Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2007, Article ID 39404, 7 pages
doi:10.1155/2007/39404

Research Article

On the Recursive Sequence $x_n = 1 + \sum_{i=1}^{k} \alpha_i x_{n-p_i} / \sum_{j=1}^{m} \beta_j x_{n-q_j}$

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Received 28 September 2006; Revised 3 December 2006; Accepted 11 December 2006

We give a complete picture regarding the behavior of positive solutions of the following important difference equation: $x_n = 1 + \sum_{i=1}^{k} \alpha_i x_{n-p_i} / \sum_{j=1}^{m} \beta_j x_{n-q_j}$, $n \in \mathbb{N}_0$, where α_i, $i \in \{1,\dots,k\}$, and β_j, $j \in \{1,\dots,m\}$, are positive numbers such that $\sum_{i=1}^{k} \alpha_i = \sum_{j=1}^{m} \beta_j = 1$, and p_i, $i \in \{1,\dots,k\}$, and q_j, $j \in \{1,\dots,m\}$, are natural numbers such that $p_1 < p_2 < \cdots < p_k$ and $q_1 < q_2 < \cdots < q_m$. The case when $\gcd(p_1,\dots,p_k,q_1,\dots,q_m) = 1$ is the most important. For the case we prove that if all p_i, $i \in \{1,\dots,k\}$, are even and all q_j, $j \in \{1,\dots,m\}$, are odd, then every positive solution of this equation converges to a periodic solution of period two, otherwise, every positive solution of the equation converges to a unique positive equilibrium.

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1. Introduction and preliminaries

In [1], we studied the behavior of positive solutions of the recursive equation

$$y_n = 1 + \frac{y_{n-k}}{y_{n-m}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

with $y_{-s}, y_{-s+1}, \dots, y_{-1} \in (0,\infty)$ and $k,m \in \{1,2,3,4,\dots\}$, where $s = \max\{k,m\}$. We proved that if 2^i is the highest power of 2 which divides m, then if $2^{i+1} \nmid k$, y_n tends to 2, exponentially, and otherwise every solution tends to a period t solution, with $t = 2\gcd(k,m)$. The method we used in [1] is a little bit complicated and its idea essentially stems from the theory of nonexpansive metrics. Since the above result is formulated in number theoretic language, we expect that the result is a particular case of a more general result, which

motivates us to investigate the following somewhat natural generalization of (1.1):

$$x_n = 1 + \frac{\sum_{i=1}^{k} \alpha_i x_{n-p_i}}{\sum_{j=1}^{m} \beta_j x_{n-q_j}}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where α_i, $i \in \{1,\ldots,k\}$, and β_j, $j \in \{1,\ldots,m\}$, are positive numbers such that $\sum_{i=1}^{k} \alpha_i = \sum_{j=1}^{m} \beta_j = 1$, and p_i, $i \in \{1,\ldots,k\}$, and q_j, $j \in \{1,\ldots,m\}$, are natural numbers such that $p_1 < p_2 < \cdots < p_k$ and $q_1 < q_2 < \cdots < q_m$.

Here, we give a complete picture regarding the asymptotic behavior of positive solutions of (1.2). For closely related results, see, for example, [1–16] and the references therein.

In the proof of the main result of this paper, we need the following result by Karakostas (see [8, 9]).

Theorem 1.1. *Let J be some interval of real numbers, let $f \in C[J^2, J]$, and let $(x_n)_{n=-1}^{\infty}$ be a bounded solution of the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}), \quad n \in \mathbb{N}_0, \tag{1.3}$$

with $I = \liminf_{n\to\infty} I_n$, $S = \limsup_{n\to\infty} x_n$ and with $I, S \in J$. Then there exist two solutions $(I_n)_{n=-\infty}^{\infty}$ and $(S_n)_{n=-\infty}^{\infty}$ of the difference equation

$$x_{n+1} = f(x_n, x_{n-1}) \tag{1.4}$$

which satisfy the equation for all $n \in \mathbb{Z}$, with $I_0 = I$, $S_0 = S$, $I_n, S_n \in [I, S]$ for all $n \in \mathbb{Z}$ and such that for every $N \in \mathbb{Z}$, I_N and S_N are limit points of $(x_n)_{n=-1}^{\infty}$. Furthermore, for every $m \le -1$, there exist two subsequences (x_{r_n}) and (x_{l_n}) of the solution $(x_n)_{n=-1}^{\infty}$ such that the following are true:

$$\lim_{n\to\infty} x_{r_n+N} = I_N, \quad \lim_{n\to\infty} x_{l_n+N} = S_N \quad \text{for every } N \ge m. \tag{1.5}$$

The solutions $(I_n)_{n=-\infty}^{\infty}$ and $(S_n)_{n=-\infty}^{\infty}$ of (1.4) are called full limiting solutions of (1.4) associated with the solution $(x_n)_{n=-1}^{\infty}$ of (1.3).

2. Main results

First, we study the boundedness character of positive solutions of (1.2). For closely related results, see, for example, [4, 6, 12–14].

Theorem 2.1. *Every positive solution of (1.2) is bounded.*

Proof. Assume that (x_n) is a positive solution of (1.2). Note that $x_n > 1$ for $n \ge 0$. Hence, it is possible to choose positive numbers l and L greater than one such that $lL = L + l$ and $l \le x_i \le L$ for $i \in \{0, 1, \ldots, s-1\}$, where $s = \max\{p_k, q_m\}$. Employing (1.2), we obtain

$$l = 1 + \frac{l}{L} \le x_s = 1 + \frac{\sum_{i=1}^{k} \alpha_i x_{s-p_i}}{\sum_{j=1}^{m} \beta_j x_{s-q_j}} \le 1 + \frac{L}{l} = L. \tag{2.1}$$

By the induction, we obtain that $x_n \in [l,L]$ for every $n \in \mathbb{N}_0$, finishing the proof of the theorem. □

We are now in a position to formulate and prove the main result of this paper.

Theorem 2.2. *Consider (1.2). Assume that*

$$G := \gcd(p_1,\dots,p_k,q_1,\dots,q_m) = 1. \tag{2.2}$$

Then if all p_i, $i \in \{1,\dots,k\}$, are even and all q_j, $j \in \{1,\dots,m\}$, are odd, every positive solution of (1.2) converges to a periodic solution of period two. Otherwise, every positive solution of (1.2) converges to a unique positive equilibrium.

Proof. Let

$$\mathcal{P} = \{p_i \mid i = 1,\dots,k\}, \qquad \mathcal{Q} = \{q_j \mid j = 1,\dots,m\}. \tag{2.3}$$

Assume first that $\mathcal{P} \cap \mathcal{Q} \neq \emptyset$. In view of Theorem 2.1, every positive solution (x_n) of (1.2) is bounded which implies that there are finite $\liminf_{n\to\infty} x_n = I$ and $\limsup_{n\to\infty} x_n = S$. Letting $n \to \infty$ in (1.2), we obtain

$$1 + \frac{I}{S} \le I \le S \le 1 + \frac{S}{I}, \tag{2.4}$$

from which it follows that

$$SI = I + S. \tag{2.5}$$

Let $(L_{-i})_{i\in\mathbb{Z}}$ be a full limiting sequence of a solution (x_n) of (1.2), such that $L_0 = S$. Since $(L_{-i})_{i\in\mathbb{Z}}$ is a solution of (1.2) belonging to the interval $[I,S]$, we have that

$$S = L_0 = 1 + \frac{\sum_{i=1}^{k} \alpha_i L_{-p_i}}{\sum_{j=1}^{m} \beta_j L_{-q_j}} \le 1 + \frac{S}{I} = S. \tag{2.6}$$

From (2.6), it follows that $L_{-p_i} = S$ for every $i \in \{1,\dots,k\}$ and $L_{-q_j} = I$ for every $j \in \{1,\dots,m\}$. Employing assumption $\mathcal{P} \cap \mathcal{Q} \neq \emptyset$, we obtain $I = S$, from which the result follows in this case.

Now we assume that $\mathcal{P} \cap \mathcal{Q} = \emptyset$. Further, assume that there is $p_{i_0} \in \mathcal{P}$ which is odd. Let $p_{i_0} = 2s+1$ and let q_{j_0} be an arbitrary element of $\mathcal{Q}$. Then, (1.2) can be written in the form

$$x_n = 1 + \frac{\alpha_{i_0} x_{n-(2s+1)} + \sum_{i=1, i\neq i_0}^{k} \alpha_i x_{n-p_i}}{\beta_{j_0} x_{n-q_{j_0}} + \sum_{j=1, j\neq j_0}^{m} \beta_j x_{n-q_j}}. \tag{2.7}$$

Let $(L_{-i})_{i\in\mathbb{Z}}$ be a full limiting sequence of a solution (x_n) of (1.2), such that $L_0 = S = \limsup_{n\to\infty} x_n$. From

$$S = L_0 = 1 + \frac{\alpha_{i_0} L_{-(2s+1)} + \sum_{i=1, i\neq i_0}^{k} \alpha_i L_{-p_i}}{\beta_{j_0} L_{-q_{j_0}} + \sum_{j=1, j\neq j_0}^{m} \beta_j L_{-q_j}}, \tag{2.8}$$

similar to (2.6), we obtain

$$L_{-(2s+1)} = S, \qquad L_{-q_{j_0}} = I. \tag{2.9}$$

From (2.9) and since $(L_{-i})_{i\in\mathbb{Z}}$ is a solution of (2.7), it follows that

$$L_{-2(2s+1)} = S, \qquad L_{-2q_{j_0}} = S. \tag{2.10}$$

Indeed, since

$$S = L_{-(2s+1)} = 1 + \frac{\alpha_{i_0} L_{-2(2s+1)} + \sum_{i=1, i\neq i_0}^{k} \alpha_i L_{-p_i-(2s+1)}}{\beta_{j_0} L_{-q_{j_0}-(2s+1)} + \sum_{j=1, j\neq j_0}^{m} \beta_j L_{-q_j-(2s+1)}} \leq 1 + \frac{S}{I} = S, \tag{2.11}$$

we obtain the first equality in (2.10). On the other hand, from

$$I = L_{-q_{j_0}} = 1 + \frac{\alpha_{i_0} L_{-q_{j_0}-(2s+1)} + \sum_{i=1, i\neq i_0}^{k} \alpha_i L_{-q_{j_0}-p_i}}{\beta_{j_0} L_{-2q_{j_0}} + \sum_{j=1, j\neq j_0}^{m} \beta_j L_{-q_{j_0}-q_j}} \geq 1 + \frac{I}{S} = I, \tag{2.12}$$

the second equality in (2.10) follows.

By induction we obtain

$$L_{-(2s+1)i} = S, \quad i \in \mathbb{N}, \tag{2.13}$$

$$L_{-q_{j_0} j} = \begin{cases} I, & j \text{ odd}, \\ S, & j \text{ even}. \end{cases} \tag{2.14}$$

If we take $i = q_{j_0}$ in (2.13) and $j = 2s+1$ in (2.14), we obtain $I = L_{-(2s+1)q_{j_0}} = S$, as desired.

Now, assume that all $p_i \in \mathcal{P}$ are even, and $\mathcal{Q}$ has odd as well as even elements. Then, (1.2) can be written in the form

$$x_n = 1 + \frac{\sum_{i=1}^{k} \alpha_i x_{n-p_i}}{\beta_{j_0} x_{n-q_{j_0}} + \beta_{j_1} x_{n-q_{j_1}} + \sum_{j=1, j\neq j_0, j_1}^{m} \beta_j x_{n-q_j}}, \tag{2.15}$$

where $q_{j_0} = 2s$ and $q_{j_1} = 2t+1$.

From a result in number theory [11], we know that the condition $G = 1$ implies that for each sufficiently large n, say, $n \geq n_0$, there are nonnegative numbers $d_i \in \mathbb{N}_0$, $i \in \{1, \ldots, k+m\}$, such that

$$\sum_{i=1}^{k} p_i d_i + \sum_{j=1}^{m} q_j d_{k+j} = n. \tag{2.16}$$

From condition $G = 1$, by using (2.15) and (2.16), and employing the procedure described above for getting formulae (2.13) and (2.14), we obtain that the subsequence $(L_{-i})_{i\geq n_0}$ of the full limiting sequence $(L_i)_{i\in\mathbb{Z}}$ with $L_0 = S$ takes values I and S.

Now we prove that the sequence $(L_{-i})_{i\in\mathbb{N}}$ is eventually periodic with periods $p_1, p_2, \ldots, p_k$ and also with periods $2q_1, \ldots, 2q_m$. Indeed, if we replace n in (2.15) by $-n_0 - l$, $l \in \{0, 1, \ldots, p_1 - 1\}$, we obtain that $L_{-n_0-l} = L_{-n_0-l-p_1 i}$ for every $i \in \mathbb{N}$ and each $l \in \{0, 1, \ldots, p_1 - 1\}$, that is, $(L_{-i})_{i\in\mathbb{N}}$ is eventually periodic with period p_1. Similarly it can be proven that $(L_{-i})_{i\in\mathbb{N}}$ is eventually periodic with periods $p_2, \ldots, p_k$. The periodicity with periods $2q_1, \ldots, 2q_m$ can be proven similar to (2.9) and (2.10) and by using induction.

Since all $p_i \in \mathcal{P}$ are even and $G = 1$, we have that

$$2 \le \gcd(p_1, p_2, \ldots, p_k, 2q_1, \ldots, 2q_m) = 2\gcd\left(\frac{p_1}{2}, \frac{p_2}{2}, \ldots, \frac{p_k}{2}, q_1, \ldots, q_m\right) \le 2G = 2, \tag{2.17}$$

that is,

$$\gcd(p_1, p_2, \ldots, p_k, 2q_1, \ldots, 2q_m) = 2. \tag{2.18}$$

Hence, the sequence $(L_{-i})_{i\in\mathbb{N}}$ is eventually periodic with period two. Since $(L_i)_{i\in\mathbb{Z}}$ is a solution of (1.2), we obtain that $(L_i)_{i\in\mathbb{Z}}$ is also periodic with period two.

Assume now that

$$\ldots, x, y, x, y, x, y, \ldots, \tag{2.19}$$

is a two-periodic solution of (2.15). Then we have

$$x = 1 + \frac{x}{cx + (1-c)y}, \qquad y = 1 + \frac{y}{cy + (1-c)x}, \tag{2.20}$$

for some $c \in (0, 1)$. Hence,

$$(c-1)xy = cx^2 - (c+1)x - (1-c)y = cy^2 - (c+1)y - (1-c)x, \tag{2.21}$$

from which it follows that $c(x - y)(x + y - 2) = 0$. If $x + y = 2$ and $x \neq y$, then we have that x and y are different positive solutions of the equation

$$x = 1 + \frac{x}{cx + (1-c)(2-x)}, \tag{2.22}$$

which implies that $(2c - 1)(x - 1)^2 = 1$. Hence, if $c \le 1/2$, then this equation does not have real roots. If $c > 1/2$, then $x = 1 \pm (1/(2c-1))^{1/2}$ are solutions. However, since $c \in (1/2, 1)$, the number $1 - (1/(2c-1))^{1/2}$ is negative. Therefore, it follows that $x = y$ as desired.

Assume now that the set $\mathcal{P}$ contains only even elements while $\mathcal{Q}$ contains only odd elements. Then, it is easy to see that (1.2) in this case has infinite prime two-periodic solutions of the form $x, y, x, y, \ldots$, such that $xy = x + y$. Similar to (2.18), it can be proven that, in this case, the full limiting sequence $(L_i)_{i\in\mathbb{Z}}$, $L_0 = S$ is periodic with period two and that

$$L_{2i} = S, \quad L_{2i-1} = I, \quad i \in \mathbb{Z}. \tag{2.23}$$

Assume that $\varepsilon, \delta \in (0, S)$ are such that

$$(S-\varepsilon)(I+\delta) = (S-\varepsilon) + (I+\delta). \tag{2.24}$$

Then, for such chosen ε and δ, there is a $k_0 \in \mathbb{Z}$ such that

$$x_{k_0+2j} > S-\varepsilon, \qquad x_{k_0+2j-1} < I+\delta, \tag{2.25}$$

for $j \in \{1,2,\ldots,[s/2]+1\}$, where $s = \max\{p_k, q_m\}$.

From (1.2) and (2.25), we have that

$$\begin{aligned} x_{k_0+2[s/2]+3} &< 1 + \frac{I+\delta}{S-\varepsilon} = I+\delta, \\ x_{k_0+2[s/2]+4} &> 1 + \frac{S-\varepsilon}{I+\delta} = S-\varepsilon. \end{aligned} \tag{2.26}$$

By induction, we obtain

$$x_{k_0+2i+1} < I+\delta, \qquad x_{k_0+2i} > S-\varepsilon, \tag{2.27}$$

for every $i \in \mathbb{N}$. From (2.27) and the fact that $\varepsilon \to 0$ implies $\delta \to 0$, it follows that $\lim_{n\to\infty} x_{2n} = S$ and $\lim_{n\to\infty} x_{2n-1} = I$, or $\lim_{n\to\infty} x_{2n} = I$ and $\lim_{n\to\infty} x_{2n-1} = S$, finishing the proof of the theorem. □

Remark 2.3. Note that the case when all p_i, $i \in \{1,\ldots,k\}$, and q_j, $j \in \{1,\ldots,m\}$, are even is excluded from the consideration in Theorem 2.1 since we assume that $G = 1$. However, this case is reduced to the cases considered in Theorem 2.1. Indeed, let 2^s be the highest power of 2 which divides G, then (1.2) can be separated into 2^s different equations of the form

$$x_n^{(t)} = 1 + \frac{\sum_{i=1}^{k} \alpha_i x_{n-p_i/2^s}^{(t)}}{\sum_{j=1}^{m} \beta_j x_{n-q_j/2^s}^{(t)}}, \quad n \in \mathbb{N}_0, \tag{2.28}$$

where $t \in \{0,1,\ldots,2^s-1\}$. Note that by the definition of 2^s, it follows that at least one of the numbers $p_i/2^s$, $i \in \{1,\ldots,k\}$, and $q_j/2^s$, $j \in \{1,\ldots,m\}$, is odd. Hence, Theorem 2.1 can be applied to the equations in (2.28).

Acknowledgment

The author would like to express his sincere thanks to the referees whose comments improved the presentation in this paper.

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