, such as those that may use game theory or optimal control where matters like game set-ups and problem frameworks, player payoffs, and cost (utility) functions are difficult to assess, along with the basic assumptions underlying these approaches. The determination of the model parameters/functions in these situations is, at best, also difficult. The model we have proposed contains parameters, which could be estimated with greater ease, and from data that may be more widely available. We intend to take the estimation of these parameters from field data in a follow-on study so that further improvements in the description of the dynamical characterization of the system can be incorporated.

The differential equations that model the system have 8 (effectively 6) parameters, and it is somewhat remarkable that, though highly nonlinear, the dynamical regimes of behavior can still be analyzed, in general, in a relatively straightforward manner. The analysis presented herein, however, does not deal with degenerate cases, such as if $g = 0$. Such degeneracies could cause fundamental changes in the behavior of the dynamical system. If $g = 0$, for example, we have a line of fixed points "born" along the y-axis. The equilibrium points can lose their hyperbolicity and a linearized analysis cannot be resorted to in order to assess their stability. While it may be unlikely that in real life the parameter $g = 0$, the gross changes in the phase portraits and the totally different dynamical behavior generated by such special parameter values point out the need for further investigation. We plan to take up these issues in our future work. Furthermore, our study here has basically concentrated on the fixed points of the dynamical system, their existence, and their stability. Several issues such as the transient behavior of the system, which may be important from a policy and strategic planning viewpoint, though touched upon, have not been emphasized, nor have the possibility of the parameters of the model being functions of time been entertained. For the sake of brevity, this initial paper has not looked into several of these aspects; they will be the focus of our attention in the future.

The dynamical behavior of our model indicates several general features, many of which point to new strategic ways of planning to quell terrorism. While some of these features may appear reasonable from an intuitive stance, there are many that do not immediately suggest themselves purely on intuitive grounds. More importantly, the study points to the various conditions under which different strategies might be meaningful and effective, and their possible effects. We remind the reader that when there is military/police intervention the parameter $b > 0$; when there is no such intervention, $b = 0$. Similarly, when there is nonviolent intervention the parameter $e > 0$; when there is no such intervention, $e = 0$. We summarize some of our observations below.

(1) There is no possibility of the populations of the terrorists and susceptibles evolving chaotically in a manner that precludes prediction because of extreme sensitivity to the small uncertainties in determining the population numbers of the two populations at any given time. From a mathematical standpoint, dynamical chaos is precluded in the model that has been proposed herein to describe the behavior of the system.

(2) In the presence of military/police intervention against terrorists with or without nonviolent intervention, the dynamical system always reaches a unique equilibrium state as long as the susceptible population keeps growing; that is, if $a, g > 0$, and $e \geq 0$, a unique (nonzero) equilibrium population of terrorists and susceptibles will emerge as long as $b > 0$ and/or $f > 0$.

(3) As long as $e(1 - c) - ab > 0$, the equilibrium population of terrorists is less when solely nonviolent action is used instead of military/police intervention. This result is somewhat unexpected and points out new directions in strategic planning for curbing terrorism.

(4) Were no intervention to be made so that $b = e = 0$, the system would become unstable if $c > 1$, or if $f + c > 1$. The first of these conditions, $c > 1$, appears intuitive for it simply states that terrorists are increasing in number faster than they are dying/self-destructing. The latter condition, however, is less intuitive and points out that it is not necessary for $c > 1$ for the terrorist population to explode; for, the increase in susceptibles caused by terrorist propaganda can also lead to such an explosion, the susceptibles being seduced into terrorist behavior through their recruitment by terrorists, as per the axy term in (2.4). On the other hand, were either $f + c > 1$ and/or $c > 1$, the terrorist situation would go out of hand unless some sort of interdiction is made.

(5) Perhaps the most critical parameter which emerges from this dynamical investigation is the value of c, the ratio of the rate of increase of terrorists either through recruitment from among their own population or importation of terrorists from other geographical areas to the rate at which terrorists die/self-destruct. (a) If this ratio is less than unity, nonviolent action will cause the populations eventually to reach a steady state. The time to reach steady state is large and the population of susceptibles may fluctuate considerably, however the orbits are seen to lie in a narrow region around the y-axis, pointing to the possible advantages of using nonviolent action in curbing terrorist populations. However, if this ratio exceeds unity ($c > 1$), nonviolent action will not be able to stop the upward increase in the terrorist population. Thus the presence of porous boundaries in an area of terrorist activity can lead to substantially different dynamical behavior, and our

model points to the fact that sealing the boundaries of a region to prevent the flow of terrorists into it should be a matter of considerable strategic importance in curbing terrorist activities. (b) Military/police response always results in the two populations reaching an eventual steady state. This occurs even if $c > 1$.

(6) The combined action of both military/police and nonviolent initiatives, in general, brings the equilibrium population of terrorists considerable lower than with only military intervention, though it appears to increase the duration of time necessary to reach the steady state.

(7) The variation of the populations of terrorists and susceptibles may show a soliton-like behavior in time, even though these populations reach steady states. This is because the dynamical flow is slow in the vicinity of the y-axis. This means that the terrorist population can decrease, and can remain decreased, for extended periods of time, before possibly increasing substantially again. Quiescent periods with low terrorist activity, because of low terrorist populations, seem to be an *inherent feature* of the dynamics of the system. They are unrelated to explanations like the increased effectiveness of military or nonviolent actions. This points to the fact that curbing terrorism has to be a long-term strategy which must continually proceed despite periods of time when terrorist activities may be seen to be of negligible occurrence. This is because the dynamics itself has this nature, that is, long "quiescent" periods of time could elapse between "explosions" of terrorist populations.

(8) The equilibrium populations, when they exist, are shown to be stable. Furthermore, from a dynamical standpoint, these equilibrium points are shown to be either stable spirals or nodes. This would mean that even in the presence of military and/or nonviolent intervention, the population of terrorists, together with their activities, can increase before the equilibrium population is eventually reached. This is controlled by the initial population numbers of the terrorists and the susceptibles in relation to the location (and stability) of the equilibrium populations. Thus, the fact that terrorist activities (especially over the short haul) may increase in spite of military/police and/or nonviolent intervention does not indicate that such intervention is ineffective.

(9) In addition to Results 3.3–3.5 that are related to the steady state population of terrorists when it exists, the steady state population of susceptibles, y_0, when it exists, has the following property:

$$y_0 \big|_{\substack{b=0 \\ e=0}} = y_0 \big|_{\substack{b=0 \\ e>0}} < y_0 \big|_{\substack{b>0 \\ e>0}} < y_0 \big|_{\substack{b>0 \\ e=0}}. \tag{4.1}$$

(10) Military/police intervention appears to be necessary for allowing terrorist populations to reach stable values in areas where the boundaries are porous and $c > 1$. The addition of nonviolent intervention appears to reduce the steady state terrorist population substantially. In areas where the relative increase in the terrorist population is such that $c < 1$ and $f + c < 1$, the use of nonviolent action might be preferable to military/police intervention if the intention is to bring down the eventual value of the terrorist population.

Appendix

(a) When $f, g > 0$, and $c > 1$, for $y_0 > 0$, we require $x_0 > (c - 1)/b$. We know that the equation $r(x) = 0$ given in (3.21) has two real positive roots. Since for $x = (c - 1)/b$, $r((c - 1)/b) = af(1 - c)/b < 0$, we find that only the larger of the two roots of $r(x) = 0$ will yield a $y_0 > 0$. Hence, while there are two values of $x_0 > 0$ for which $r(x_0) = 0$, only the larger of two values of x_0 will result in a $y_0 > 0$.

(b) When $g > 0$, $f = 0$, the Jacobian evaluated at the fixed point $(x_0^{(1)}, y_0^{(1)})$ is given by

$$J_{(x_{01}, y_{01})} = \begin{bmatrix} ay - 2bx - (1 - c) & ax \\ -ay & g - ax \end{bmatrix}_{(x_{01}, y_{01})} = \begin{bmatrix} -\dfrac{gb}{a} & g \\ c - \left(1 + \dfrac{bg}{a}\right) & 0 \end{bmatrix}. \tag{A.1}$$

Its eigenvalues, λ, are given by

$$\lambda^2 - \left(-\frac{gb}{a}\right)\lambda + g\left(1 + \frac{bg}{a}\right) - cg = 0. \tag{A.2}$$

Since the trace of the Jacobian is negative and its determinant is positive if $c < 1 + (bg/a)$, the fixed point $(x_0^{(1)}, y_0^{(1)})$ is either a stable node or a stable spiral.

The Jacobian at $(x_0^{(2)}, y_0^{(2)})$, when $c > 1$ and $(bg/a + 1 - c) > 0$, is given by

$$J_{(x_{02}, y_{02})} = \begin{bmatrix} ay - 2bx - (1 - c) & ax \\ -ay & g - ax \end{bmatrix}_{(x_{02}, y_{02})} = \begin{bmatrix} 1 - c & \dfrac{a(c - 1)}{b} \\ 0 & \dfrac{a}{b}\left(\dfrac{bg}{a} + 1 - c\right) \end{bmatrix}, \tag{A.3}$$

which shows that $(x_0^{(2)}, y_0^{(2)})$ is unstable saddle.

The Jacobian at $(\tilde{x}_0^{(1)}, \tilde{y}_0^{(1)})$ is given by

$$J_{(x_0, y_0)} = \begin{bmatrix} ay - 2bx - (1 - c) & ax \\ -ay & g - ax \end{bmatrix}_{(x_0, y_0)} = \begin{bmatrix} 1 - c & \dfrac{a(c - 1)}{b} \\ 0 & \dfrac{a}{b}\left(\dfrac{bg}{a} + 1 - c\right) \end{bmatrix}, \tag{A.4}$$

whose eigenvalues are both negative since $1 < gb/a + 1 < c$. Hence the fixed point is a stable node.

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Firdaus Udwadia: Departments of Civil Engineering, Aerospace and Mechanical Engineering, Mathematics, Systems Architecture Engineering, and Information and Operations Management, 430K Olin Hall, University of Southern California, Los Angeles, CA 90089-1453, USA
E-mail address: fudwadia@usc.edu

George Leitmann: Graduate School, College of Engineering, University of California at Berkeley, Berkeley, CA 94720, USA
E-mail address: gleit@uclink4.berkeley.edu

Luca Lambertini: Department of Economics, University of Bologna, Bologna 40125, Italy
E-mail address: lamberti@spbo.unibo.it