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Research Article

Trench's Perturbation Theorem for Dynamic Equations

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We consider a nonoscillatory second-order linear dynamic equation on a time scale together with a linear perturbation of this equation and give conditions on the perturbation that guarantee that the perturbed equation is also nonoscillatory and has solutions that behave asymptotically like a recessive and dominant solutions of the unperturbed equation. As the theory of time scales unifies continuous and discrete analysis, our results contain as special cases results for corresponding differential and difference equations by William F. Trench.

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1. Introduction

We consider the second-order linear dynamic equation

$$\left(r(t)x^{\Delta}\right)^{\Delta} + p(t)x^{\sigma} = 0, \quad t \in \mathbb{T}, \tag{1.1}$$

together with its linear perturbation

$$\left(r(t)y^{\Delta}\right)^{\Delta} + p(t)y^{\sigma} = f(t)y^{\sigma}, \quad t \in \mathbb{T}, \tag{1.2}$$

where we assume that $\mathbb{T}$ is a time scale, that is, a nonempty closed subset of the real numbers $\mathbb{T}$ that is unbounded above, $p : \mathbb{T} \to \mathbb{R}$, $r : \mathbb{T} \to \mathbb{R}^{+}$, $f : \mathbb{T} \to \mathbb{C}$, are rd-continuous, and (1.1) is nonoscillatory, that is, $rxx^{\sigma} > 0$ eventually for all solutions of (1.1). For the theory of time scales, we refer the reader to [1, 2] and we mention here only that the forward shift and the derivative of a function $z : \mathbb{T} \to \mathbb{R}$ are given by $z^{\sigma}(t) = z(t)$ and $z^{\Delta}(t) = z'(t)$ if $\mathbb{T} = \mathbb{R}$ and $z^{\sigma}(t) = z(t+1)$ and $z^{\Delta}(t) = z(t+1) - z(t)$ if $\mathbb{T} = \mathbb{Z}$, and that the product rule

and the quotient rule for differentiable $z_1, z_2 : \mathbb{T} \to \mathbb{R}$ read [1, Theorem 1.20]

$$(z_1 z_2)^{\Delta} = z_1^{\Delta} z_2 + z_1^{\sigma} z_2^{\Delta}, \qquad \left(\frac{z_1}{z_2}\right)^{\Delta} = \frac{z_1^{\Delta} z_2 - z_1 z_2^{\Delta}}{z_2 z_2^{\sigma}}, \tag{1.3}$$

where $z^{\sigma} = z \circ \sigma$ for a function $z : \mathbb{T} \to \mathbb{R}$. By [1, Theorem 4.61], since (1.1) is nonoscillatory, there exists a solution x_1 of (1.1) such that $\lim_{t\to\infty}(u(t)/x_1(t)) = \infty$ for all of x_1 linearly independent solutions u of (1.1).

Let u be any solution of (1.1) that is linearly independent of x_1, that is,

$$r\left(u^{\Delta} x_1 - u x_1^{\Delta}\right) \equiv c \neq 0, \tag{1.4}$$

and define $x_2 = u/c$ so that

$$r\left(x_2^{\Delta} x_1 - x_2 x_1^{\Delta}\right) = 1, \qquad \lim_{t\to\infty} \frac{x_2(t)}{x_1(t)} = \infty, \tag{1.5}$$

which implies that

$$\varphi := \frac{x_2}{x_1} \quad \text{satisfies } \varphi^{\Delta} = \frac{1}{r x_1 x_1^{\sigma}} > 0, \qquad \lim_{t\to\infty} \varphi(t) = \infty. \tag{1.6}$$

Note that x_1 and x_2 are called *recessive* and *dominant* solutions of the nonoscillatory dynamic equation (1.1).

The main result of this paper gives conditions on f, x_1, and x_2 that guarantee that the perturbed equation (1.2) is also nonoscillatory and has solutions y_1 and y_2 that behave asymptotically like x_1 and x_2. Our results unify corresponding results by William F. Trench for differential equations [3, 4] and for difference equations [5] and extend them to other dynamic equations, for example, to q-difference equations [6]. Note that the unification process forces us to give many of the calculations from [5] in a "shifted form" for the discrete case.

In the next section, we will derive some preparatory results while the main theorem is stated and proved in Section 3. The paper concludes with some corollaries given in Section 4.

2. Some auxiliary results

Suppose now and in the remainder of this paper that

$$\psi(t) := \int_t^{\infty} f(\tau) x_1(\sigma(\tau)) x_2(\sigma(\tau)) \Delta\tau \quad \text{converges for all } t \in \mathbb{T}. \tag{2.1}$$

Then

$$\phi(t) := \sup_{\tau \geq t} \left|\psi(\tau)\right| \longrightarrow 0 \quad \text{monotonically as } t \longrightarrow \infty. \tag{2.2}$$

Lemma 2.1. *Using notation (2.2) and condition (2.1),*

$$G(t) := \int_t^\infty f(\tau)x_1^2(\sigma(\tau))\Delta\tau \quad \text{is well defined for all } t \in \mathbb{T} \tag{2.3}$$

and satisfies

$$|G| \le \frac{2\phi}{\varphi^\sigma} \quad \text{on } \mathbb{T}. \tag{2.4}$$

Proof. We use (1.6), the product rule, the definition of the integral, and [1, Theorems 1.75 and 1.16(iv)] to find

$$\begin{aligned} G(t) &= \int_t^\infty f(\tau)x_1(\sigma(\tau))x_2(\sigma(\tau))\frac{1}{\varphi(\sigma(\tau))}\Delta\tau = -\int_t^\infty \psi^\Delta(\tau)\frac{1}{\varphi(\sigma(\tau))}\Delta\tau \\ &= -\int_t^\infty \left[\left(\psi\cdot\frac{1}{\varphi}\right)^\Delta(\tau) - \psi(\tau)\left(\frac{1}{\varphi}\right)^\Delta(\tau)\right]\Delta\tau = \frac{\psi(t)}{\varphi(t)} + \int_t^\infty \psi(\tau)\left(\frac{1}{\varphi}\right)^\Delta(\tau)\Delta\tau \\ &= \frac{\psi(t)}{\varphi(t)} + \int_{\sigma(t)}^\infty \psi(\tau)\left(\frac{1}{\varphi}\right)^\Delta(\tau)\Delta\tau + \psi(t)\left[\left(\frac{1}{\varphi}\right)(\sigma(t)) - \left(\frac{1}{\varphi}\right)(t)\right] \\ &= \frac{\psi(t)}{\varphi^\sigma(t)} + \int_{\sigma(t)}^\infty \psi(\tau)\left(\frac{1}{\varphi}\right)^\Delta(\tau)\Delta\tau, \end{aligned} \tag{2.5}$$

and therefore

$$\begin{aligned} |G(t)| &\le \frac{\phi(t)}{\varphi^\sigma(t)} + \phi(t)\int_{\sigma(t)}^\infty \left|\left(\frac{1}{\varphi}\right)^\Delta(\tau)\right|\Delta\tau \\ &= \frac{\phi(t)}{\varphi^\sigma(t)} - \phi(t)\int_{\sigma(t)}^\infty \left(\frac{1}{\varphi}\right)^\Delta(\tau)\Delta\tau = \frac{\phi(t)}{\varphi^\sigma(t)} + \frac{\phi(t)}{\varphi^\sigma(t)}, \end{aligned} \tag{2.6}$$

where we have used (1.6), (2.2), and its consequence $\phi^\sigma \le \phi$. □

In the sequel, we use the Landau "O" symbol defined in the standard way for asymptotic behavior as $t\to\infty$ and consider the collection of differentiable functions

$$\mathcal{B} := \left\{z : \mathbb{T} \longrightarrow \mathbb{R} : z = \mathrm{O}(\phi),\ z^\Delta = \mathrm{O}\left(\frac{\phi\varphi^\Delta}{\varphi^\sigma}\right)\right\}, \tag{2.7}$$

which can easily be seen to be a Banach space when equipped with the norm

$$\|z\| = \|z\|_{\mathcal{B}} := \left\|\max\left\{\frac{|z|}{\phi}, \frac{|z^\Delta|\varphi^\sigma}{\phi\varphi^\Delta}\right\}\right\|_\infty \quad \text{for } z \in \mathcal{B}. \tag{2.8}$$

Lemma 2.2. *Using notations (2.2), (2.3), (2.7), (2.8) and under condition (2.1),*

$$u := \psi - \varphi G \quad \text{satisfies } u \in \mathcal{B}, \quad \|u\|_{\mathcal{B}} \le 2. \tag{2.9}$$

Proof. Note that the fourth equal sign of the first calculation in the proof of Lemma 2.1 implies

$$u(t) = -\varphi(t)\int_t^\infty \psi(\tau)\left(\frac{1}{\varphi}\right)^\Delta(\tau)\Delta\tau, \tag{2.10}$$

and therefore

$$|u(t)| \le \varphi(t)\phi(t)\int_t^\infty \left|\left(\frac{1}{\varphi}\right)^\Delta(\tau)\right|\Delta\tau = -\varphi(t)\phi(t)\int_t^\infty \left(\frac{1}{\varphi}\right)^\Delta(\tau)\Delta\tau = \phi(t) \tag{2.11}$$

due to (2.2) and (1.6). By the product rule and (1.6),

$$\begin{aligned} u^\Delta(t) &= \psi^\Delta(t) - \varphi^\Delta(t)G(t) - \varphi(\sigma(t))G^\Delta(t) \\ &= -f(t)x_1(\sigma(t))x_2(\sigma(t)) - \varphi^\Delta(t)G(t) + \varphi(\sigma(t))f(t)x_1^2(\sigma(t)) = -\varphi^\Delta(t)G(t), \end{aligned} \tag{2.12}$$

and therefore

$$|u^\Delta(t)| = \varphi^\Delta(t)|G(t)| \le \frac{2\varphi^\Delta(t)\phi(t)}{\varphi^\sigma(t)} \tag{2.13}$$

due to (2.4) from Lemma 2.1. □

For $z \in \mathcal{B}$, let us define the operator

$$(\mathcal{L}z)(t) := \int_t^\infty [x_2(\sigma(\tau)) - x_1(\sigma(\tau))\varphi(t)]f(\tau)x_1(\sigma(\tau))z(\sigma(\tau))\Delta\tau. \tag{2.14}$$

Lemma 2.3. *Using notations (2.2), (2.3), (2.7), (2.8), (2.14), and under the conditions (2.1) and*

$$\widetilde{\phi}(t) := \int_t^\infty |G(\tau)|\phi(\tau)\varphi^\Delta(\tau)\Delta\tau < \infty, \qquad \nu := \limsup_{t\to\infty}\frac{\widetilde{\phi}(t)}{\phi(t)} < \frac{1}{2}, \tag{2.15}$$

we have that, on $[T,\infty)$ *for sufficiently large* $T \in \mathbb{T}$,

$$\mathcal{L} : \mathcal{B} \longrightarrow \mathcal{B} \text{ is a contraction.} \tag{2.16}$$

Proof. Let $z \in \mathcal{B}$. For $T \ge t$, consider

$$w(z,T)(t) := \int_t^T [x_2(\sigma(\tau)) - x_1(\sigma(\tau))\varphi(t)]f(\tau)x_1(\sigma(\tau))z(\sigma(\tau))\Delta\tau. \tag{2.17}$$

Then by (1.6), (2.3), and the product rule,

$$\begin{aligned} w(z,T)(t) &= \int_t^T [\varphi(\sigma(\tau)) - \varphi(t)] x_1(\sigma(\tau)) f(\tau) x_1(\sigma(\tau)) z(\sigma(\tau)) \Delta\tau \\ &= -\int_t^T [\varphi(\sigma(\tau)) - \varphi(t)] z(\sigma(\tau)) G^{\Delta}(\tau) \Delta\tau \\ &= -\int_t^T \{[(\varphi - \varphi(t)) z G]^{\Delta}(\tau) - [(\varphi - \varphi(t)) z]^{\Delta}(\tau) G(\tau)\} \Delta\tau \\ &= -[\varphi(T) - \varphi(t)] z(T) G(T) + \int_t^T [\varphi(\sigma(\tau)) - \varphi(t)] z^{\Delta}(\tau) G(\tau) \Delta\tau \\ &\quad + \int_t^T \varphi^{\Delta}(\tau) z(\tau) G(\tau) \Delta\tau. \end{aligned} \tag{2.18}$$

We estimate each of these three terms separately

$$\begin{aligned} |[\varphi(T) - \varphi(t)] z(T) G(T)| &\le |\varphi(T) - \varphi(t)|\, |z(T)|\, \frac{2\phi(T)}{\varphi(\sigma(T))} \\ &\le 2\varphi(T)\|z\|\phi(T)\frac{\phi(T)}{\varphi(\sigma(T))} \le 2\|z\|\phi^2(T) \end{aligned} \tag{2.19}$$

due to (1.6), (2.4), and (2.8);

$$|\varphi^{\Delta}(\tau) z(\tau) G(\tau)| \le \varphi^{\Delta}(\tau)\|z\|\phi(\tau)\, |G(\tau)| \tag{2.20}$$

due to (1.6) and (2.8); and

$$\begin{aligned} |[\varphi(\sigma(\tau)) - \varphi(t)] z^{\Delta}(\tau) G(\tau)| &\le |\varphi(\sigma(\tau)) - \varphi(t)|\, \|z\| \frac{\phi(\tau)\varphi^{\Delta}(\tau)}{\varphi^{\sigma}(\tau)} |G(\tau)| \\ &\le \|z\|\phi(\tau)\varphi^{\Delta}(\tau)|G(\tau)| \quad \text{for } t \le \sigma(\tau), \end{aligned} \tag{2.21}$$

due to (1.6) and (2.8). Altogether,

$$|w(z,T)(t)| \le 2\|z\| \left\{ \phi^2(T) + \int_t^T |G(\tau)|\, \phi(\tau)\varphi^{\Delta}(\tau) \Delta\tau \right\}. \tag{2.22}$$

Thus, using (2.2) and (2.15),

$$|(\mathcal{L}z)(t)| \le 2\|z\| \widetilde{\phi}(t) \tag{2.23}$$

so that with $\theta := 2\nu < 1$,

$$\left| \frac{(\mathcal{L}z)(t)}{\phi(t)} \right| \le 2\|z\| \frac{\widetilde{\phi}(t)}{\phi(t)} \le 2\|z\|\nu = \theta\|z\|. \tag{2.24}$$

Next, from the first equal sign in the first calculation of this proof,

$$(\mathcal{L}z)(t) = \int_t^{\infty} [\varphi(\sigma(\tau)) - \varphi(t)] f(\tau) x_1^2(\sigma(\tau)) z(\sigma(\tau)) \Delta\tau, \tag{2.25}$$

so that by [1, Theorem 1.117(ii)] and the product rule,

$$(\mathcal{L}z)^{\Delta}(t) = -\int_t^{\infty} \varphi^{\Delta}(t) f(\tau) x_1^2(\sigma(\tau)) z(\sigma(\tau))\Delta\tau = \varphi^{\Delta}(t)\int_t^{\infty} G^{\Delta}(\tau) z(\sigma(\tau))\Delta\tau$$
$$= \varphi^{\Delta}(t)\int_t^{\infty} \{(zG)^{\Delta}(\tau) - z^{\Delta}(\tau)G(\tau)\}\Delta\tau = -\varphi^{\Delta}(t)\left\{z(t)G(t) + \int_t^{\infty} G(\tau) z^{\Delta}(\tau)\Delta\tau\right\}. \tag{2.26}$$

Thus, using (2.7), (2.8), (2.4), (1.6), (2.15),

$$\begin{aligned} \left|(\mathcal{L}z)^{\Delta}(t)\right| &\le \varphi^{\Delta}(t)\left\{\|z\|\phi(t)\frac{2\phi(t)}{\varphi^{\sigma}(t)} + \int_t^{\infty} \left|G(\tau)\right| \|z\| \frac{\phi(\tau)\varphi^{\Delta}(\tau)}{\varphi^{\sigma}(\tau)}\Delta\tau\right\} \\ &\le \frac{\|z\|\varphi^{\Delta}(t)}{\varphi^{\sigma}(t)}\{2\phi^2(t) + \widetilde{\phi}(t)\} \end{aligned} \tag{2.27}$$

and therefore,

$$\left|\frac{(\mathcal{L}z)^{\Delta}(t)}{\phi(t)\varphi^{\Delta}(t)/\varphi^{\sigma}(t)}\right| \le \|z\|\left\{2\phi(t) + \frac{\widetilde{\phi}(t)}{\phi(t)}\right\} \le \|z\|\{2\phi(t) + \nu\} < \theta\|z\| \tag{2.28}$$

eventually due to (2.2) and (2.15). Hence, $\mathcal{L}z \in \mathcal{B}$ and

$$\|\mathcal{L}z\| \le \theta\|z\| \quad \forall z \in \mathcal{B} \tag{2.29}$$

on $[T, \infty)$ for sufficiently large $T \in \mathbb{T}$. □

3. The perturbation result

In this section, we use the auxiliary results from Section 2 to prove the following main theorem of this paper.

Theorem 3.1. *Let x_1 and x_2 be the recessive and dominant solutions of the unperturbed linear second-order dynamic equation (1.1) satisfying (1.6). Using notations (2.2) and (2.3) and under the conditions (2.1) and (2.15), there exist solutions y_1 and y_2 of the perturbed dynamic equation (1.2) such that*

$$\frac{y_1}{x_1} = 1 + \mathrm{O}(\phi), \qquad \left(\frac{y_1}{x_1}\right)^{\Delta} = \mathrm{O}\left(\frac{\phi\varphi^{\Delta}}{\varphi^{\sigma}}\right), \tag{3.1}$$

$$\frac{y_2}{x_2} = 1 + \mathrm{O}(\widehat{\phi}), \qquad \left(\frac{y_2}{x_2}\right)^{\Delta} = \mathrm{O}\left(\frac{\widehat{\phi}\varphi^{\Delta}}{\varphi^{\sigma}}\right), \tag{3.2}$$

where $\widehat{\phi}$ is defined by

$$\widehat{\phi}(t) := \frac{1}{\varphi(t)}\int_{t_0}^{t} \phi(\tau)\varphi^{\Delta}(\tau)\Delta\tau. \tag{3.3}$$

Proof. We use the notation (2.7), (2.8), (2.9), and (2.14) and define

$$\mathcal{T}z := u + \mathcal{L}z \quad \text{for } z \in \mathcal{B}. \tag{3.4}$$

By Lemmas 2.2 and 2.3, $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ is a contraction. By Banach's fixed point theorem, $\mathcal{T}$ has a unique fixed point $\zeta \in \mathcal{B}$. Hence, $\zeta = \mathcal{T}\zeta = u + \mathcal{L}\zeta$. Define

$$y_1 := x_1(1+\zeta). \tag{3.5}$$

We show that y_1 solves (1.2). First, from the proofs of Lemmas 2.2 and 2.3,

$$-r\zeta^{\Delta} = -r\left[u^{\Delta} + (\mathcal{L}\zeta)^{\Delta}\right] = r\varphi^{\Delta}G + r\varphi^{\Delta}(\zeta G + \widetilde{\zeta}) = \frac{G(1+\zeta)+\widetilde{\zeta}}{x_1 x_1^{\sigma}}, \tag{3.6}$$

where $\widetilde{\zeta}^{\Delta} := -G\zeta^{\Delta}$. So we have

$$y_1^{\Delta} = x_1^{\Delta}(1+\zeta) + x_1^{\sigma}\zeta^{\Delta} \tag{3.7}$$

and hence,

$$ry_1^{\Delta} = rx_1^{\Delta}(1+\zeta) - \frac{G(1+\zeta)+\widetilde{\zeta}}{x_1} \tag{3.8}$$

so that, by using (3.6),

$$\begin{aligned}
\left(ry_1^{\Delta}\right)^{\Delta} &= \left(rx_1^{\Delta}\right)^{\Delta}(1+\zeta)^{\sigma} + rx_1^{\Delta}\zeta^{\Delta} - \left(\frac{G(1+\zeta)+\widetilde{\zeta}}{x_1}\right)^{\Delta} \\
&= -py_1^{\sigma} + r\zeta^{\Delta}x_1^{\Delta} - \frac{\left(G^{\Delta}(1+\zeta)^{\sigma} + G\zeta^{\Delta} + \widetilde{\zeta}^{\Delta}\right)x_1 - \left[G(1+\zeta)+\widetilde{\zeta}\right]x_1^{\Delta}}{x_1 x_1^{\sigma}} \\
&= -py_1^{\sigma} + r\zeta^{\Delta}x_1^{\Delta} + fy_1^{\sigma} + \frac{G(1+\zeta)+\widetilde{\zeta}}{x_1 x_1^{\sigma}}x_1^{\Delta} = -py_1^{\sigma} + fy_1^{\sigma},
\end{aligned} \tag{3.9}$$

and y_1 indeed solves (1.2). From (2.7) and since $\zeta \in \mathcal{B}$, (3.1) holds true. Also, by (2.2) and since $\zeta \in \mathcal{B}$, $ry_1y_1^{\sigma} = rx_1x_1^{\sigma}(1+\zeta)(1+\zeta^{\sigma}) > 0$ eventually, say without loss of generality on $[t_0, \infty)$. So we may define

$$y_2 := y_1(c+\xi) \quad \text{with } c = \frac{x_2(t_0)}{x_1(t_0)}, \qquad \xi(t) = \int_{t_0}^{t} \frac{\Delta\tau}{r(\tau)y_1(\tau)y_1(\sigma(\tau))}. \tag{3.10}$$

We have

$$y_2^{\Delta} = y_1^{\Delta}(c+\xi) + y_1^{\sigma}\xi^{\Delta} = y_1^{\Delta}(c+\xi) + \frac{1}{ry_1} \tag{3.11}$$

and hence,

$$\left(ry_2^{\Delta}\right)^{\Delta} = \left(ry_1^{\Delta}\right)^{\Delta}(c+\xi)^{\sigma} + ry_1^{\Delta}\xi^{\Delta} - \frac{y_1^{\Delta}}{y_1y_1^{\sigma}} = \left(-py_1^{\sigma} + fy_1^{\sigma}\right)(c+\xi)^{\sigma} = -py_2^{\sigma} + fy_2^{\sigma}, \tag{3.12}$$

so that y_2 is another (nonoscillatory) solution of (1.2). Next, by (1.6),

$$x_2 = x_1\varphi, \quad \varphi(t) = c + \int_{t_0}^{t} \frac{\Delta\tau}{r(\tau)x_1(\tau)x_1(\sigma(\tau))} \tag{3.13}$$

and therefore,

$$\frac{y_2}{x_2} = \frac{y_1}{x_1}\frac{c+\xi}{\varphi} = \frac{y_1}{x_1}(1+\gamma), \tag{3.14}$$

where

$$\begin{aligned}\gamma(t) &:= \frac{c+\xi(t)-\varphi(t)}{\varphi(t)} = \frac{\xi(t) - \int_{t_0}^{t} \Delta\tau/r(\tau)x_1(\tau)x_1(\sigma(\tau))}{\varphi(t)} \\ &= -\frac{1}{\varphi(t)}\int_{t_0}^{t}\left[1 - \frac{x_1(\tau)x_1(\sigma(\tau))}{y_1(\tau)y_1(\sigma(\tau))}\right]\frac{\Delta\tau}{r(\tau)x_1(\tau)x_1(\sigma(\tau))} \\ &= -\frac{1}{\varphi(t)}\int_{t_0}^{t}\left(1 - \frac{x_1x_1^{\sigma}}{y_1y_1^{\sigma}}\right)(\tau)\varphi^{\Delta}(\tau)\Delta\tau.\end{aligned} \tag{3.15}$$

Now the first equation in (3.1) and (2.2) imply (see (3.3))

$$1 - \frac{x_1x_1^{\sigma}}{y_1y_1^{\sigma}} = \mathrm{O}(\phi) \quad \text{and thus } \gamma = \mathrm{O}(\widehat{\phi}). \tag{3.16}$$

Then (2.2) together with (3.3) imply $\phi = \mathrm{O}(\widehat{\phi})$ so that

$$\frac{y_2}{x_2} = \frac{y_1}{x_1}(1+\gamma) = \big(1+\mathrm{O}(\phi)\big)\big(1+\mathrm{O}(\widehat{\phi})\big) = 1+\mathrm{O}(\widehat{\phi}), \tag{3.17}$$

which proves the first equation in (3.2). Finally,

$$\left(\frac{y_2}{x_2}\right)^{\Delta} = \left(\frac{y_1}{x_1}\right)^{\Delta}(1+\gamma) + \left(\frac{y_1}{x_1}\right)^{\sigma}\gamma^{\Delta} = \mathrm{O}\left(\frac{\phi\varphi^{\Delta}}{\varphi^{\sigma}}\right)\big(1+\mathrm{O}(\widehat{\phi})\big) + \big(1+\mathrm{O}(\phi)\big)\mathrm{O}\left(\frac{\widehat{\phi}\varphi^{\Delta}}{\varphi^{\sigma}}\right), \tag{3.18}$$

because of (2.2), (3.1), and

$$\begin{aligned}\gamma^{\Delta} &= \frac{(c+\xi-\varphi)^{\Delta}\varphi - (c+\xi-\varphi)\varphi^{\Delta}}{\varphi\varphi^{\sigma}} = \frac{\xi^{\Delta}-\varphi^{\Delta}}{\varphi^{\sigma}} - \gamma\frac{\varphi^{\Delta}}{\varphi^{\sigma}} \\ &= -\frac{\varphi^{\Delta}}{\varphi^{\sigma}}\left(\gamma + 1 - \frac{\xi^{\Delta}}{\varphi^{\Delta}}\right) = \mathrm{O}\left(\frac{\varphi^{\Delta}}{\varphi^{\sigma}}\right)\big[\mathrm{O}(\widehat{\phi}) + \mathrm{O}(\phi)\big] = \mathrm{O}\left(\frac{\widehat{\phi}\varphi^{\Delta}}{\varphi^{\sigma}}\right).\end{aligned} \tag{3.19}$$

So (3.2) is established and this completes the proof. □

4. Applications

In this last section, we state and prove some simple consequences of Theorem 3.1.

Corollary 4.1. *The conclusions of Theorem 3.1 hold if*

$$\int^{\infty} |f(\tau)|\,x_1(\sigma(\tau))x_2(\sigma(\tau))\Delta\tau < \infty. \tag{4.1}$$

Proof. Clearly, (4.1) implies (2.1). Define

$$\hat{\psi}(t) := \int_t^{\infty} |f(\tau)|\,x_1(\sigma(\tau))x_2(\sigma(\tau))\Delta\tau, \qquad \hat{G} := \int_t^{\infty} |f(\tau)|\,x_1^2(\sigma(\tau))\Delta\tau. \tag{4.2}$$

Then

$$|G(t)| \le \hat{G}(t) = \int_t^{\infty} \frac{|f(\tau)|\,x_1(\sigma(\tau))x_2(\sigma(\tau))}{\varphi(\sigma(\tau))}\Delta\tau \le \frac{\hat{\psi}(t)}{\varphi(\sigma(t))}. \tag{4.3}$$

For $T \ge t$, we have by the product rule that

$$\begin{aligned}
&\int_t^T |f(\tau)|\,x_1(\sigma(\tau))x_2(\sigma(\tau))\Delta\tau \\
&\quad = \int_t^T |f(\tau)|\,x_1^2(\sigma(\tau))\varphi(\sigma(\tau))\Delta\tau = -\int_t^T \hat{G}^{\Delta}(\tau)\varphi(\sigma(\tau))\Delta\tau \\
&\quad = -\int_t^T [(\hat{G}\varphi)^{\Delta}(\tau) - \hat{G}(\tau)\varphi^{\Delta}(\tau)]\Delta\tau = -(\hat{G}\varphi)(T) + (\hat{G}\varphi)(t) + \int_t^T \hat{G}(\tau)\varphi^{\Delta}(\tau)\Delta\tau.
\end{aligned} \tag{4.4}$$

Note now that $(\hat{G}\varphi)(T) \to 0$ as $T \to \infty$ since

$$0 \le \hat{G}(T)\varphi(T) \le \frac{\varphi(T)\hat{\psi}(T)}{\varphi(\sigma(T))} \le \hat{\psi}(T) \longrightarrow 0 \quad \text{as } T \longrightarrow \infty \tag{4.5}$$

due to (4.3) and (4.1). Note also that the quantity on the right of the last equal sign in the above calculation must have a finite limit as $T \to \infty$ since the quantity on the left of the first equal sign converges as $T \to \infty$ due to (4.1). Therefore,

$$\int^{\infty} \hat{G}(\tau)\varphi^{\Delta}(\tau)\Delta\tau < \infty, \quad \text{and thus by (4.3)}, \qquad \int^{\infty} |G(\tau)|\,\varphi^{\Delta}(\tau)\Delta\tau < \infty. \tag{4.6}$$

Hence, using (2.2),

$$\frac{\widetilde{\phi}(t)}{\phi(t)} = \int_t^{\infty} |G(\tau)|\,\frac{\phi(\tau)}{\phi(t)}\varphi^{\Delta}(\tau)\Delta\tau \le \int_t^{\infty} |G(\tau)|\,\varphi^{\Delta}(\tau)\Delta\tau \longrightarrow 0 \quad \text{as } t \longrightarrow \infty \tag{4.7}$$

so that (2.15) holds with $\nu = 0$. □

Corollary 4.2. *Assume (2.1). If ϕ satisfies (2.2) and*

$$\limsup_{t\to\infty} \left\{ \frac{1}{\phi(t)} \int_t^{\infty} \frac{\phi^2(\tau)\varphi^{\Delta}(\tau)}{\varphi(\sigma(\tau))}\Delta\tau \right\} =: \widetilde{\nu} < \frac{1}{4}, \tag{4.8}$$

then the conclusions of Theorem 3.1 hold.

Proof. From (2.4), we have

$$\frac{\widetilde{\phi}(t)}{\phi(t)} = \int_t^\infty |G(\tau)| \frac{\phi(\tau)}{\phi(t)} \varphi^\Delta(\tau)\Delta\tau \le \int_t^\infty \frac{2\phi^2(\tau)}{\varphi(\sigma(\tau))\phi(t)} \varphi^\Delta(\tau)\Delta\tau \tag{4.9}$$

so that (4.8) implies

$$\limsup_{t\to\infty} \frac{\widetilde{\phi}(t)}{\phi(t)} \le 2\widetilde{\nu} < \frac{1}{2}, \tag{4.10}$$

and therefore (2.15) is satisfied. □

Corollary 4.3. *Assume that α is a positive function tending monotonically to ∞ and define*

$$S(t) := \int_{t_0}^t \alpha(\sigma(\tau)) f(\tau) x_1(\sigma(\tau)) x_2(\sigma(\tau))\Delta\tau. \tag{4.11}$$

If $|S(t)| \le A < \infty$ and

$$\limsup_{t\to\infty} \left\{ \alpha(\sigma(t)) \int_{t_0}^\infty \frac{\varphi^\Delta(\tau)}{(\alpha^2\varphi)(\sigma(\tau))}\Delta\tau \right\} =: \widehat{\nu} < \frac{1}{4A}, \tag{4.12}$$

then the conclusions of Theorem 3.1 hold with

$$\phi(t) = \frac{1}{\alpha(\sigma(t))}, \qquad \widehat{\phi}(t) = \frac{1}{\varphi(t)} \int_{t_0}^t \frac{\varphi^\Delta(\tau)}{\alpha(\sigma(\tau))}\Delta\tau. \tag{4.13}$$

Proof. Using the definition of S, we find that

$$\psi(t) = \int_t^\infty \frac{S^\Delta(\tau)}{\alpha(\sigma(\tau))}\Delta\tau, \qquad G(t) = \int_t^\infty \frac{S^\Delta(\tau)}{(\alpha\varphi)(\sigma(\tau))}\Delta\tau. \tag{4.14}$$

Similar to the proof of Lemma 2.1, using integration by parts, we get

$$|\psi(t)| \le \frac{2A}{\alpha(\sigma(\tau))}, \qquad |G(t)| \le \frac{2A}{(\alpha\varphi)(\sigma(\tau))}. \tag{4.15}$$

Hence, we can choose ϕ as in the statement of the corollary in place of (2.2) and all results as presented in this paper still hold with this ϕ. Now

$$\frac{\widetilde{\phi}(t)}{\phi(t)} \le \frac{1}{\phi(t)} \int_t^\infty \frac{2A\phi(\tau)\varphi^\Delta(\tau)}{(\alpha\varphi)(\sigma(\tau))}\Delta\tau = \alpha(\sigma(t)) \int_t^\infty \frac{2A\varphi^\Delta(\tau)}{(\alpha^2\varphi)(\sigma(\tau))}\Delta\tau, \tag{4.16}$$

so that (4.12) implies

$$\limsup_{t\to\infty} \frac{\widetilde{\phi}(t)}{\phi(t)} \le 2A\widehat{\nu} < \frac{1}{2}, \tag{4.17}$$

and therefore, (2.15) is satisfied. □

References

[1] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, Mass, USA, 2001.
[2] M. Bohner and A. Peterson, Eds., *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2003.
[3] W. F. Trench, "Linear perturbations of a nonoscillatory second order equation," *Proceedings of the American Mathematical Society*, vol. 97, no. 3, pp. 423–428, 1986.
[4] W. F. Trench, "Linear perturbations of a nonoscillatory second order differential equation. II," *Proceedings of the American Mathematical Society*, vol. 131, no. 5, pp. 1415–1422, 2003.
[5] W. F. Trench, "Linear perturbations of a nonoscillatory second order difference equation," *Journal of Mathematical Analysis and Applications*, vol. 255, no. 2, pp. 627–635, 2001.
[6] V. Kac and P. Cheung, *Quantum Calculus*, Universitext, Springer, New York, NY, USA, 2002.

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