MULTIPLE POSITIVE SOLUTIONS OF STRUM-LIOUVILLE EQUATIONS WITH SINGULARITIES

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Received 20 October 2005; Accepted 8 January 2006

The existence of multiple positive solutions for Strum-Liouville boundary value problems with singularities is investigated. By applying a fixed point theorem of cone map, some existence and multiplicity results of positive solutions are derived. Our results improve and generalize those in some well-known results.

1. Introduction

In this paper, we consider the following second-order Strum-Liouville boundary value problem with singularities (BVP):

$$\begin{gathered}(p(t)u'(t))' + g(t)F(t,u) = 0, \quad \forall t \in (0,1), \\ \alpha u(0) - \beta \lim_{t\to 0^+} p(t)u'(t) = 0, \\ \gamma u(1) - \delta \lim_{t\to 1^-} p(t)u'(t) = 0,\end{gathered} \tag{1.1}$$

where $\alpha,\beta,\gamma,\delta \geq 0$, and

$$\rho = \beta\gamma + \alpha\gamma \int_0^1 \frac{dr}{p(r)} + \alpha\delta > 0. \tag{1.2}$$

We always assume that the following hypotheses hold:

(A_1) $p \in C([0,1], [0,+\infty])$, and $\int_0^1 dt/p(t) < +\infty$;

(A_2) $g \in L(0,1)$, and $g(s) \geq 0$, a.e. and there exists $[a,b] \subset (0,1)$, such that $0 < \int_a^b g(s)ds$;

(A_3) $F(t,u) \in C([0,1] \times [0,+\infty], [0,+\infty])$.

The BVP(1.1) arises in many different areas of applied mathematics and physics, and only its positive solution is significant in some practice. For the special case as follows with

Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2006, Article ID 32018, Pages 1–11
DOI 10.1155/DDNS/2006/32018

$p(t) \equiv 1$ and $F(t,u) = f(u)$, BVP(1.1) was studied by many authors (see [1, 4–8]). Erbe and Wang [1] investigated this problem by using norm-type cone expansion and compression theorems; Lan and Webb [4] get the existence of positive solution for BVP(1.1) by using a well-known nonzero fixed point theorem. This problem is also studied by Ma [6] by using compact operator approximation.

Recently the authors [10] investigated the Strum-Liouville equation which has singularities at 0 and 1, the existence of one positive solution is established by applying the fixed point index theory.

However, the authors [1, 4, 6, 8, 10] only investigated the existence of positive solution for BVP(1.1). Liu and Li [5] get the existence of multiple positive solutions in the special case as follows with $p(t) \equiv 1$ and $g(t) \equiv 1$ by using the fixed pointed theorem. When we choose $g(t) \equiv 1$, Wang et al. [11] established some nonexistence, existence, and multiplicity results for the BVP(1.1) which are based on the Schauder fixed point theorem, the method of upper and lower solutions, and the Leray-Schauder degree theory. Furthermore, Ma and Thompson [7] studied the existence of multiple positive solutions by applying bifurcation techniques. The purpose of this paper is to consider BVP(1.1) in which $g(t) \in L^1(0,1)$ and $F(t,u)$ satisfies weaker conditions than those in [1, 4–11], the existence of multiple positive solutions for BVP(1.1) is obtained by using a fixed point theorem. Our method is different from the ones in those papers and our results are often new even when $p(t) \equiv 1$, $F(t,u) = f(u)$.

A map $u \in C([0,1],R^+) \cap C^1((0,1),R^+)$, $p(t)u'(t) \in C^1((0,1),R^+)$ is called a positive solution of BVP(1.1) if $x(t) > 0$, for all $t \in (0,1)$ and $x(t)$ satisfies BVP(1.1).

2. Some lemmas

We will need the following well-known result (see, e.g., [3, Theorems 2.1 and 2.2].

Lemma 2.1. *Let K be a cone in a Banach space X. Assume that Ω is a bounded open subset of X with $\theta \in \Omega$ and let $A : K \cap (\overline{\Omega}) \to K$ be a completely continuous operator. If $Au \neq \lambda x$ for $x \in K \cap \partial\Omega$, $\lambda \geq 1$, then $i(A, K \cap \Omega, K) = 1$.*

Lemma 2.2. *Assume that $A : K \cap (\overline{\Omega}) \to K$ is completely continuous, and there exists $B : K \cap \partial\Omega \to K$ which is completely continuous, such that*

(i) $\inf_{x\in\cap\partial\Omega} \|Bu\| > 0$;

(ii) *$x - Ax \neq \lambda Bx$ for $x \in K \cap \partial\Omega$, and $\lambda \geq 0$,*

then $i(A, K \cap \Omega, K) = 0$.

We denote by $G(t,s)$ Green's function for the homogeneous boundary value problem

$$\begin{gathered}
\big(p(t)u'(t)\big)' = 0, \quad \forall t \in (0,1), \\
\alpha u(0) - \beta \lim_{t\to 0^+} p(t)u'(t) = 0, \\
\gamma u(1) - \delta \lim_{t\to 1^-} p(t)u'(t) = 0.
\end{gathered} \tag{2.1}$$

We know that $G(t,s)$ is nonnegative on $[0,1]\times[0,1]$ and is expressed by

$$G(t,s)=\begin{cases}\dfrac{1}{\rho}\left(\beta+\alpha\displaystyle\int_0^s\frac{dr}{p(r)}\right)\left(\delta+\gamma\displaystyle\int_t^1\frac{dr}{p(r)}\right), & 0\le s\le t\le 1,\\ \dfrac{1}{\rho}\left(\beta+\alpha\displaystyle\int_0^t\frac{dr}{p(r)}\right)\left(\delta+\gamma\displaystyle\int_s^1\frac{dr}{p(r)}\right), & 0\le t\le s\le 1,\end{cases} \tag{2.2}$$

where $\rho=\alpha\delta+\alpha\gamma\int_0^1(dr/p(r))+\beta\gamma$.

Lemma 2.3 [2]. *Green's function $G(t,s)$ has the following properties:*

(i) $G(t,s)\le G(s,s)\le(1/\rho)(\beta+\alpha\int_0^1(dr/p(r)))(\delta+\gamma\int_0^1(dr/p(r))):=\theta<\infty$;

(ii) *for all $t\in[a,b]\subset(0,1)$, $s\in[0,1]$, there is $G(t,s)\ge\sigma G(s,s)$,*

where

$$\sigma=\min\left\{\frac{\delta+\gamma\int_b^1(dr/p(r))}{\delta+\gamma\int_0^1(dr/p(r))},\frac{\beta+\alpha\int_0^a(dr/p(r))}{\beta+\alpha\int_0^1(dr/p(r))}\right\}. \tag{2.3}$$

It is obvious that $0<\sigma<1$. Let $X=C[0,1]$, $\Omega_h=\{u\in X:\|u\|\le h\}$ for any $h>0$, and $K=\{u\in C[0,1]:u\ge 0$, and $x(t)\ge\sigma x(\tau)$ for $t\in[a,b]$, $\tau\in[0,1]\}$. Then K is a positive cone in X.

Define an operator A by

$$(Au)(t)=\int_0^1 G(t,s)g(s)F(s,u(s))\,ds. \tag{2.4}$$

It is well known that $u\in C[0,1]\cap C^1(0,1)$ is a positive solution of BVP(1.1) if and only if u is a fixed point of the operator A in K.

Lemma 2.4. *$A:K\to K$ is completely continuous.*

Proof. Similar to the proof of [10, Lemma 2.1], we can prove that A is a completely continuous operator. □

3. Main result

Theorem 3.1. *Assume that (A_1)–(A_3) hold, if*

(H_1) *there exists $p\in(0,1)$ such that $0<\liminf_{u\to 0^+}\min_{t\in[0,1]}(F(t,u)/u^p)\le+\infty$,*

(H_2) *there exists $q\in(0,1]$ such that $0\le\limsup_{u\to+\infty}\max_{t\in[0,1]}(F(t,u)/u^q)<+\infty$,*

then BVP(1.1) has at least one positive solution.

Proof. Without loss of generality, we assume that there exists $\epsilon>0$ such that $x\ne Ax$ for $x\in K$ with $0<\|x\|<\epsilon$, otherwise there is a fixed point in K and this would complete the proof.

By virtue of (H_1), there exists $\tau>0$ and $\epsilon_1>0$ such that

$$F(t,u)\ge\tau u^p,\quad \text{for } 0\le u\le\epsilon_1. \tag{3.1}$$

Define $B : C[0,1] \to C[0,1]$ by

$$Bu = \phi, \quad \forall u \in C[0,1], \tag{3.2}$$

where $\phi(t) \equiv 1$, $\phi \in C[0,1]$. Then it is easy to verify that $B : K \cap \partial\Omega \to K$ is completely continuous and $\inf_{K\cap\partial\Omega_r} \|Bu\| > 0$, $\phi \in K\backslash\theta$ with $\|\phi\| = 1$.

Choose

$$\epsilon_2 = \min\left\{\epsilon, \epsilon_1, \left(\sigma\delta\int_a^b G(s,s)g(s)ds\right)^{1/(1-p)}\right\} \tag{3.3}$$

and $r \in (0,\epsilon_2]$; we now prove that

$$u - Au \neq \lambda Bu, \quad \text{for } u \in K \cap \partial\Omega_r,\ \lambda \geq 0. \tag{3.4}$$

In fact, if not, there are $\lambda_0 \geq 0$ and $u_0 \in K \cap \partial\Omega_r$ such that $u_0 - Au_0 = \lambda_0 Bu_0$. So $\lambda_0 > 0$, then we have $u_0 = Au_0 + \lambda_0 Bu_0 \geq \lambda_0\phi$. Let $\lambda^* = \sup\{\lambda : u_0(s) \geq \lambda\phi(s),\ s \in [a,b]\}$, then $\lambda^* \in [\lambda_0, +\infty)$ and $u_0(s) \geq \lambda^*$ for $s \in [a,b]$. So the inequality

$$\lambda^* \leq u_0(s) \leq ||u_0|| = r \leq \left(\sigma\delta\int_a^b G(s,s)g(s)ds\right)^{1/(1-p)}, \quad \forall s \in [a,b]. \tag{3.5}$$

Then if $t \in [a,b]$, we have

$$\begin{aligned} u_0(t) &= \int_0^1 G(t,s)g(s)F(s,u_0(s))ds + \lambda_0\phi(t) \\ &\geq \int_a^b G(t,s)g(s)F(s,u_0(s))ds + \lambda_0 \\ &\geq \sigma\int_a^b G(s,s)g(s)\delta(u_0(s))^p ds + \lambda_0 \\ &\geq \sigma\delta(\lambda^*)^p\int_a^b G(s,s)g(s)ds + \lambda_0 \geq \lambda^* + \lambda_0 \end{aligned} \tag{3.6}$$

which contradicts the definition of λ^*, hence (3.4) holds, by Lemma 2.2

$$i(A, K \cap \partial\Omega_r, K) = 0. \tag{3.7}$$

In view of (H_2), there exists $\eta > 0$ and $N > 0$ such that $F(t,u) \leq \eta u^q$, for $u \geq C_0$, then

$$0 \leq F(t,u) \leq M + \eta u^q, \quad \text{for } u \in [0,+\infty], \tag{3.8}$$

where $M = \max\{F(t,u) : 0 \leq u \leq C_0\}$.

Choose sufficiently large $R > 0$ such that

$$MR^{-1} + \eta R^{q-1} < \left(\theta\int_0^1 g(s)ds\right)^{-1}. \tag{3.9}$$

We will prove that

$$Au \neq \lambda u, \quad \text{for } u \in K \cap \partial\Omega_R,\ \lambda \geq 1. \tag{3.10}$$

In fact, if not, there exist $u_1 \in K \cap \partial\Omega_R$ and $\lambda_1 \geq 1$ such that $Au_1 = \lambda_1 u_1$, then if $t \in [0,1]$, we have

$$\begin{aligned} u_1(t) \leq \lambda_1 u_1(t) &= \int_0^1 G(t,s)g(s)F(s,u_1(s))\,ds \\ &\leq \int_0^1 G(t,s)g(s)\left(M+\eta u_1(s)^q\right)ds \\ &\leq \left(M+\eta R^q\right)\int_0^1 G(t,s)g(s)\,ds \\ &\leq \left(M+\eta R^q\right)\theta\int_0^1 g(s)\,ds. \end{aligned} \tag{3.11}$$

Hence $R \leq (M+\eta R^q)\theta\int_0^1 g(s)ds$. That is,

$$R \leq \left(M+\eta R^q\right)\theta\int_0^1 g(s)\,ds \tag{3.12}$$

which yields

$$MR^{-1}+\eta R^{q-1} \geq \left(\theta\int_0^1 g(s)\,ds\right)^{-1}, \tag{3.13}$$

which is a contradiction to (3.9), so (3.10) holds. By Lemma 2.1, we have

$$i\left(A, K\cap\partial\Omega_R, K\right) = 1, \tag{3.14}$$

(3.7) and (3.14) together imply

$$i\left(A, K\cap\left(\Omega_R\backslash\overline{\Omega_r}\right), K\right) = i\left(K\cap\partial\Omega_R, K\right) - i\left(A, K\cap\partial\Omega_r, K\right) = 0-1 = -1. \tag{3.15}$$

Consequently, according to [2, Theorem 2.3.2], A has a fixed point $u^* \in K\cap(\Omega_R\backslash\Omega_r)$, so BVP(1.1) has at least one positive solution u^*. This completes the proof. □

Theorem 3.2. *Assume that* (A_1)–(A_3) *hold, if*

(H_3) *there exists* $k \in (1,+\infty)$ *such that* $0 < \limsup_{u\to+\infty}\min_{t\in[0,1]}(F(t,u)/u^k) \leq +\infty$,

(H_4) *there exists* $l \in (1,+\infty)$ *such that* $0 \leq \liminf_{u\to 0^+}\max_{t\in[0,1]}(F(t,u)/u^l) < +\infty$,

then BVP(1.1) has at least one positive solution.

Proof. By virtue of (H_3), there exist $\xi > 0$ and $C_0 > 0$ such that $F(t,u) \geq \xi u^k$, for $u \geq C_0$. Choose

$$R > \max\left\{N\sigma^{-1}, \left[\min_{t\in[a,b]}\int_a^b G(t,s)g(s)\,ds\,\xi\right]^{(-1/k-1)}\right\}. \tag{3.16}$$

Without loss of generality, we assume that $u \neq Au$ for $u \in K \cap \partial\Omega_R$, otherwise the conclusion holds. Define $B : C[0,1] \to C[0,1]$ by

$$Bu = \psi, \quad \forall u \in C[0,1], \tag{3.17}$$

where $\psi(t) = \min_{t\in[a,b]} \int_a^b G(t,s)g(s)ds$, $\psi \in C[0,1]$. Then it is easy to verify $B : K \cap \partial\Omega \to K$ is completely continuous and $\inf_{K\cap\partial\Omega_R} \|Bu\| > 0$, $\phi \in K\backslash\theta$ with $\|\psi\| = C$, C is a const.

We now prove that

$$u - Au \neq \lambda Bu, \quad \text{for } u \in K \cap \partial\Omega_R,\ \lambda \geq 0. \tag{3.18}$$

In fact, if not, there are $\lambda_1 \geq 0$ and $u_1 \in K \cap \partial\Omega_r$ such that $u_1 - Au_1 = \lambda_1 Bu_1$. So $\lambda_1 > 0$, then we have $u_1 = Au_1 + \lambda_1 Bu_1 \geq \lambda_1\psi$. Let $\lambda^* = \sup\{\lambda : u_1(s) \geq \lambda\psi(s),\ s \in [a,b]\}$, then $\lambda^* \in [\lambda_1, +\infty]$ and $u_1(s) \geq \lambda^*\psi$ for $s \in [a,b]$. Then if $t \in [a,b]$, we have

$$\begin{aligned}
u_1(t) &= \int_0^1 G(t,s)g(s)F(s,u_1(s))\,ds + \lambda_1\psi(t)\\
&\geq \int_a^b G(t,s)g(s)F(s,u_1(s))\,ds + \lambda_1\psi\\
&\geq \int_a^b G(t,s)g(s)\xi(u_1(s))^p\,ds + \lambda_1\psi\\
&\geq \xi(\lambda^*\psi)^k \min_{t\in[a,b]} \int_a^b G(t,s)g(s)ds + \lambda_1\psi \geq (\lambda^* + \lambda_1)\psi,
\end{aligned} \tag{3.19}$$

which contradicts the definition of λ^*, hence (3.18) holds, by Lemma 2.2

$$i(A, K \cap \partial\Omega_R, K) = 0. \tag{3.20}$$

By virtue of (H_4), there are $\mu > 0$ and $\epsilon > 0$ such that $0 \leq F(t,u) \leq \mu u^l$ for $0 \leq u \leq \epsilon$. Take

$$0 \leq r \leq \min\left\{\epsilon, R, \left(\mu \max_{t\in[0,1]} \int_0^1 G(t,s)g(s)ds\right)^{-1/(l-1)}\right\}, \tag{3.21}$$

then we now prove that

$$Au \neq \lambda u, \quad \text{for } u \in K \cap \partial\Omega_r,\ \mu \geq 1. \tag{3.22}$$

In fact, if it is not true, there exist $u_0 \in K \cap \partial\Omega_r$ and $\lambda_0 \geq 1$ such that $Au_0 = \lambda_0 u_0$. Then if $t \in [0,1]$,

$$\begin{aligned}
u_0(t) \leq \lambda_0 u_0(t) &= \int_0^1 G(t,s)g(s)F(s,u_0(s))ds\\
&\leq \int_0^1 G(t,s)g(s)\mu(u_0(s))^l\,ds\\
&\leq r^l\left(\mu \max_{t\in[0,1]} \int_0^1 G(t,s)g(s)ds\right),
\end{aligned} \tag{3.23}$$

that is,

$$r \leq \mu r^{l} \max_{t\in[0,1]} G(t,s)g(s)ds \tag{3.24}$$

which contradicts the definition of r, so (3.22) holds. According to Lemma 2.1,

$$i(A, K \cap \partial\Omega_r, K) = 1, \tag{3.25}$$

(3.20) and (3.25) together imply

$$i(A, K \cap (\Omega_R \backslash \overline{\Omega_r}), K) = i(K \cap \partial\Omega_R, K) - i(A, K \cap \partial\Omega_r, K) = 0 - 1 = -1. \tag{3.26}$$

Consequently, according to [2, Theorem 2.3.2], A has a fixed point $u^* \in K \cap (\Omega_R \backslash \Omega_r)$, so BVP(1.1) has at least one positive solution u^*. This completes the proof. □

Theorem 3.3. *Assume that* (A_1)–(A_3) *hold,* (H_1) *and* (H_3) *are satisfied. In addition if*
(H_5) *there exists* T_1, *such that*

$$T_1 > \max_{(t,u)\in[0,1]\times[\sigma T_1, T_1]} F(t,u) \max_{t\in[0,1]} \int_0^1 G(t,s)g(s)ds, \tag{3.27}$$

then BVP(1.1) has at least two positive solutions.

Proof. By the proof of Theorems 3.1 and 3.2, there exist $0 < r < T_1 < R$ such that (3.7) and (3.20) hold, respectively.

We now prove that

$$Au \neq \lambda u, \quad \text{for } u \in K \cap \partial\Omega_{T_1},\ \lambda \geq 1. \tag{3.28}$$

Otherwise, there are $u_2 \in K \cap \partial\Omega_{T_1}$ and $\lambda_2 \geq 1$ such that $Au_2 = \lambda u_2$. Then if $t \in [0,1]$,

$$\begin{aligned} u_2(t) \leq \lambda_2 u_2(t) &= \int_0^1 G(t,s)g(s)F(s,u_2(s))ds \\ &\leq \max_{(t,u)\in[0,1]\times[\sigma T_1, T_1]} F(t,u) \int_0^1 G(t,s)g(s)ds \\ &\leq \max_{(t,u)\in[0,1]\times[\sigma T_1, T_1]} F(t,u) \max_{t\in[0,1]} \int_0^1 G(t,s)g(s)ds, \end{aligned} \tag{3.29}$$

that is,

$$T_1 \leq \max_{(t,u)\in[0,1]\times[\sigma T_1, T_1]} F(t,u) \max_{t\in[0,1]} \int_0^1 G(t,s)g(s)ds, \tag{3.30}$$

which contradicts with (H_5), hence (3.28) holds. According to Lemma 2.1,

$$i(A, K \cap \partial\Omega_{T_1}, K) = 1, \tag{3.31}$$

(3.7), (3.20), and (3.31) together imply

$$\begin{aligned} i(A, K \cap (\Omega_R \backslash \overline{\Omega}_{T_1}), K) &= i(K \cap \partial\Omega_R, K) - i(A, K \cap \partial\Omega_{T_1}, K) = 0 - 1 = -1, \\ i(A, K \cap (\Omega_{T_1} \backslash \overline{\Omega}_r), K) &= i(K \cap \partial\Omega_{T_1}, K) - i(A, K \cap \partial\Omega_r, K) = 1 - 0 = 1. \end{aligned} \tag{3.32}$$

Then according to [2, Theorem 2.3.2], A has two fixed points $u_1^* \in K \cap (\Omega_R \backslash \Omega_{T_1})$ and $u_2^* \in K \cap (\Omega_{T_1} \backslash \Omega_r)$, so BVP(1.1) has at least two positive solutions u_1^*, u_2^*. This completes the proof. □

Theorem 3.4. *Assume that (A_1)–(A_3) hold, (H_2) and (H_4) are satisfied. In addition if*
(H_6) *there exists T_2, such that*

$$0 < T_2 < \min_{(t,u) \in [a,b] \times [\sigma T_2, T_2]} F(t,u) \min_{t \in [a,b]} \int_a^b G(t,s) g(s) ds, \tag{3.33}$$

then BVP(1.1) has at least two positive solutions.

Proof. By the proof of Theorems 3.1 and 3.2, there exist $0 < r < T_2 < R$ such that (3.14) and (3.25) hold, respectively.

Define $B : C[0,1] \to C[0,1]$ by

$$Bu = u, \quad \forall u \in C[0,1]. \tag{3.34}$$

Then it is easy to verify $B : K \cap \partial\Omega \to K$ is completely continuous and $\inf_{K \cap \partial\Omega_r} \|Bu\| > 0$.

We now prove that

$$u - Au \neq \lambda Bu, \quad \text{for } u \in K \cap \partial\Omega_{T_2},\ \lambda \geq 0. \tag{3.35}$$

Otherwise, there are $u_3 \in K \cap \partial\Omega_{T_2}$ and $\lambda_3 \geq 1$ such that $u_3 - Au_3 = \lambda_3 Bu_3$. Then if $t \in [a,b]$,

$$\begin{aligned} u_3(t) &= Au_3(t) + \lambda_3 Bu_3 \\ &\geq \int_a^b G(t,s) g(s) F(s, u_3(s)) ds \\ &\geq \min_{(t,u) \in [a,b] \times [\sigma T_2, T_2]} F(t,u) \int_a^b G(t,s) g(s) ds \\ &\geq \min_{(t,u) \in [a,b] \times [\sigma T_2, T_2]} F(t,u) \min_{t \in [a,b]} \int_a^b G(t,s) g(s) ds, \end{aligned} \tag{3.36}$$

that is,

$$T_2 > \min_{(t,u) \in [a,b] \times [\sigma T_2, T_2]} F(t,u) \min_{t \in [a,b]} \int_a^b G(t,s) g(s) ds, \tag{3.37}$$

which contradicts with (H_6), hence (3.35) holds. According to Lemma 2.2,

$$i(A, K \cap \partial\Omega_{T_2}, K) = 0, \tag{3.38}$$

(3.14), (3.25), and (3.38) together imply

$$\begin{aligned} i(A, K \cap (\Omega_R \backslash \overline{\Omega_{T_2}}), K) &= i(K \cap \partial\Omega_R, K) - i(A, K \cap \partial\Omega_{T_1}, K) = 1 - 0 = 1, \\ i(A, K \cap (\Omega_{T_2} \backslash \overline{\Omega_r}), K) &= i(K \cap \partial\Omega_2, K) - i(A, K \cap \partial\Omega_r, K) = 0 - 1 = -1. \end{aligned} \tag{3.39}$$

Then according to [2, Theorem 2.3.2], A has two fixed points $u_1^* \in K \cap (\Omega_R \backslash \Omega_{T_2})$ and $u_2^* \in K \cap (\Omega_{T_2} \backslash \Omega_r)$, so BVP(1.1) has at least two positive solutions u_1^*, u_2^*. This completes the proof. □

Theorem 3.5. *Assume that (A_1)–(A_3) hold, $T_1 < T_2$, (H_1), (H_2), (H_5), and (H_6) are satisfied. Then BVP(1.1) admits at least three positive solutions.*

Proof. By the proof of Theorem 3.1, there exist $0 < r < T_1$ and $R > T_2$ such that (3.7) and (3.14) hold, respectively. By Theorems 3.3 and 3.4, (3.31) and (3.38) are valid. Therefore BVP(1.1) has at least three positive solutions $u_1^* \in K \cap (\Omega_R \backslash \overline{\Omega}_{T_2})$, $u_2^* \in K \cap (\Omega_{T_2} \backslash \overline{\Omega}_{T_1})$, $u_3^* \in K \cap (\Omega_{T_1} \backslash \overline{\Omega}_r)$. This completes the proof. □

Similar to the proof of Theorem 3.5, we can get the following theorem.

Theorem 3.6. *Assume that (A_1)–(A_3) hold, $T_1 > T_2$, (H_3), (H_4), (H_5), and (H_6) are satisfied. Then BVP(1.1) admits at least three positive solutions.*

Remark 3.7. In fact, if $T_1 = T_2 = T$, the conditions of Theorems 3.5 and 3.6 both cannot ensure that BVP(1.1) has at least three positive solutions or even one positive solution. The reason is that the conditions (H_5) and (H_6) imply that the inequalities

$$\begin{aligned} Au &\ngeq u, \quad \text{for } u \in K \cap \Omega_T, \\ Au &\nleq u, \quad \text{for } u \in K \cap \Omega_T, \end{aligned} \tag{3.40}$$

hold, respectively, however, the latter two contradict each other.

Remark 3.8. In this paper, if $p(t) = 1$, $F(t,u) = f(u)$, all the theorems above still hold and the results are new. Here the function $f(u)$ and the boundary conditions are more general than in [1, 6, 7] where $f(u)$ only satisfies $\lim_{u\to 0+} f(u)/u = 0$ (or ∞), $\lim_{u\to\infty} f(u)/u = \infty$ (or 0) and only the cases $\beta = 0$, $\delta = 0$ are considered. In addition, our method is different from those methods in [1, 6, 7].

Remark 3.9. In the proof of theorems, one of the key steps is to find the operator B. We note that it is more general than the ones in [6, 8–11]. We think not only about the superlinear, sublinear cases but also the general cases. Hence, our results improve and generalize those in some well-known papers.

4. Examples

In this section, we provide some examples to illustrate the validity of the results established in Section 2.

Example 4.1. Consider the following boundary value problem:

$$u'' + \frac{1}{\sqrt{t(1-t)}}\left[\frac{1}{4}(1+t)u^{3/2}(t) + u^{1/2}(t)\right], \quad t \in (0,1), \qquad u(0) = u(1) = 0. \tag{4.1}$$

Conclusion. BVP (4.1) has at least two positive solutions.

Proof. Let

$$p(t) = 1, \qquad g(t) = \frac{1}{\sqrt{t(1-t)}}, \qquad F(t,u) = \frac{1}{4}(1+t)u^{3/2}(t) + u^{1/2}(t), \tag{4.2}$$

and we choose $[a,b] = [1/4, 3/4] \subset (0,1)$.

It is easy to verify that the conditions (A_1)–(A_3) of Theorem 3.3 are satisfied. In term of (2.1) and (2.2), the corresponding Green function is

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases} \qquad \sigma = \frac{1}{4}, \qquad G(s,s) = s(1-s), \qquad \int_0^1 \frac{dt}{\sqrt{t(1-t)}} = \pi. \tag{4.3}$$

We first verify the conditions (H_1) and (H_3). In fact let $p = 1/2$, $k = 3/2$, we have

$$\lim_{u\to 0^+}\min_{t\in[0,1]} \frac{F(t,u)}{u^p} = \lim_{u\to 0^+}\min_{t\in[0,1]} \frac{(1/4)(1+t)u^{3/2}(t) + u(t)}{u^{1/2}} = \frac{1}{4}, \qquad \lim_{u\to +\infty}\min_{t\in[0,1]} \frac{F(t,u)}{u^k} = \lim_{u\to +\infty}\min_{t\in[0,1]} \frac{(1/4)(1+t)u^{3/2}(t) + u^{1/2}(t)}{u^{3/2}} = 1. \tag{4.4}$$

Choosing $T_1 = 4$, we have

$$\max_{(t,u)\in[0,1]\times[1,4]} F(t,u) \max_{t\in[0,1]} \int_0^1 G(s,s)g(s)ds = \max_{(t,u)\in[0,1]\times[1,4]} \left[\frac{1}{4}(1+t)u^{3/2}(t) + u^{1/2}(t)\right]\frac{1}{8}\pi = 6\cdot\frac{1}{8}\pi = \frac{3}{4}\pi < 4. \tag{4.5}$$

So condition (H_5) holds. Consequently by Theorem 3.3, BVP (4.1) has at least two positive solutions. □

Acknowledgments

The first and second authors were supported financially by the National Natural Science Foundation of China (10471075) and the Natural Science Foundation of Shandong Province of China (Y2003A01).

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