PERMANENCE FOR A CLASS OF NONLINEAR DIFFERENCE SYSTEMS

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A class of nonlinear difference systems is considered in this paper. By exploring the relationship between this system and a correspondent first-order difference system, some permanence results are obtained.

1. Introduction

Consider the following system of nonlinear difference equations:

$$x_{n+1} = \lambda x_n + f(\alpha_1 y_n - \beta_1 y_{n-1}), \qquad y_{n+1} = \lambda y_n + f(\alpha_2 x_n - \beta_2 x_{n-1}), \tag{1.1}$$

where $\lambda \in (0,1)$, α_i, β_i $(i = 1,2)$ are given positive constants, and $f : \mathbb{R} \to \mathbb{R}$ is a real function. System (1.1) can be regarded as the discrete analog of the following neural network of two neurons with dynamical threshold effects:

$$\begin{aligned} \frac{dx(t)}{dt} &= -\mu x(t) + f(\alpha_1 y(t) - \beta_1 y(t-\tau)), \\ \frac{dy(t)}{dt} &= -\mu y(t) + f(\alpha_2 x(t) - \beta_2 x(t-\tau)). \end{aligned} \tag{1.2}$$

System (1.2) has found interesting applications in, for example, temporal evolution of sublattice magnetization (see [3]). Recently, the dynamics of (1.2) and some related models have been discussed in [1, 2, 5].

System (1.1) can also be viewed as an extension to two dimensions of the equation

$$x_{n+1} = \lambda x_n + f(x_n - x_{n-1}), \tag{1.3}$$

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which has been studied by Sedaghat [6] and other authors (see [4, 7]). By exploring the relationship between (1.3) and the following first-order initial value problem:

$$v_{n+1} = f(v_n), \qquad v_1 = x_1 - x_0, \tag{1.4}$$

some sufficient conditions for the permanence of (1.3) are obtained in [6]. It is natural to expect that similar results in [6] can be extended from (1.3) to system (1.1). This is the goal of this paper.

As usual, system (1.1) is said to be permanent, if there exists a compact set Ω in the interior of $\mathbb{R} \times \mathbb{R}$ such that any solution of (1.1) will ultimately stay in Ω.

The organization of this paper is as follows. In Section 2, we discuss the following difference system:

$$u_{n+1} = f(\alpha_1 v_n), \quad v_{n+1} = f(\alpha_2 u_n), \quad n = 1,2,\ldots, \tag{1.5}$$

and give some propositions which address the permanence of system (1.5), and therefore which themselves are of some interest and importance. In Section 3, by setting up a useful relationship between systems (1.1) and (1.5), we obtain some sufficient conditions for the permanence of system (1.1). An important example is given in Section 4.

2. Basic propositions

In this section, we discuss some properties of system (1.5). For convenience, we will adopt some notations as follows:

$$g := \alpha_1 f, \qquad h := \alpha_2 f, \qquad F^2 := F \circ F, \qquad F^n := F \circ F^{n-1}, \quad n = 2,3,\ldots, \tag{2.1}$$

where $F \circ G(x) = F(G(x))$.

It is easy to have the following proposition.

Proposition 2.1. *Every solution $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ of system (1.5) satisfies*

$$\begin{aligned} u_{n+1} &= \begin{cases} f \circ (g \circ h)^{k-1} \circ g(\alpha_2 u_1), & \text{if } n = 2k, \\ f \circ (g \circ h)^{k}(\alpha_1 v_1), & \text{if } n = 2k+1, \end{cases} \\ v_{n+1} &= \begin{cases} f \circ (h \circ g)^{k-1} \circ g(\alpha_1 v_1), & \text{if } n = 2k, \\ f \circ (h \circ g)^{k}(\alpha_2 u_1), & \text{if } n = 2k+1. \end{cases} \end{aligned} \tag{2.2}$$

Proposition 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function. Assume that the following condition holds.*

(H_1) *There exist $\delta_i \in (0,1)$ and $M_1 > 0$ such that for all $x \geq M_1$,*

$$f(\alpha_i x) \leq \delta_i x, \quad i = 1,2. \tag{2.3}$$

Then every solution of (1.5) is eventually bounded from above (independent of initial conditions).

Proof. Let $\{(u_n, v_n)\}$ be a solution of (1.5). We claim that there exists a positive integer m such that

$$u_m < M_1, \qquad v_m < M_1. \tag{2.4}$$

First we can prove that there is an m_1 such that $u_{m_1} < M_1$. Otherwise, for any $n > 0$, we have $u_n \geq M_1$. Then

$$\begin{aligned}
v_{n+1} &= f(\alpha_2 u_n) \leq \delta_2 u_n < u_n, \\
u_{n+2} &= f(\alpha_1 v_{n+1}) \leq f(\alpha_1 u_n) \leq \delta_1 u_n, \\
v_{n+3} &= f(\alpha_2 u_{n+2}) \leq \delta_2 u_{n+2} < u_{n+2}, \\
u_{n+4} &= f(\alpha_1 v_{n+3}) \leq f(\alpha_1 u_{n+2}) \leq \delta_1 u_{n+2} \leq \delta_1^2 u_n.
\end{aligned} \tag{2.5}$$

It follows, by induction, that

$$u_{n+2k} \leq \delta_1^k u_n, \quad k = 1, 2, \ldots. \tag{2.6}$$

Now, fix n and take $k \to \infty$ in (2.6) and note that $0 < \delta_1 < 1$, we then get

$$\lim_{k \to \infty} u_{n+2k} = 0, \tag{2.7}$$

which contradicts to $u_n \geq M_1 > 0$.

Next we distinguish two cases.

Case 1. If $v_{m_1} < M_1$, then (2.4) holds.

Case 2. If $v_{m_1} \geq M_1$, we show that there exists k_1 such that

$$v_{m_1+2k_1} < M_1. \tag{2.8}$$

Assume that (2.8) is not true, then $v_{m_1+2k} \geq M_1$ for all k. Similar to the proof of (2.6), we have

$$0 < M_1 \leq v_{m_1+2k} \leq \delta_2^k v_{m_1} \longrightarrow 0 \quad (\text{as } k \longrightarrow \infty) \tag{2.9}$$

which is contradiction.

Noting $u_{m_1} < M_1$ implies that $u_{m_1+2k} < M_1$ for all k, then take $m = m_1 + 2k_1$, and (2.4) holds.

Now, by (1.5), we have

$$\begin{aligned}
u_{m+1} &= f(\alpha_1 v_m) \leq f(\alpha_1 M_1) \leq \delta_1 M_1 < M_1, \\
v_{m+1} &= f(\alpha_2 u_m) \leq f(\alpha_2 M_1) \leq \delta_2 M_1 < M_1.
\end{aligned} \tag{2.10}$$

Thus, by induction, we obtain

$$u_n < M_1, \qquad v_n < M_1 \tag{2.11}$$

for all $n \geq m$.This completes the proof. □

Letting $u'_n = -u_n$, $v'_n = -v_n$, $F(x) = -f(-x)$, we then have the following proposition which comes directly from Proposition 2.2.

Proposition 2.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function. Assume that the following condition holds.*

(H_2) *There exist $\delta_i \in (0,1)$ and $M_2 > 0$ such that for all $x \leq -M_2$,*

$$f(\alpha_i x) \geq \delta_i x, \quad i = 1,2. \tag{2.12}$$

Then every solution of (1.5) is eventually bounded from below (independent of initial conditions).

Propositions 2.2 and 2.3 can be combined to give the following proposition.

Proposition 2.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function. If there exist $\delta_i \in (0,1)$ such that*

$$\lim_{x \to \infty} \frac{f(\alpha_i x)}{x} = \delta_i, \quad i = 1,2, \tag{2.13}$$

then (1.5) is permanent.

3. Permanence of (1.1)

In this section, we are concerned with the permanence of system (1.1). To this end, we need to establish the following lemma which gives a useful link between the solutions of (1.1) and (1.5).

Lemma 3.1. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function. Let $\{(x_n, y_n)\}$ be a nonnegative solution of the following difference inequalities:*

$$x_{n+1} \leq \lambda x_n + f(\alpha_1 y_n - \beta_1 y_{n-1}), \qquad y_{n+1} \leq \lambda y_n + f(\alpha_2 x_n - \beta_2 x_{n-1}), \tag{3.1}$$

with initial conditions (x_0, y_0) and (x_1, y_1), and $\{(u_n, v_n)\}$ is the solution of (1.5) with the initial values u_1, v_1 satisfying

$$\alpha_2 u_1 = \alpha_2 x_1 - \beta_2 x_0, \qquad \alpha_1 v_1 = \alpha_1 y_1 - \beta_1 y_0. \tag{3.2}$$

If the following condition holds:

(H_3) $\alpha_i \lambda - \beta_i \leq 0$, $i = 1,2$,

then for all $n \geq 1$,

$$\alpha_2 x_n \leq \lambda^{n-1} \beta_2 x_0 + \sum_{k=1}^{n} \lambda^{n-k} \alpha_2 u_k, \qquad \alpha_1 y_n \leq \lambda^{n-1} \beta_1 y_0 + \sum_{k=1}^{n} \lambda^{n-k} \alpha_1 v_k. \tag{3.3}$$

Proof. We first observe that

$$\alpha_2 x_1 = \beta_2 x_0 + \alpha_2 u_1, \qquad \alpha_1 y_1 = \beta_1 y_0 + \alpha_1 v_1, \tag{3.4}$$

and that

$$\begin{aligned}\alpha_2 x_2 &\le \alpha_2(\lambda x_1 + f(\alpha_1 y_1 - \beta_1 y_0))\\ &= \lambda(\beta_2 x_0 + \alpha_2 u_1) + \alpha_2 f(\alpha_1 v_1) = \lambda\beta_2 x_0 + \lambda\alpha_2 u_1 + \alpha_2 u_2,\\ \alpha_1 y_2 &\le \alpha_1(\lambda y_1 + f(\alpha_2 x_1 - \beta_2 x_0))\\ &= \lambda(\beta_1 y_0 + \alpha_1 v_1) + \alpha_1 f(\alpha_2 u_1) = \lambda\beta_1 y_0 + \lambda\alpha_1 v_1 + \alpha_1 v_2.\end{aligned} \tag{3.5}$$

Hence, (3.3) holds for $n = 1,2$. Next we assume that (3.3) holds for all integers less than or equal to some integer n. Then

$$\begin{aligned}\alpha_2 x_{n+1} &\le \alpha_2(\lambda x_n + f(\alpha_1 y_n - \beta_1 y_{n-1}))\\ &\le \lambda^n\beta_2 x_0 + \sum_{k=1}^{n}\lambda^{n-k+1}\alpha_2 u_k + \alpha_2 f(\alpha_1 y_n - \beta_1 y_{n-1}),\\ \alpha_1 y_{n+1} &\le \alpha_1(\lambda y_n + f(\alpha_2 y_n - \beta_2 x_{n-1}))\\ &\le \lambda^n\beta_1 y_0 + \sum_{k=1}^{n}\lambda^{n-k+1}\alpha_1 v_k + \alpha_1 f(\alpha_2 x_n - \beta_2 x_{n-1}).\end{aligned} \tag{3.6}$$

So it remains to show that

$$f(\alpha_1 y_n - \beta_1 y_{n-1}) \le u_{n+1}, \qquad f(\alpha_2 x_n - \beta_2 x_{n-1}) \le v_{n+1}. \tag{3.7}$$

To this end, we note that

$$\begin{aligned}\alpha_1 y_n - \beta_1 y_{n-1} &\le (\alpha_1\lambda - \beta_1)y_{n-1} + \alpha_1 f(\alpha_2 x_{n-1} - \beta_2 x_{n-2})\\ &\le \alpha_1 f(\alpha_2 x_{n-1} - \beta_2 x_{n-2}) = g(\alpha_2 x_{n-1} - \beta_2 x_{n-2}),\\ \alpha_2 x_n - \beta_2 x_{n-1} &\le (\alpha_2\lambda - \beta_2)x_{n-1} + \alpha_2 f(\alpha_1 y_{n-1} - \beta_1 y_{n-2})\\ &\le \alpha_2 f(\alpha_1 y_{n-1} - \beta_1 y_{n-2}) = h(\alpha_1 y_{n-1} - \beta_1 y_{n-2}),\end{aligned} \tag{3.8}$$

which, together with the assumption that f is nondecreasing, implies that

$$\begin{aligned}f(\alpha_1 y_n - \beta_1 y_{n-1}) &\le f\circ g(\alpha_2 x_{n-1} - \beta_2 x_{n-2}),\\ f(\alpha_2 x_n - \beta_2 x_{n-1}) &\le f\circ h(\alpha_1 y_{n-1} - \beta_1 y_{n-2}).\end{aligned} \tag{3.9}$$

Following this fashion, we can get

$$\begin{aligned}f(\alpha_1 y_n - \beta_1 y_{n-1}) &\le \begin{cases} f\circ(g\circ h)^{k-1}\circ g(\alpha_2 u_1), & \text{if } n = 2k,\\ f\circ(g\circ h)^{k}(\alpha_1 v_1), & \text{if } n = 2k+1,\end{cases}\\ f(\alpha_2 x_n - \beta_2 x_{n-1}) &\le \begin{cases} f\circ(h\circ g)^{k-1}\circ g(\alpha_1 v_1), & \text{if } n = 2k,\\ f\circ(h\circ g)^{k}(\alpha_2 u_1), & \text{if } n = 2k+1.\end{cases}\end{aligned} \tag{3.10}$$

Then (3.7) follows from Proposition 2.1 and thus the proof is complete. □

Similar to the proof of Lemma 3.1, we have the following.

Lemma 3.2. *Suppose that $f:\mathbb{R}\to\mathbb{R}$ is a nondecreasing function. Let $\{(x_n,y_n)\}$ be a nonpositive solution of the following difference inequalities:*

$$x_{n+1}\geq \lambda x_n+f(\alpha_1 y_n-\beta_1 y_{n-1}),\qquad y_{n+1}\geq \lambda y_n+f(\alpha_2 x_n-\beta_2 x_{n-1}),\tag{3.11}$$

with initial conditions (x_0,y_0) and (x_1,y_1), and $\{(u_n,v_n)\}$ is the solution of (1.5) with the initial values u_1,v_1 satisfying (3.2). If the condition (H_3) holds, then for all $n\geq 1$,

$$\alpha_2 x_n\geq \lambda^{n-1}\beta_2 x_0+\sum_{k=1}^{n}\lambda^{n-k}\alpha_2 u_k,\ \alpha_1 y_n\geq \lambda^{n-1}\beta_1 y_0+\sum_{k=1}^{n}\lambda^{n-k}\alpha_1 v_k.\tag{3.12}$$

We are now able to state and prove our permanence results for system (1.1).

Theorem 3.3. *Let f be nondecreasing and bounded from below on $\mathbb{R}$. Suppose that (H_1) and (H_3) hold. Assume further that*
(H_4) $\alpha_i\geq\beta_i$, $i=1,2$.
Then (1.1) is permanent.

Proof. If we define $X_n=f(\alpha_2 x_n-\beta_2 x_{n-1})$, $Y_n=f(\alpha_1 y_n-\beta_1 y_{n-1})$ for all $n\geq 1$, then it follows inductively from (1.1) that

$$x_n=\lambda^{n-1}x_1+\sum_{k=1}^{n-1}\lambda^{n-k-1}Y_k,\qquad y_n=\lambda^{n-1}y_1+\sum_{k=1}^{n-1}\lambda^{n-k-1}X_k.\tag{3.13}$$

Let L_0 be a lower bound for $f(t)$ and without loss of generality we assume that $L_0\leq 0$. As $X_k\geq L_0$ and $Y_k\geq L_0$ for all k, we conclude from (3.13) that for all n,

$$x_n\geq \lambda^{n-1}x_1+\frac{(1-\lambda^{n-1})L_0}{1-\lambda},\qquad y_n\geq \lambda^{n-1}y_1+\frac{(1-\lambda^{n-1})L_0}{1-\lambda},\tag{3.14}$$

and therefore $\{(x_n,y_n)\}$ is bounded from below. In fact, it is clear that there is a positive integer n_0 such that for all $n\geq n_0$,

$$x_n\geq L,\qquad y_n\geq L,\tag{3.15}$$

where $L=L_0/(1-\lambda)-1<0$. We next show that $\{(x_n,y_n)\}$ is bounded from above as well. Define

$$\phi_n=x_{n+n_0}-L,\qquad \varphi_n=y_{n+n_0}-L\tag{3.16}$$

for all $n\geq 0$, so that $\phi_n\geq 0$, $\varphi_n\geq 0$ for all n. Now for each $n\geq 1$, we have

$$\begin{aligned}\phi_{n+1}&=\lambda x_{n+n_0}+f(\alpha_1 y_{n+n_0}-\beta_1 y_{n+n_0-1})-L=\lambda\phi_n+f(\alpha_1 y_{n+n_0}-\beta_1 y_{n+n_0-1})-(1-\lambda)L,\\ \varphi_{n+1}&=\lambda y_{n+n_0}+f(\alpha_2 x_{n+n_0}-\beta_2 x_{n+n_0-1})-L=\lambda\varphi_n+f(\alpha_2 x_{n+n_0}-\beta_2 x_{n+n_0-1})-(1-\lambda)L.\end{aligned}\tag{3.17}$$

Note that

$$\begin{aligned}\alpha_1 y_{n+n_0} - \beta_1 y_{n+n_0-1} &= \alpha_1\varphi_n - \beta_1\varphi_{n-1} + (\alpha_1-\beta_1)L \le \alpha_1\varphi_n - \beta_1\varphi_{n-1},\\ \alpha_2 x_{n+n_0} - \beta_2 x_{n+n_0-1} &= \alpha_2\phi_n - \beta_2\phi_{n-1} + (\alpha_2-\beta_2)L \le \alpha_2\phi_n - \beta_2\phi_{n-1},\end{aligned} \tag{3.18}$$

which, together with the assumption that f is nondecreasing, implies that

$$\begin{aligned}f\left(\alpha_1 y_{n+n_0} - \beta_1 y_{n+n_0-1}\right) &\le f\left(\alpha_1\varphi_n - \beta_1\varphi_{n-1}\right),\\ f\left(\alpha_2 x_{n+n_0} - \beta_2 x_{n+n_0-1}\right) &\le f\left(\alpha_2\phi_n - \beta_2\phi_{n-1}\right).\end{aligned} \tag{3.19}$$

Define $F(x) := f(x) - (1-\lambda)L$. By (3.17) and (3.19), we get

$$\phi_{n+1} \le \lambda\phi_n + F\left(\alpha_1\varphi_n - \beta_1\varphi_{n-1}\right), \qquad \varphi_{n+1} \le \lambda\varphi_n + F\left(\alpha_2\phi_n - \beta_2\phi_{n-1}\right). \tag{3.20}$$

Let $\delta_i^* \in (\delta_i, 1)$, $i = 1,2$, and $M_1^* = \max\{M_1, -(1-\lambda)L/(\delta_1^*-\delta_1), -(1-\lambda)L/(\delta_2^*-\delta_2)\}$. It is readily verified that for all $x \ge M_1^*$,

$$F(\alpha_i x) \le \delta_i^* x \quad (i = 1,2). \tag{3.21}$$

Consider the following initial value problem:

$$\begin{aligned}u_{n+1} &= F(\alpha_1 v_n), &\qquad u_1 &= \frac{\alpha_2\phi_1 - \beta_2\phi_0}{\alpha_2},\\ v_{n+1} &= F(\alpha_2 u_n), &\qquad v_1 &= \frac{\alpha_1\varphi_1 - \beta_1\varphi_0}{\alpha_1}.\end{aligned} \tag{3.22}$$

From Proposition 2.2 we know that there exist integer $m \ge 0$ and constant $M_0 > 0$ such that for all $n \ge m$, $u_n \le M_0$, $v_n \le M_0$. Applying Lemma 3.1 to (3.20), we obtain that for all $n \ge m$,

$$\begin{aligned}\alpha_2\phi_n &\le \lambda^{n-1}\beta_2\phi_0 + \sum_{k=1}^{m-1}\lambda^{n-k}\alpha_2 u_k + \sum_{k=m}^{n}\lambda^{n-k}\alpha_2 u_k\\ &\le \lambda^{n-m+1}\left(\lambda^{m-2}\beta_2\phi_0 + \lambda^{m-2}\alpha_2 u_1 + \cdots + \alpha_2 u_{m-1}\right) + \alpha_2 M_0\sum_{k=0}^{n-m}\lambda^k\\ &= \lambda^{n-m+1}M^* + \alpha_2 M_0(1-\lambda)^{-1}\left(1-\lambda^{n-m+1}\right),\\ \alpha_1\varphi_n &\le \lambda^{n-1}\beta_1\varphi_0 + \sum_{k=1}^{m-1}\lambda^{n-k}\alpha_1 v_k + \sum_{k=m}^{n}\lambda^{n-k}\alpha_1 v_k\\ &\le \lambda^{n-m+1}\left(\lambda^{m-2}\beta_1\varphi_0 + \lambda^{m-2}\alpha_1 v_1 + \cdots + \alpha_1 v_{m-1}\right) + \alpha_1 M_0\sum_{k=0}^{n-m}\lambda^k\\ &= \lambda^{n-m+1}N^* + \alpha_1 M_0(1-\lambda)^{-1}\left(1-\lambda^{n-m+1}\right),\end{aligned} \tag{3.23}$$

where $M^* = \lambda^{m-2}\beta_2\phi_0 + \lambda^{m-2}\alpha_2 u_1 + \cdots + \alpha_2 u_{m-1}$, $N^* = \lambda^{m-2}\beta_1\varphi_0 + \lambda^{m-2}\alpha_1 v_1 + \cdots + \alpha_1 v_{m-1}$. Thus there exists $n_1 \geq m$ such that for all $n \geq n_1$,

$$\phi_n \leq \frac{M_0}{1-\lambda} + 1, \qquad \varphi_n \leq \frac{M_0}{1-\lambda} + 1. \tag{3.24}$$

Hence, for all $n \geq n_0 + n_1$, we have

$$(x_n, y_n) \in [L, M] \times [L, M], \tag{3.25}$$

where

$$M = \frac{M_0}{1-\lambda} + 1 + L. \tag{3.26}$$

This shows that (1.1) is permanent. The proof is completed. □

Similarly, we have the following.

Theorem 3.4. *Let f be nondecreasing and bounded from above on $\mathbb{R}$. Suppose that (H_2), (H_3), and (H_4) hold. Then (1.1) is permanent.*

From the proof of Theorem 3.3, we can easily establish the following assertion.

Corollary 3.5. *Let f be bounded from below (from above) on $\mathbb{R}$. Then every solution of (1.1) is bounded from below (from above). In particular, if f is bounded, then every solution of (1.1) is bounded.*

4. An example

Consider the following system of two difference equations:

$$X_{n+1} = \lambda X_n + \alpha_1 f(Y_n) - \beta_1 f(Y_{n-1}), \qquad Y_{n+1} = \lambda Y_n + \alpha_2 f(X_n) - \beta_2 f(X_{n-1}), \tag{4.1}$$

where $\lambda \in [0,1)$, α_i, β_i $(i = 1,2)$ are given positive constants with , and $f : \mathbb{R} \to \mathbb{R}$ is a real function.

Let $\{(X_n, Y_n)\}$ be a solution of (4.1), and for $n \geq 1$, define

$$\begin{aligned} x_n &= \left(\frac{\beta_2}{\alpha_2}\right)^n x_0 + \sum_{k=0}^{n-1}\left(\frac{\beta_2}{\alpha_2}\right)^{n-k-1}\frac{1}{\alpha_2}Y_k, \\ y_n &= \left(\frac{\beta_1}{\alpha_1}\right)^n y_0 + \sum_{k=0}^{n-1}\left(\frac{\beta_1}{\alpha_1}\right)^{n-k-1}\frac{1}{\alpha_1}X_k, \end{aligned} \tag{4.2}$$

for some real numbers x_0, y_0. We will show that $\{(x_n, y_n)\}$ satisfies (1.1) for some choice of (x_0, y_0). Note that

$$X_n = \alpha_1 y_{n+1} - \beta_1 y_n, \qquad Y_n = \alpha_2 x_{n+1} - \beta_2 x_n, \tag{4.3}$$

$$\begin{aligned} x_2 &= \left(\frac{\beta_2}{\alpha_2}\right)^2 x_0 + \frac{\beta_2}{\alpha_2^2} Y_0 + \frac{1}{\alpha_2} Y_1, \\ y_2 &= \left(\frac{\beta_1}{\alpha_1}\right)^2 y_0 + \frac{\beta_1}{\alpha_1^2} X_0 + \frac{1}{\alpha_1} X_1. \end{aligned} \tag{4.4}$$

In order for $\{(x_n, y_n)\}$ to satisfy (1.1), x_0 and y_0 must be chosen such that

$$\begin{aligned} \lambda x_1 + f(\alpha_1 y_1 - \beta_1 y_0) &= \lambda\left(\frac{\beta_2}{\alpha_2} x_0 + \frac{1}{\alpha_2} Y_0\right) + f(X_0), \\ \lambda y_1 + f(\alpha_2 x_1 - \beta_2 x_0) &= \lambda\left(\frac{\beta_1}{\alpha_1} y_0 + \frac{1}{\alpha_1} X_0\right) + f(Y_0). \end{aligned} \tag{4.5}$$

Solving for x_0 and y_0 we obtain

$$\begin{aligned} x_0 &= -\frac{1}{\beta_2} Y_0 - \frac{\alpha_2}{\beta_2(\beta_2 - \lambda\alpha_2)} Y_1 + \frac{\alpha_2^2}{\beta_2(\beta_2 - \lambda\alpha_2)} f(X_0), \\ y_0 &= -\frac{1}{\beta_1} X_0 - \frac{\alpha_1}{\beta_1(\beta_1 - \lambda\alpha_1)} X_1 + \frac{\alpha_1^1}{\beta_1(\beta_1 - \lambda\alpha_1)} f(Y_0). \end{aligned} \tag{4.6}$$

Thus,

$$x_2 = \lambda x_1 + f(\alpha_1 y_1 - \beta_1 y_0), \qquad y_2 = \lambda y_1 + f(\alpha_2 x_1 - \beta_2 x_0). \tag{4.7}$$

Now, for any $n \geq 1$, from (4.1) and (4.3), we have

$$\begin{aligned} \alpha_2[x_{n+2} - \lambda x_{n+1} - f(\alpha_1 y_{n+1} - \beta_1 y_n)] &= \beta_2[x_{n+1} - \lambda x_n - f(\alpha_1 y_n - \beta_1 y_{n-1})], \\ \alpha_1[y_{n+2} - \lambda y_{n+1} - f(\alpha_2 x_{n+1} - \beta_2 x_n)] &= \beta_1[y_{n+1} - \lambda y_n - f(\alpha_2 x_n - \beta_2 x_{n-1})]. \end{aligned} \tag{4.8}$$

By (4.7) and (4.8), we can get inductively that $\{(x_n, y_n)\}$ is the solution of (1.1). From (4.3), we know

$$|X_n| \leq \alpha_1 |y_{n+1}| + \beta_1 |y_n|, \qquad |Y_n| \leq \alpha_2 |x_{n+1}| + \beta_2 |x_n|. \tag{4.9}$$

Therefore, by Theorems 3.3 and 3.4 , we obtain the following result on permanence in system (4.1).

Corollary 4.1. *Let f be nondecreasing and bounded from below (or from above) on $\mathbb{R}$. Suppose that conditions (H_1) (or (H_2)), (H_3), and (H_4) hold. Then system (4.1) is permanent.*

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