

A Note on the Global Attractivity of a Discrete Model of Nicholson's Blowflies*

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In this paper, we further study the global attractivity of the positive equilibrium of the discrete Nicholson's blowflies model

$$N_{n+1} - N_n = -\delta N_n + pN_{n-k}e^{-aN_{n-k}}, \quad n = 0, 1, 2, \dots$$

We obtain a new criterion for the positive equilibrium N^* to be a global attractor, which improve the corresponding results obtained by So and Yu (*J. Math. Anal. Appl.* 193 (1995), 233–244).

Keywords: Attractivity, Positive equilibrium, Discrete Nicholson's blowflies model

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I. INTRODUCTION

The delay difference equation

$$\begin{aligned} N_{n+1} - N_n &= -\delta N_n + pN_{n-k}e^{-aN_{n-k}}, \\ n &= 0, 1, 2, \dots, \end{aligned} \quad (1)$$

is a discrete analogue of the delay differential equation

$$N'(t) = -\delta N(t) + pN(t-\tau)e^{-aN(t-\tau)}, \quad t \geq 0,$$

which has been used in describing the dynamics of Nicholson's blowflies [2,4–6].

By the biology consideration, we assume that $\delta \in (0, 1)$, $p, a \in (0, +\infty)$, and $k \in N = \{0, 1, 2, \dots\}$. The initial condition is

$$N_j = \varphi_j \geq 0, \quad j \in \{-k, -k+1, \dots, 0\}, \quad (2)$$

and $\varphi_j > 0$, for some $j \in \{-k, -k+1, \dots, 0\}$.

By a solution of (1) and (2) we mean a sequence $\{N_n\}$ which satisfies (1) for $n = 0, 1, 2, \dots$ as well as the initial condition (2). Clearly, the unique solution $\{N_n\}$ of the above initial value problem is positive for all large n [1].

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If $p > \delta$, then Eq. (1) has a unique positive equilibrium N^* and

$$N^* = \frac{1}{a} \ln\left(\frac{p}{\delta}\right). \quad (3)$$

The global attractivity of N^* was studied by Kocic and Lada [3] and So and Yu [1] respectively.

The recent result is the following [1].

THEOREM A Assume that $p > \delta$ and that

$$[(1 - \delta)^{-k-1} - 1] \ln\left(\frac{p}{\delta}\right) \leq 1. \quad (4)$$

Then any nontrivial solution N_n of (1) and (2) satisfies

$$\lim_{n \rightarrow \infty} N_n = N^*.$$

In this note, our purpose is to improve condition (4). Exactly speaking, we will show some conditions for the global attractivity of N^* when (4) does not hold. Our results are discrete analogues of the results in [2].

To prove our main results, we need some known results.

LEMMA 1 [1] Let $\{N_n\}$ be a solution of (1) and (2). Then

$$\limsup_{n \rightarrow \infty} N_n \leq \frac{p}{ae\delta}. \quad (5)$$

As in [2], the following system of inequalities

$$\begin{cases} y + \ln(1 + (y/aN^*)) \leq M(e^{-x} - 1), \\ x + \ln(1 + (x/aN^*)) \geq M(e^{-y} - 1), \end{cases} \quad (6)$$

play an important role in our analysis, where $M = aN^*[(1 - \delta)^{-k-1} - 1] = [(1 - \delta)^{-k-1} - 1] \ln(p/\delta)$.

Let

$$D = \{(x, y): -aN^* < x \leq 0 \leq y < \infty\}. \quad (7)$$

LEMMA 2 [2] If one of the following conditions holds:

- (i) $M \leq 1$;
- (ii) $M < 1 + (1/aN^*)$ and $aN^* \geq (\sqrt{5} - 1)/2$;

$$(iii) \quad M \leq 1 + (1/aN^*) \text{ and } aN^* > (\sqrt{1 + 4\sqrt{3}} - 1)/2,$$

then (6) has a unique solution $x = y = 0$ in D .

II. MAIN RESULTS

The following theorem provides a new sufficient condition for the equilibrium $N^* = (1/a) \ln(p/\delta)$ to be a global attractor.

THEOREM 1 Assume that $p > \delta$ and the assumption in Lemma 2 holds. Then any nontrivial solution $\{N_n\}$ of (1) and (2) satisfies

$$\lim_{n \rightarrow \infty} N_n = N^*.$$

Proof Let

$$N_n = N^* + \frac{1}{a} x_n.$$

Then $\{x_n\}$ is a solution of the equation

$$\begin{aligned} x_{n+1} - x_n + \delta x_n + a\delta N^*(1 - e^{-x_{n-k}}) \\ - \delta x_{n-k} e^{-x_{n-k}} = 0, \quad n = 0, 1, 2, \dots \end{aligned} \quad (8)$$

Since $N_n > 0$ for all large n , it follows that $x_n > -aN^*$ for all large n .

To prove this theorem, it is sufficient to prove $\lim_{n \rightarrow \infty} x_n = 0$. Lemma 1 implies that $\{x_n\}$ is bounded above. Let

$$\mu = \limsup_{n \rightarrow \infty} x_n \quad \text{and} \quad \lambda = \liminf_{n \rightarrow \infty} x_n. \quad (9)$$

Then $-aN^* \leq \lambda \leq \mu < \infty$. We claim that $\lambda = \mu = 0$. For the case $\{x_n\}$ is eventually nonnegative or eventually nonpositive, this has been proved in the proof of Theorem 2 in [3]. Therefore it is sufficient to consider the case that $\{x_n\}$ is an oscillatory solution of (8).

Our purpose is to prove that $\lambda = \mu = 0$ under the assumptions. There are four possible cases:

- (1) $\lambda = \mu = 0$;
- (2) $\mu > 0$ and $\lambda = 0$;

- (3) $\mu = 0$ and $\lambda < 0$;
 (4) $\mu > 0$ and $\lambda < 0$.

The cases 2 and 3 can be considered to be special cases of case 4. Now we consider case 4.

In this case, there exists a sequence $\{n_i\}$ of positive integers such that

$$k < n_1 < n_2 < \dots < n_i < n_{i+1} \rightarrow \infty \quad \text{as } i \rightarrow \infty.$$

$$x_{n_i} < 0 \quad \text{and} \quad x_{n_{i+1}} \geq 0, \quad \text{for } i = 1, 2, \dots,$$

and for each $i = 1, 2, \dots$, the terms of the finite sequence x_j for $n_i < j < n_{i+1}$ assume both positive and negative values. Let m_i and M_i be integers in (n_i, n_{i+1}) such that for $i = 1, 2, \dots$

$$x_{M_i} = \max\{x_j : n_i < j < n_{i+1}\},$$

and

$$x_{m_i} = \min\{x_j : n_i < j < n_{i+1}\}.$$

We can assume without loss of generality that for $i = 1, 2, \dots$

$$\begin{aligned} x_{M_i} &> 0, \quad x_{M_i} - x_{M_{i-1}} \geq 0 \quad \text{and} \\ \limsup_{i \rightarrow \infty} x_{M_i} &= \mu > 0, \end{aligned}$$

while

$$\begin{aligned} x_{m_i} &< 0, \quad x_{m_i} - x_{m_{i-1}} \leq 0 \quad \text{and} \\ \liminf_{i \rightarrow \infty} x_{m_i} &= \lambda < 0. \end{aligned}$$

Then there exist subsequence $\{q_i\}$ of $\{m_i\}$ and subsequence $\{Q_i\}$ of $\{M_i\}$ such that

$$\begin{aligned} x_{Q_i} &> 0, \quad x_{Q_i} - x_{Q_{i-1}} \geq 0 \quad \text{and} \\ \lim_{i \rightarrow \infty} x_{Q_i} &= \mu > 0, \end{aligned} \tag{10}$$

while

$$\begin{aligned} x_{q_i} &< 0, \quad x_{q_i} - x_{q_{i-1}} \leq 0 \quad \text{and} \\ \lim_{i \rightarrow \infty} x_{q_i} &= \lambda < 0. \end{aligned} \tag{11}$$

It follows from (8) and (10) that

$$x_{Q_{i-1}} + aN^* \leq [x_{Q_{i-k-1}} + aN^*]e^{-x_{Q_{i-k-1}}},$$

thus

$$\begin{aligned} x_{Q_i} + aN^* &= (1 - \delta)(x_{Q_{i-1}} + aN^*) \\ &\quad + \delta(x_{Q_{i-k-1}} + aN^*)e^{-x_{Q_{i-k-1}}} \\ &\leq (1 - \delta)(x_{Q_{i-k-1}} + aN^*)e^{-x_{Q_{i-k-1}}} \\ &\quad + \delta(x_{Q_{i-k-1}} + aN^*)e^{-x_{Q_{i-k-1}}} \\ &= (x_{Q_{i-k-1}} + aN^*)e^{-x_{Q_{i-k-1}}} \end{aligned}$$

that is

$$x_{Q_i} + aN^* \leq (x_{Q_{i-k-1}} + aN^*)e^{-x_{Q_{i-k-1}}}. \tag{12}$$

Now let us prove

$$x_{Q_{i-k-1}} < 0, \tag{13}$$

assume the contrary, then $x_{Q_{i-k-1}} = 0$ or $x_{Q_{i-k-1}} > 0$. If $x_{Q_{i-k-1}} = 0$, then $x_{Q_i} \leq 0$, which contradicts (10). If $x_{Q_{i-k-1}} > 0$, then $x_{Q_{i-k-1}} > x_{Q_i}$, thus

$$\liminf_{i \rightarrow \infty} x_{Q_{i-k-1}} \geq \liminf_{i \rightarrow \infty} x_{Q_i} = \mu,$$

on the other hand, we have

$$\limsup_{i \rightarrow \infty} x_{Q_{i-k-1}} \leq \limsup_{i \rightarrow \infty} x_{M_i} = \mu,$$

so we get

$$\lim_{i \rightarrow \infty} x_{Q_{i-k-1}} = \mu, \tag{14}$$

then taking the limit in (12), we obtain

$$\mu + aN^* \leq (\mu + aN^*)e^{-\mu},$$

which implies $\mu \leq 0$ that contradicts (10), so (13) holds.

From (12) and (13), we have

$$x_{Q_i} + aN^* < aN^*e^{-x_{Q_{i-k-1}}},$$

therefore

$$x_{Q_i-k-1} < -\ln\left(1 + \frac{x_{Q_i}}{aN^*}\right). \quad (15)$$

For given $\varepsilon > 0$, by (9), there exists a positive integer n^* such that

$$\lambda - \varepsilon < x_n < \mu + \varepsilon, \quad \text{for } n \geq n^* - k,$$

this induce $x_{n-k}e^{-x_{n-k}} < \mu + \varepsilon$, for $n \geq n^*$.

Rewriting Eq. (8) into the following form:

$$\begin{aligned} & (1-\delta)^{-n-1}x_{n+1} - (1-\delta)^{-n}x_n \\ & + a\delta N^*(1-\delta)^{-n-1}(1-e^{-x_{n-k}}) \\ & - \delta(1-\delta)^{-n-1}x_{n-k}e^{-x_{n-k}} = 0. \end{aligned} \quad (16)$$

Now summing (16) up from $n = Q_i - k - 1$ (assuming $Q_i - k - 1 \geq n^*$) to $n = Q_i - 1$. we have

$$\begin{aligned} (1-\delta)^{-Q_i}x_{Q_i} &= (1-\delta)^{-Q_i+k+1}x_{Q_i-k-1} - a\delta N^* \\ & \times \sum_{n=Q_i-k-1}^{Q_i-1} (1-\delta)^{-n-1}(1-e^{-x_{n-k}}) \\ & + \delta \sum_{n=Q_i-k-1}^{Q_i-1} (1-\delta)^{-n-1}x_{n-k}e^{-x_{n-k}} \\ & < (1-\delta)^{-Q_i+k+1}x_{Q_i-k-1} + a\delta N^* \\ & \times \sum_{n=Q_i-k-1}^{Q_i-1} (1-\delta)^{-n-1}(e^{-\lambda+\varepsilon} - 1) \\ & + \delta \sum_{n=Q_i-k-1}^{Q_i-1} (1-\delta)^{-n-1}(\mu + \varepsilon) \\ & = (1-\delta)^{-Q_i+k+1}x_{Q_i-k-1} \\ & + [(\mu + \varepsilon) + aN^*(e^{-\lambda+\varepsilon} - 1)] \\ & \times (1-\delta)^{-Q_i}[1 - (1-\delta)^{k+1}]. \end{aligned}$$

Substituting (15) into the above inequality, we get

$$\begin{aligned} (1-\delta)^{-Q_i}x_{Q_i} &< -(1-\delta)^{-Q_i+k+1}\ln\left(1 + \frac{x_{Q_i}}{aN^*}\right) \\ & + [(\mu + \varepsilon) + aN^*(e^{-\lambda+\varepsilon} - 1)] \\ & \times (1-\delta)^{-Q_i}[1 - (1-\delta)^{k+1}], \end{aligned}$$

and

$$\begin{aligned} & x_{Q_i} + (1-\delta)^{k+1}\ln\left(1 + \frac{x_{Q_i}}{aN^*}\right) \\ & < [(\mu + \varepsilon) + aN^*(e^{-\lambda+\varepsilon} - 1)][1 - (1-\delta)^{k+1}], \end{aligned}$$

let $i \rightarrow \infty$, $\varepsilon \rightarrow 0$, we get

$$\begin{aligned} & \mu + (1-\delta)^{k+1}\ln\left(1 + \frac{\mu}{aN^*}\right) \\ & \leq [\mu + aN^*(e^{-\lambda} - 1)][1 - (1-\delta)^{k+1}]. \end{aligned}$$

We rewrite the above inequality:

$$\mu + \ln\left(1 + \frac{\mu}{aN^*}\right) \leq M(e^{-\lambda} - 1). \quad (17)$$

In a similar way, we have

$$\lambda + \ln\left(1 + \frac{\lambda}{aN^*}\right) \geq M(e^{-\mu} - 1). \quad (18)$$

Then we establish the following system of inequalities:

$$\begin{cases} \mu + \ln(1 + (\mu/aN^*)) \leq M(e^{-\lambda} - 1), \\ \lambda + \ln(1 + (\lambda/aN^*)) \geq M(e^{-\mu} - 1). \end{cases} \quad (19)$$

For case 2, the system of inequalities corresponding to (19) is

$$\begin{cases} \mu + \ln(1 + (\mu/aN^*)) \leq M(e^{-\lambda} - 1), \\ \lambda = 0. \end{cases} \quad (20)$$

It is obvious that (20) holds iff $\lambda = \mu = 0$.

For case 3, the system of inequalities corresponding to (19) is

$$\begin{cases} \mu = 0, \\ \lambda + \ln(1 + (\lambda/aN^*)) \geq M(e^{-\mu} - 1). \end{cases} \quad (21)$$

Similarly, (21) holds iff $\lambda = \mu = 0$.

Thus it will suffice to consider case 4, for (19) in case 4, by Lemma 2, we get $\lambda = \mu = 0$. So the proof is complete.

Remark 1 In cases 2 and 3 in Theorem 1, we add some reasonable conditions to aN^* . We know

$$M = aN^*[(1 - \delta)^{-k-1} - 1] \leq 1 + \frac{1}{aN^*},$$

on the right side of which there is nothing to do with δ and k . While $1 + (1/aN^*) \rightarrow \infty$ as $aN^* \rightarrow 0+$, properly choosing the values of $[(1 - \delta)^{-k-1} - 1]$, we can let M equal or infinitely tend to the value of $1 + (1/aN^*)$, then M can be changed to arbitrarily large. Obviously this is not reasonable.

Remark 2 Theorem 4.1 in [1] only applies to the case $M \leq 1$, while Theorem 1 in this paper not only applies to $M \leq 1$ but also to $M > 1$. So the results in this paper improve those in [1].

Example Consider the delay difference equation

$$N_{n+1} - N_n = -\frac{1}{4}N_n + \frac{1}{4}e^{(\sqrt{5}-1)/2}N_{n-3}e^{-2N_{n-3}}, \quad (22)$$

then we can calculate

$$aN^* = \frac{\sqrt{5}-1}{2} \quad \text{and} \quad [(1 - \delta)^{-k-1} - 1] = \frac{175}{81},$$

thus,

$$M \approx 1.335 \quad \text{and} \quad 1 + \frac{1}{aN^*} = \frac{\sqrt{5}+3}{2} \approx 2.618.$$

The conditions in Theorem 1 are satisfied. Thus

$$N^* = \frac{\sqrt{5}-1}{4}$$

is a global attractor or (22). But Theorem 4.1 in [1] cannot apply to this case.

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