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Research Article

Three Solutions for Forced Duffing-Type Equations with Damping Term

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Using the variational principle of Ricceri and a local mountain pass lemma, we study the existence of three distinct solutions for the following resonant Duffing-type equations with damping and perturbed term $u''(t) + \sigma u'(t) + f(t, u(t)) + \lambda g(t, u(t)) = p(t)$, a.e. $t \in [0, \omega]$, $u(0) = 0 = u(\omega)$ and without perturbed term $u''(t) + \sigma u'(t) + f(t, u(t)) = p(t)$, a.e. $t \in [0, \omega]$, $u(0) = 0 = u(\omega)$.

1. Introduction

In this paper, we consider the following resonant Duffing-type equations with damping and perturbed term:

$$u'' + \sigma u'(t) + f(t, u(t)) + \lambda g(t, u(t)) = p(t), \quad \text{a.e. } t \in [0, \omega],$$
$$u(0) = 0 = u(\omega), \tag{1.1}$$

where $\sigma, \lambda \in \mathbf{R}$, $f, g : [0, \omega] \times \mathbf{R} \to \mathbf{R}$, and $p : [0, \omega] \to \mathbf{R}$ are continuous. Letting $\lambda = 0$ in problem (1.1) leads to

$$u''(t) + \sigma u'(t) + f(t, u(t)) = p(t), \quad \text{a.e. } t \in [0, \omega],$$
$$u(0) = 0 = u(\omega), \tag{1.2}$$

which is a common Duffing-type equation without perturbation.

The Duffing equation has been used to model the nonlinear dynamics of special types of mechanical and electrical systems. This differential equation has been named after the studies of Duffing in 1918 [1], has a cubic nonlinearity, and describes an oscillator. It is the simplest oscillator displaying catastrophic jumps of amplitude and phase when the frequency of the forcing term is taken as a gradually changing parameter. It has drawn extensive attention due to the richness of its chaotic behaviour with a variety of interesting bifurcations, torus and Arnolds tongues. The main applications have been in electronics, but it can also have applications in mechanics and in biology. For example, the brain is full of oscillators at micro and macro levels [2]. There are applications in neurology, ecology, secure communications, cryptography, chaotic synchronization, and so on. Due to the rich behaviour of these equations, recently there have been also several studies on the synchronization of two coupled Duffing equations [3, 4]. The most general forced form of the Duffing-type equation is

$$u''(t) + \sigma u'(t) + f(t, u(t)) = p(t). \tag{1.3}$$

Recently, many authors have studied the existence of periodic solutions of the Duffing-type equation (1.3). By using various methods and techniques, such as polar coordinates, the method of upper and lower solutions and coincidence degree theory and a series of existence results of nontrivial solutions for the Duffing-type equations such as (1.3) have been obtained; we refer to [5–11] and references therein. There are also authors who studied the Duffing-type equations by using the critical point theory (see [12, 13]). In [12], by using a saddle point theorem, Tomiczek obtained the existence of a solution of the following Duffing-type system:

$$\begin{gathered} u''(t) + \sigma u'(t) + \left(m^2 - \frac{\sigma^2}{4}\right)u(t) + f(t, u(t)) = p(t), \quad \text{a.e. } t \in [0, \omega], \\ u(0) = 0 = u(\omega), \end{gathered} \tag{1.4}$$

which is a special case of problems (1.1)-(1.2). However, to the best of our knowledge, there are few results for the existence of multiple solutions of (1.3).

Our aim in this paper is to study the variational structure of problems (1.1)-(1.2) in an appropriate space of functions and the existence of solutions for problems (1.1)-(1.2) by means of some critical point theorems. The organization of this paper is as follows. In Section 2, we shall study the variational structure of problems (1.1)-(1.2) and give some important lemmas which will be used in later section. In Section 3, by applying some critical point theorems, we establish sufficient conditions for the existence of three distinct solutions to problems (1.1)-(1.2).

2. Variational Structure

In the Sobolev space $H := H_0^1(0, \omega)$, consider the inner product

$$\langle u, v \rangle_H = \int_0^\omega u'(s)v'(s)\mathrm{d}s \quad \forall u, v \in H, \tag{2.1}$$

inducing the norm

$$\|u\|_H = \sqrt{\langle u, v\rangle_H} = \left(\int_0^\omega |u'(s)|^2 \mathrm{d}s\right)^{1/2} \quad \forall u \in H. \tag{2.2}$$

We also consider the inner product

$$\langle u, v\rangle = \int_0^\omega e^{\sigma s} u'(s) v'(s) \mathrm{d}s \quad \forall u, v \in H, \tag{2.3}$$

and the norm

$$\|u\| = \sqrt{\langle u, v\rangle} = \left(\int_0^\omega e^{\sigma s} |u'(s)|^2 \mathrm{d}s\right)^{1/2} \quad \forall u \in H. \tag{2.4}$$

Obviously, the norm $\|\cdot\|$ and the norm $\|\cdot\|_H$ are equivalent. So H is a Hilbert space with the norm $\|\cdot\|$.

By Poincaré's inequality,

$$\begin{aligned} \|u\|_2^2 := \int_0^\omega |u(s)|^2 \mathrm{d}s &\le \frac{1}{\lambda_1} \int_0^\omega |u'(s)|^2 \mathrm{d}s \\ &\le \frac{1}{\lambda_1 \min\{1, e^{\sigma\omega}\}} \int_0^\omega e^{\sigma s} |u'(s)|^2 \mathrm{d}s := \lambda_0 \|u\|^2 \quad \forall u \in H, \end{aligned} \tag{2.5}$$

where $\lambda_0 := 1/\lambda_1 \min\{1, e^{\sigma\omega}\}$, $\lambda_1 := \pi^2/\omega^2$ is the first eigenvalue of the problem

$$\begin{aligned} -u''(t) &= \lambda u(t), \quad t \in [0, \omega], \\ u(0) &= 0 = u(\omega). \end{aligned} \tag{2.6}$$

Usually, in order to find the solution of problems (1.1)-(1.2), we should consider the following functional Φ, Ψ defined on H:

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \int_0^\omega e^{\sigma s} |u'(s)|^2 \mathrm{d}s + \int_0^\omega e^{\sigma s} p(s) u(s) \mathrm{d}s - \int_0^\omega e^{\sigma s} F(s, u(s)) \mathrm{d}s \\ \Psi(u) &= -\int_0^\omega e^{\sigma s} G(s, u(s)) \mathrm{d}s, \end{aligned} \tag{2.7}$$

where $F(s, u) = \int_0^u f(s, \mu) \mathrm{d}\mu$, $G(s, u) = \int_0^u g(s, \mu) \mathrm{d}\mu$.

Finding solutions of problem (1.1) is equivalent to finding critical points of $I := \Phi + \lambda\Psi$ in H and

$$\begin{aligned}\langle I'(u), v\rangle = &\int_0^\omega e^{\sigma s}u'(s)v'(s)\mathrm{d}s + \int_0^\omega e^{\sigma s}p(s)v(s)\mathrm{d}s\\ &- \int_0^\omega e^{\sigma s}f(s,u)v(s)\mathrm{d}s - \int_0^\omega e^{\sigma s}\lambda g(s,u)v(s)\mathrm{d}s, \quad \forall u, v \in H.\end{aligned} \tag{2.8}$$

Lemma 2.1 (Hölder Inequality). *Let $f, g \in C([a,b])$, $p > 1$, and q the conjugate number of p. Then*

$$\int_a^b |f(s)g(s)|\mathrm{d}s \le \left(\int_a^b |f(s)|^p\mathrm{d}s\right)^{1/p} \cdot \left(\int_a^b |g(s)|^q\mathrm{d}s\right)^{1/q}. \tag{2.9}$$

Lemma 2.2. *Assume the following condition holds.*

(f1) *There exist positive constants α, β, and $\gamma \in [0,1)$ such that*

$$|f(s,x)| \le \alpha + \beta|x|^\gamma \quad \forall(s,x) \in [0,\omega] \times \mathbf{R}. \tag{2.10}$$

Then Φ is coercive.

Proof. Let $\{u_n\}_{n\in\mathbf{N}} \subset H$ be a sequence such that $\lim_{n\to+\infty}\|u_n\| = +\infty$. It follows from (f1) and Hölder inequality that

$$\begin{aligned}\Phi(u_n) &= \frac{1}{2}\int_0^\omega e^{\sigma s}|u_n'(s)|^2\mathrm{d}s + \int_0^\omega e^{\sigma s}p(s)u_n(s)\mathrm{d}s - \int_0^\omega e^{\sigma s}F(s,u_n(s))\mathrm{d}s\\ &\ge \frac{1}{2}\|u_n\|^2 - \left(\int_0^\omega e^{2\sigma s}|p(s)|^2\mathrm{d}s\right)^{1/2}\|u_n\|_2 - \max\{1, e^{\sigma\omega}\}\int_0^\omega\left(\alpha|u_n| + \beta|u_n|^{\gamma+1}\right)\mathrm{d}s\\ &\ge \frac{1}{2}\|u_n\|^2 - \left(\int_0^\omega e^{2\sigma s}|p(s)|^2\mathrm{d}s\right)^{1/2}\|u_n\|_2\\ &\quad - \alpha\sqrt{\omega}\max\{1, e^{\sigma\omega}\}\|u_n\|_2 - \beta\sqrt{\omega^{1-\gamma}}\max\{1, e^{\sigma\omega}\}\|u_n\|_2^{\gamma+1}\\ &\ge \frac{1}{2}\|u_n\|^2 - \sqrt{\lambda_0}\left[\left(\int_0^\omega e^{2\sigma s}|p(s)|^2\mathrm{d}s\right)^{1/2} + \alpha\sqrt{\omega}\max\{1, e^{\sigma\omega}\}\right]\|u_n\|\\ &\quad - \sqrt{\lambda_0^{\gamma+1}}\beta\sqrt{\omega^{1-\gamma}}\max\{1, e^{\sigma\omega}\}\|u_n\|^{\gamma+1},\end{aligned} \tag{2.11}$$

which implies from $\gamma \in [0,1)$ that $\lim_{n\to+\infty}\Phi(u_n) = +\infty$. This completes the proof. □

From the proof of Lemma 2.2, we can show the following Lemma.

Lemma 2.3. *Assume that* $2\beta\lambda_0 \max\{e^{\sigma\omega}, 1\} < 1$ *and the following condition holds.*

(f2) *There exist positive constants* α_0 *and* β_0 *such that*

$$\left|f(s,x)\right| \leq \alpha_0 + \beta_0|x| \quad \forall (s,x) \in [0,\omega] \times \mathbf{R}. \tag{2.12}$$

Then Φ *is coercive.*

Lemma 2.4. *Assume the following condition holds.*

(f3) $\lim_{|x|\to+\infty} \int_0^x f(s,\mu)\mathrm{d}\mu \leq 0$ *for all* $s \in [0,\omega]$.

Then Φ *is coercive.*

Proof. Let $\{u_n\}_{n\in\mathbf{N}} \subset H$ be a sequence such that $\lim_{n\to+\infty}\|u_n\| = +\infty$. Fix $\epsilon > 0$, from (f3), there exists $K = K(\epsilon) > 0$ such that

$$F(s,x) \leq -\epsilon \quad \forall s \in [0,\omega],\ |x| > K. \tag{2.13}$$

Denote by $\{|u| \leq K\}$ the set $\{s \in [0,\omega] : |u(s)| \leq K\}$ and by $\{|u| > K\}$ its complement in $[0,\omega]$. Put $\phi_K(s) := \sup_{|x|\leq K}|F(s,x)|$ for all $s \in [0,\omega]$. By the continuity of f, we know that $\sup_{s\in[0,\omega]}\phi_K(s) < +\infty$. Then one has

$$\begin{aligned}
\Phi(u_n) &= \frac{1}{2}\int_0^\omega e^{\sigma s}\left|u_n'(s)\right|^2 \mathrm{d}s + \int_0^\omega e^{\sigma s} p(s) u_n(s)\mathrm{d}s \\
&\quad - \int_{\{|u_n|\leq K\}} e^{\sigma s} F(s,u_n(s))\mathrm{d}s - \int_{\{|u_n|>K\}} e^{\sigma s} F(s,u_n(s))\mathrm{d}s \\
&\geq \frac{1}{2}\|u_n\|^2 - \sqrt{\lambda_0}\left(\int_0^\omega e^{2\sigma s}\left|p(s)\right|^2 \mathrm{d}s\right)^{1/2}\|u_n\| - \int_0^\omega e^{\sigma s}\phi_K(s)\mathrm{d}s,
\end{aligned} \tag{2.14}$$

which implies that $\lim_{n\to+\infty}\Phi(u_n) = +\infty$. This completes the proof. □

Based on Ricceri's variational principle in [14, 15], Fan and Deng [16] obtained the following result which is a main tool used in our paper.

Lemma 2.5 (see [16]). *Suppose that* D *is a bounded convex open subset of* H, $v_1, v_2 \in D$, $\Phi(v_1) = \inf_D \Phi = c_0$, $\inf_{\partial D}\Phi = b > c_0$, v_2 *is a strict local minimizer of* Φ, *and* $\Phi(v_2) = c_1 > c_0$. *Then, for* $\epsilon > 0$ *small enough and any* $\rho_2 > c_1$, $\rho_1 \in (c_0, \min\{b, c_1\})$, *there exists* $\lambda_* > 0$ *such that for each* $\lambda \in (0, \lambda_*)$, $\Phi + \lambda\Psi$ *has at least two local minima* u_1 *and* u_2 *lying in* D, *where* $u_1 \in \Phi^{-1}((-\infty, \rho_0)) \cap D$, $u_2 \in \Phi^{-1}((-\infty, \rho_1)) \cap B(u_1, \epsilon)$, *where* $B(u_1, \epsilon) = \{u \in H : \|u - u_1\| < \epsilon\}$, *and* $u_2 \overline{\in}\ \overline{B(u_1, \epsilon)}$.

3. Main Results

In this section, we will prove that problems (1.1)-(1.2) have three distinct solutions by using the variational principle of Ricceri and a local mountain pass lemma.

Theorem 3.1. *Assume that (f1) holds. Suppose further that*

(f4) *there exists $\delta > 0$ such that*

$$\frac{x^2}{2\lambda_0 e^{\sigma s}} + p(s)x > \int_0^x f(s,\mu)\mathrm{d}\mu \quad \forall (s,x) \in [0,\omega] \times [-\delta, 0) \cup (0,\delta], \tag{3.1}$$

(f5) *there exists $x_0 \in H$ such that $\Phi(x_0) < 0$.*

Then there exist $\lambda^* > 0$ and $r > 0$ such that, for every $\lambda \in (-\lambda^*, \lambda^*)$, problem (1.1) *admits at least three distinct solutions which belong to $B(0,r) \subseteq H$.*

Proof. By Lemma 2.2, condition (f1) implies that the functional Φ is coercive. Since Φ is sequentially weakly lower semicontinuous (see [16, Propositions 2.5 and 2.6]), Φ has a global minimizer v_1. By (f5), we obtain $\Phi(v_1) = \inf_H \Phi = c_0 < 0$. Let $D := B(0,\eta) = \{u \in H : \|u\| < \eta\}$. Since Φ is coercive, we can choose a large enough η such that

$$v_1 \in D, \quad \Phi(v_1) = \inf_D \Phi = c_0 < 0, \quad \inf_{\partial D} \Phi = b > 0 > c_0. \tag{3.2}$$

Now we prove that Φ has a strict local minimum at $v_2 = 0$. By the compact embedding of H into $C(0,\omega;\mathbf{R})$, there exists a constant $c_1 > 0$ such that

$$\max_{s\in[0,\omega]} |u(s)| \le c_1 \|u\| \quad \forall u \in H. \tag{3.3}$$

Choosing $r_\delta < \delta/c_1$, it results that

$$\overline{B(0,r_\delta)} = \{u \in H : \|u\| \le r_\delta\} \subseteq \left\{u \in H : \max_{s\in[0,\omega]} |u(s)| < \delta\right\}. \tag{3.4}$$

Therefore, for every $u \in B(0,r_\delta) \setminus \{0\}$, it follows from (f4) that

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\int_0^\omega e^{\sigma s}\left|u'(s)\right|^2 \mathrm{d}s + \int_0^\omega e^{\sigma s} p(s)u(s)\mathrm{d}s - \int_0^\omega e^{\sigma s} F(s,u(s))\mathrm{d}s \\
&\ge \frac{1}{2\lambda_0}\int_0^\omega |u(s)|^2 \mathrm{d}s + \int_0^\omega e^{\sigma s} p(s)u(s)\mathrm{d}s - \int_0^\omega e^{\sigma s} F(s,u(s))\mathrm{d}s \\
&= \int_0^\omega e^{\sigma s}\left(\frac{|u(s)|^2}{2\lambda_0 e^{\sigma s}} + p(s)u(s) - F(s,u(s))\right)\mathrm{d}s \\
&> \Phi(0) = 0,
\end{aligned} \tag{3.5}$$

which implies that $v_2 = 0$ is a strict local minimum of Φ in H with $c_1 := \Phi(v_2) = 0 > c_0$.

At this point, we can apply Lemma 2.5 taking Ψ and $-\Psi$ as perturbing terms. Then, for $\epsilon \in (0, r_\delta]$ small enough and any $\rho_1 \in (c_0, \min\{b, c_1\})$, $\rho_2 \in (0,+\infty)$, we can obtain the following.

(i) There exists $\widehat{\lambda} > 0$ such that, for each $\lambda \in (-\widehat{\lambda}, \widehat{\lambda})$, $\Phi + \lambda\Psi$ has two distinct local minima u_1 and u_2 satisfying

$$u_1 \in \Phi^{-1}\big((-\infty, \rho_1)\big), \qquad u_2 \in \Phi^{-1}\big((-\infty, \rho_2)\big) \cap B(0, \epsilon). \tag{3.6}$$

(ii) $\theta := \inf_{\|u\|=\epsilon}\Phi(u) > 0$ (see [16, Theorem 3.6])

Let $r_1 > 0$ be such that

$$\Phi^{-1}\big((-\infty, \rho_1)\big) \cup B(0, \epsilon) \subseteq B(0, r_1), \tag{3.7}$$

and put $b = \sup_{\|u\|\le r_1}|\Phi(u)|$. Owing to the coerciveness of Φ, there exists $r_2 > r_1$ such that $\inf_{\|u\|=r_2}\Phi(u) = d > b$. Since $g : [0, \omega] \times \mathbf{R} \to \mathbf{R}$ is continuous, then

$$\sup_{\|u\|\le r_2} |\Psi(u)| < +\infty. \tag{3.8}$$

Choosing $\lambda < (d-b)/2\sup_{\|u\|\le r_2}|\Psi(u)|$, hence, for every $u \in H$ with $\|u\| = r_2$, one has

$$\Phi(u) + \lambda\Psi(u) \ge d - |\lambda| \sup_{\|u\|\le r_2} |\Psi(u)| > \frac{b+d}{2}, \tag{3.9}$$

and when $\|u\| \le r_1$

$$\Phi(u) + \lambda\Psi(u) \le b + |\lambda| \sup_{\|u\|\le r_2} |\Psi(u)| < b + \frac{d-b}{2} := \frac{d+b}{2}. \tag{3.10}$$

Further, from (3.6), we have that $-\infty < \Phi(u_2) < \rho_2$. Since $\rho_2 \in (0, +\infty)$ is arbitrary, letting $\rho_2 := \theta/4 > 0$, we can obtain that

$$\Phi(u_2) < \frac{\theta}{4}. \tag{3.11}$$

Therefore, by (3.6) and (3.11), $\widehat{\lambda}$ can be chosen small enough that

$$\Phi(u_1) + \lambda\Psi(u_1) \le 0, \qquad \Phi(u_2) + \lambda\Psi(u_2) < \frac{\theta}{2}, \qquad \inf_{\|u\|=\epsilon} (\Phi(u) + \lambda\Psi(u)) \ge \frac{\theta}{2}, \tag{3.12}$$

and (3.9)-(3.10) hold, for every $\lambda \in (-\widehat{\lambda}, \widehat{\lambda})$.

For a given λ in the interval above, define the set of paths going from u_1 to u_2

$$\mathcal{A} = \{\varphi \in C([0,1], H) : \varphi(0) = u_1, \varphi(1) = u_2\}, \tag{3.13}$$

and consider the real number $c := \inf_{\varphi\in\mathcal{A}} \sup_{s\in[0,1]} (\Phi(\varphi(s)) + \lambda\Psi(\varphi(s)))$. Since $u_1 \overline{\in} B(0,\epsilon)$ and each path φ goes through $\partial B(0,\epsilon)$, one has $c \geq \theta/2$.

By (3.9) and (3.10), in the definition of c, there is no need to consider the paths going through $\partial B(0, r_2)$. Hence, there exists a sequence of paths $\{\varphi_n\} \subset \mathcal{A}$ such that $\varphi_n([0,1]) \subset B(0, r_2)$ and

$$\sup_{s\in[0,1]} (\Phi(\varphi_n(s)) + \lambda\Psi(\varphi_n(s))) \longrightarrow c \quad \text{as } n \longrightarrow +\infty. \tag{3.14}$$

Applying a general mountain pass lemma without the (PS) condition (see [17, Theorem 2.8]), there exists a sequence $\{u_n\} \subset B(0, r_2)$ such that $\Phi(u_n) + \lambda\Psi(u_n) \to c$ and $\Phi'(u_n) + \lambda\Psi'(u_n) \to 0$ as $n \to +\infty$. Hence $\{u_n\}$ is a bounded $(\mathrm{PS})_c$ sequence and, taking into account the fact that $\Phi' + \lambda\Psi'$ is an (S_+) type mapping, admits a convergent subsequence to some u_3. So, such u_3 turns to be a critical point of $\Phi + \lambda\Psi$, with $\Phi(u_3) + \lambda\Psi(u_3) = c$, hence different from u_1 and u_2 and $u_3 \neq 0$. This completes the proof. □

Taking $\lambda = 0$ in Theorem 3.1, we can obtain the existence of three distinct solutions for the Duffing-type equation without perturbation (1.2) as following.

Theorem 3.2. *Assume that (f1), (f4), and (f5) hold; then problem* (1.2) *admits at least three distinct solutions.*

Together with Lemma 2.3 and Lemma 2.4, we can easily show that the following corollary.

Corollary 3.3. *Assume that (f2), (f4), and (f5) hold; then there exist $\lambda^* > 0$ and $r > 0$ such that, for every $\lambda \in (-\lambda^*, \lambda^*)$, problem* (1.1) *admits at least three distinct solutions which belong to $B(0, r) \subseteq H$. Furthermore, problem* (1.2) *admits at least three distinct solutions.*

Corollary 3.4. *Assume that (f3), (f4), and (f5); hold, then there exist $\lambda^* > 0$ and $r > 0$ such that, for every $\lambda \in (-\lambda^*, \lambda^*)$, problem* (1.1) *admits at least three distinct solutions which belong to $B(0, r) \subseteq H$. Furthermore, problem* (1.2) *admits at least three distinct solutions.*

4. Some Examples

Example 4.1. Consider the following resonant Duffing-type equations with damping and perturbed term

$$\begin{gathered} u''(t) + \sigma u'(t) + f(t, u(t)) + \lambda g(t, u(t)) = p(t), \quad \text{a.e. } t \in [0, 2\pi], \\ u(0) = 0 = u(2\pi), \end{gathered} \tag{4.1}$$

where $\sigma = 1$, $\lambda \in \mathbf{R}$, $g(s,x) = sx^4$, $p(s) = 20\cos^2 s$, and

$$f(t,x) = \begin{cases} 20\cos^2 s + x^{1/3} & \text{for } (s,x) \in [0,2\pi] \times (-\infty,-1), \\ 20\cos^2 s + Q_1(x) & \text{for } (s,x) \in [0,2\pi] \times [-1,-0.001), \\ 20\cos^2 s - x^{1/3} & \text{for } (s,x) \in [0,2\pi] \times [-0.001,0.001], \\ 20\cos^2 s + Q_2(x) & \text{for } (s,x) \in [0,2\pi] \times (0.001,1], \\ 20\cos^2 s + x^{1/3} & \text{for } (s,x) \in [0,2\pi] \times (1,+\infty), \end{cases} \tag{4.2}$$

in which $Q_1 \in C([-1,-0.001])$ and $Q_2 \in C\ ([0.001,1])$ satisfy

$$Q_1(-1) = -1, \quad Q_1(-0.001) = 0.1, \qquad Q_2(0.001) = -0.1, \quad Q_2(1) = 1, \qquad \int_{0.001}^{1} Q_2(s)\mathrm{d}s > 1. \tag{4.3}$$

Then there exists $\lambda^* > 0$, for every $\lambda \in (-\lambda^*, \lambda^*)$, problem (8) admits at least three distinct solutions.

Proof. Obviously, from the definitions of Q_1 and Q_2, it is easy to see that $f : [0,\omega] \times \mathbf{R} \to \mathbf{R}$ is continuous and (f1) holds. Taking $\delta = 0.001$, for $(s,x) \in [0,2\pi] \times [-0.001,0) \cup (0,0.001]$, we have that

$$\begin{aligned} \frac{x^2}{2\lambda_0 e^{\sigma s}} + p(s)x - \int_0^x f(s,\mu)\mathrm{d}\mu &\geq \frac{x^2}{8e^{2\pi}} + 20\left(\cos^2 s\right)x - \left[20\left(\cos^2 s\right)x - \frac{3}{4}x^{4/3}\right] \\ &= \frac{x^2}{8e^{2\pi}} + \frac{3}{4}x^{4/3} \\ &> 0, \end{aligned} \tag{4.4}$$

which implies that (f4) is satisfied. Define

$$x_0(s) = \begin{cases} 0, & \text{for } s = 0, \\ 10^4 + \sin s, & \text{for } s \in (0,2\pi), \\ 0, & \text{for } s = 2\pi. \end{cases} \tag{4.5}$$

Clearly, $x_0 \in H$. Then we obtain that

$$\begin{aligned}\Phi(x_0(s)) &= \frac{1}{2}\int_0^{2\pi} e^s \cos^2 s\, \mathrm{d}s + 20\cos^2 t\int_0^{2\pi} e^s\left(10^4+\sin s\right)\mathrm{d}s \\ &\quad - \int_0^{2\pi} e^s\left(\int_0^{0.001} + \int_{0.001}^{1} + \int_1^{10^4+\sin s}\right) f(\mathrm{s},\mu)\,\mathrm{d}\mu\,\mathrm{d}s \\ &= \frac{e^{2\pi}\pi}{2} + \int_0^{2\pi} e^s \int_0^{0.001} \mu^{1/3}\mathrm{d}\mu\,\mathrm{d}s \\ &\quad - \int_0^{2\pi} e^s \int_{0.001}^{1} Q_2(\mu)\,\mathrm{d}\mu\,\mathrm{d}s - \int_0^{2\pi} e^s\int_1^{10^4+\sin s} \mu^{1/3}\mathrm{d}\mu\,\mathrm{d}s \\ &\le \frac{e^{2\pi}\pi}{2} - 10^4 \\ &< 0.\end{aligned} \tag{4.6}$$

So $\Phi(x_0) < 0$, which implies that (f5) is satisfied. To this end, all assumptions of Theorem 3.1 hold. By Theorem 3.1, there exists $\lambda^* > 0$, for every $\lambda \in (-\lambda^*, \lambda^*)$, problem (8) admits at least three distinct solutions. □

Example 4.2. Let $\lambda = 0$. From Example 4.1, we can obtain that the following resonant Duffing-type equations with damping:

$$\begin{gathered} u''(t) + u'(t) + 100e^{2\pi}\sqrt{x} = 10, \quad \text{a.e. } t \in [0, 2\pi], \\ u(0) = 0 = u(2\pi) \end{gathered} \tag{4.7}$$

admits at least three distinct solutions.

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